Appendix G
Useful formulae for the sum rules

G.1 Laplace sum rule

Let the Laplace transform operator ($Q^2 \equiv -q^2 > 0$):

$$\mathcal{L} \equiv \lim_{\substack{n,\, Q^2 \to \infty \\ n/Q^2 \equiv \tau \text{ fixed}}} (-1)^n \frac{(Q^2)^n}{(n-1)!} \frac{\partial^n}{(\partial Q^2)^n} \,. \tag{G.1}$$

Then, one has the properties:

$$\begin{aligned}
\mathcal{L}\left[\frac{1}{(Q^2+m^2)^\alpha}\right] &= \frac{1}{\Gamma(\alpha)}\