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THE SUBGROUP COMMUTATIVITY DEGREE OF FINITE *P*-GROUPS

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Abstract

The subgroup commutativity degree of a group G is the probability that two subgroups of G commute, or equivalently that the product of two subgroups is again a subgroup. For the dihedral, quasi-dihedral and generalised quaternion groups (all of 2-power cardinality), the subgroup commutativity degree tends to 0 as the size of the group tends to infinity. This also holds for the family of projective special linear groups over fields of even characteristic and for the family of the simple Suzuki groups. In this short note, we show that the family of finite P-groups also has this property.

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1. Introduction

The subgroup commutativity degree, $sd(G)$, also called *the subgroup permutability degree*, of a finite group G is defined as the probability that two subgroups of G commute, or equivalently that the product of two subgroups is again a subgroup:

$$sd(G) = \frac{1}{|L(G)|^2}|\{(H, K)\in L(G)^2 \mid HK = KH\}|,$$

where $L(G)$ is the subgroup lattice of G. Many problems related to this concept have been formulated in [6, 8]. We recall here only the problem of finding some natural families of groups G_n, $n \in \mathbb{N}$, whose subgroup commutativity degree vanishes asymptotically, that is,

$$\lim_{n\to\infty} sd(G_n) = 0. \tag{1.1}$$

It is known that the dihedral groups D_{2^n}, the quasi-dihedral groups S_{2^n}, the generalised quaternion groups Q_{2^n} (see [6]), the projective special linear groups $\mathrm{PSL}_2(2^n)$ and the simple Suzuki groups $\mathrm{Sz}(2^{2n+1})$ (see [1, 2]) satisfy (1.1). Our main result shows that the family $G_{n,p}$ of finite P-groups (defined below) also have this property.

THEOREM 1.1. *The subgroup commutativity degree of $G_{n,p}$ vanishes asymptotically, that is,*

$$\lim_{n\to\infty} sd(G_{n,p}) = 0.$$

Most of our notation is standard and will usually not be repeated here. Basic definitions and results on groups can be found in [4]. For subgroup lattice concepts, we refer the reader to [3].

2. Preliminaries

We first recall the notion of a P-group, according to [3], and explain the subgroup structure of these groups.

Let p be a prime, $n \geq 2$ be a cardinal number and G be a group. We say that G belongs to the class $P(n, p)$ if it is either elementary abelian of order p^n, or a semidirect product of an elementary abelian normal subgroup H of order p^{n-1} by a group of prime order $q \neq p$ which induces a nontrivial power automorphism on H. The group G is called a *P-group* if $G \in P(n, p)$ for some prime p and some cardinal number $n \geq 2$. It is well known that the class $P(n, 2)$ consists only of the elementary abelian group of order 2^n. Also, for $p > 2$ the class $P(n, p)$ contains the elementary abelian group of order p^n and, for every prime divisor q of $p - 1$, exactly one nonabelian P-group with elements of order q. Moreover, the order of this group is $p^{n-1}q$ if n is finite. The most important property of the groups in the class $P(n, p)$ is that they are all lattice-isomorphic (see [3, Theorem 2.2.3]). This played an essential role in [6] to produce examples of finite lattice-isomorphic groups with different subgroup commutativity degrees.

Since the subgroup commutativity degree concept is defined only for finite groups and it is trivial in the abelian case, we will focus only on finite nonabelian P-groups. So, let us suppose that $p > 2$ and $n \in \mathbb{N}$ are fixed, and take a divisor q of $p - 1$. The nonabelian group of order $p^{n-1}q$ in the class $P(n, p)$ will be denoted by $G_{n,p}$. By [3, Remark 2.2.1], it is of type

$$G_{n,p} = H\langle x\rangle,$$

where $H \cong \mathbb{Z}_p^{n-1}$ (that is, the direct product of $n - 1$ copies of $\mathbb{Z}_p$), $o(x) = q$ and there exists an integer r such that $x^{-1}hx = h^r$ for all $h \in H$.

In order to describe the subgroups of $G_{n,p}$, we need some information about the subgroups of a finite elementary abelian p-group. First of all, we recall the following well-known theorem (see, for example, [5, 7]).

THEOREM 2.1. *The number of subgroups of order p^k of the finite elementary abelian p-group $\mathbb{Z}_p^n$ is* 1 *if $k = 0$ or $k = n$, and*

$$a_{n,p}(k) = \sum_{1\leq i_1<i_2<\cdots<i_k\leq n} p^{i_1+i_2+\cdots+i_k-(k(k+1)/2)}$$

if $1 \leq k \leq n - 1$. In particular, the total number of subgroups of $\mathbb{Z}_p^n$ is

$$a_{n,p} = 2 + \sum_{k=1}^{n-1} a_{n,p}(k) = 2 + \sum_{k=1}^{n-1} \sum_{1\leq i_1<i_2<\cdots<i_k\leq n} p^{i_1+i_2+\cdots+i_k-(k(k+1)/2)}.$$

Note that an alternative way of writing the numbers $a_{n,p}(k)$, $k = 0, 1, \ldots, n$, is

$$a_{n,p}(k) = \frac{(p^n - 1)\cdots(p-1)}{(p^k - 1)\cdots(p-1)(p^{n-k} - 1)\cdots(p-1)}.$$

Properties of the above numbers will be very useful in determining $sd(G_{n,p})$.

Remark 2.2. The numbers $a_{n,p}(k)$ and $a_{n,p}$ in Theorem 2.1 satisfy the following recurrence relations:

(1) $a_{n,p}(k) = a_{n-1,p}(k) + p^{n-k} a_{n-1,p}(k-1)$ for $k = 1, 2, \ldots, n-1$;
(2) $a_{n,p} = 2a_{n-1,p} + (p^{n-1} - 1)a_{n-2,p}$.

We also remark that $a_{n,p}$ can be written as $a_{n,p} = f_n(p)$, where the polynomial $f_n \in \mathbb{Z}[X]$ is of degree $[n^2/4]$. The leading coefficient x_n of this polynomial is 1 if n is even and 2 if n is odd. Moreover, by using a computer algebra program, from (2) we can easily obtain the first terms of the integer sequence $(a_{n,p})_{n\in\mathbb{N}^*}$. Thus,

$$\begin{aligned}
a_{1,p} &= 2, \\
a_{2,p} &= p + 3, \\
a_{3,p} &= 2p^2 + 2p + 4, \\
a_{4,p} &= p^4 + 3p^3 + 4p^2 + 3p + 5, \\
a_{5,p} &= 2p^6 + 2p^5 + 6p^4 + 6p^3 + 6p^2 + 4p + 6
\end{aligned}$$

and so on.

A subgroup of $G_{n,p}$ is either cyclic if it is included in H, or a semidirect product of the same type as $G_{n,p}$ if it possesses some elements of order q. So, we can give an enumerative description of these subgroups. They are:

- one of order 1, namely the trivial subgroup H_1^1;
- $a_{n-1,p}(1)$ of order p, say H_i^p, $i = 1, \ldots, a_{n-1,p}(1)$;
- $a_{n-1,p}(2)$ of order p^2, say $H_i^{p^2}$, $i = 1, \ldots, a_{n-1,p}(2)$;

  $\vdots$
- $a_{n-1,p}(n-2)$ of order p^{n-2}, say $H_i^{p^{n-2}}$, $i = 1, \ldots, a_{n-1,p}(n-2)$;
- one of order p^{n-1}, namely $H_1^{p^{n-1}} = H$;
- p^{n-1} of order q, say H_i^q, $i = 1, \ldots, p^{n-1}$;
- $a_{n-1,p}(1)p^{n-2}$ of order pq, say H_i^{pq}, $i = 1, \ldots, a_{n-1,p}(1)p^{n-2}$;
- $a_{n-1,p}(2)p^{n-3}$ of order p^2q, say $H_i^{p^2q}$, $i = 1, \ldots, a_{n-1,p}(2)p^{n-3}$;

  $\vdots$
- $a_{n-1,p}(n-2)p$ of order $p^{n-2}q$, say $H_i^{p^{n-2}q}$, $i = 1, \ldots, a_{n-1,p}(n-2)p$;
- one of order $p^{n-1}q$, namely $H_1^{p^{n-1}q} = G_{n,p}$.

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We observe that

$$|L(G_{n,p})| = a_{n,p}$$

since $G_{n,p}$ and $\mathbb{Z}_p^n$ are lattice-isomorphic. On the other hand, by [3, Lemma 2.2.2] we infer that the normal subgroups of $G_{n,p}$ are $G_{n,p}$ itself and all subgroups contained in H. Therefore,

$$|N(G_{n,p})| = 1 + |L(H)| = 1 + |L(\mathbb{Z}_p^{n-1})| = 1 + a_{n-1,p},$$

where $N(G_{n,p})$ denotes the normal subgroup lattice of $G_{n,p}$.

3. Proof of Theorem 1.1

First, we will prove the following inequality:

$$sd(G_{n,p}) \leq \frac{a_{n-1,p}}{a_{n,p}}\Big(2 + \frac{1}{a_{n,p}}\Big). \tag{3.1}$$

For every subgroup K of $G_{n,p}$, let us denote by $C(K)$ the set of all subgroups of $G_{n,p}$ which commute with K. Then

$$sd(G_{n,p}) = \frac{1}{|L(G_{n,p})|^2} \sum_{K\in L(G_{n,p})} |C(K)| = \frac{1}{a_{n,p}^2} \sum_{K\in L(G_{n,p})} |C(K)|. \tag{3.2}$$

Moreover,

$$|C(H_i^{p^k})| = a_{n,p} \quad \text{for } k = 0, 1, \ldots, n-1 \text{ and } i = 1, 2, \ldots, a_{n-1,p}(k),$$

because all p-subgroups of $G_{n,p}$ are normal. Then (3.2) becomes

$$sd(G_{n,p}) = \frac{1}{a_{n,p}^2}\Big(a_{n-1,p}\, a_{n,p} + \sum_{k=0}^{n-1} \sum_{i=1}^{a_{n-1,p}(k)} |C(H_i^{p^k q})|\Big). \tag{3.3}$$

Assume that $k \in \{0, 1, \ldots, n-1\}$ and $i \in \{1, 2, \ldots, a_{n-1,p}(k)\}$ are fixed and take a subgroup $S \in C(H_i^{p^k q})$. Then either $S \in N(G_{n,p})$ or $q \mid |S|$. In the second case, by the equality

$$|S H_i^{p^k q}| = \frac{|S|\,|H_i^{p^k q}|}{|S \cap H_i^{p^k q}|}$$

it follows that q must divide $|S \cap H_i^{p^k q}|$ and so there is a subgroup H^q of order q of $G_{n,p}$ contained both in S and in $H_i^{p^k q}$. Thus,

$$\begin{aligned} C(H_i^{p^k q}) &= N(G_{n,p}) \cup \Big(\bigcup_{H^q\in Q} \{S \in L(G_{n,p}) \mid H^q \subseteq S\}\Big) \\ &= N(G_{n,p}) \cup \Big(\bigcup_{H^q\in Q} \{H^q T \mid T \in L(H)\}\Big), \end{aligned}$$

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where Q denotes the set of all subgroups of order q in $H_i^{p^k q}$. Since $|Q| = p^k$,

$$|C(H_i^{p^k q})| \leq |N(G_{n,p})| + |L(H)|p^k = 1 + a_{n-1,p}\,(1 + p^k).$$

This implies that

$$\sum_{k=0}^{n-1} \sum_{i=1}^{a_{n-1,p}(k)} |C(H_i^{p^k q})| \leq a_{n-1,p}\Big(1 + \sum_{k=0}^{n-1} \sum_{i=1}^{a_{n-1,p}(k)} (1 + p^k)\Big)$$
$$= a_{n-1,p}\Big(1 + a_{n-1,p} + \sum_{k=0}^{n-1} a_{n-1,p}(k)p^k\Big) = a_{n-1,p}(1 + a_{n,p}),$$

where the last equality has been obtained from the recurrence relation (1). Then (3.3) shows that

$$sd(G_{n,p}) \leq \frac{a_{n-1,p}}{a_{n,p}}\Big(2 + \frac{1}{a_{n,p}}\Big),$$

giving (3.1), as desired.

Since $a_{n,p}$ can be written as a polynomial in p of degree $[n^2/4]$ and leading coefficient $x_n \in \{1, 2\}$,

$$\lim_{n\to\infty} \frac{a_{n-1,p}}{a_{n,p}} = \lim_{n\to\infty} \frac{x_{n-1}}{x_n}\, p^{[(n-1)^2/4]-[n^2/4]} = \lim_{n\to\infty} \frac{x_{n-1}}{x_n}\, p^{-[n/2]} = 0,$$

which together with (3.1) yields

$$\lim_{n\to\infty} sd(G_{n,p}) = 0.$$

This completes the proof.

References

[1] S. Aivazidis, 'The subgroup permutability degree of projective special linear groups over fields of even characteristic', *J. Group Theory* **16** (2013), 383–396.

[2] S. Aivazidis, 'On the subgroup permutability degree of the simple Suzuki groups', *Monatsh. Math.* **176** (2015), 335–358.

[3] R. Schmidt, *Subgroup Lattices of Groups*, de Gruyter Expositions in Mathematics, 14 (de Gruyter, Berlin, 1994).

[4] M. Suzuki, *Group Theory, I, II* (Springer, Berlin, 1982, 1986).

[5] M. Tărnăuceanu, 'A new method of proving some classical theorems of abelian groups', *Southeast Asian Bull. Math.* **31** (2007), 1191–1203.

[6] M. Tărnăuceanu, 'Subgroup commutativity degrees of finite groups', *J. Algebra* **321** (2009), 2508–2520.

[7] M. Tărnăuceanu, 'An arithmetic method of counting the subgroups of a finite abelian group', *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* **53**(101) (2010), 373–386.

[8] M. Tărnăuceanu, 'Addendum to "Subgroup commutativity degrees of finite groups"', *J. Algebra* **337** (2011), 363–368.

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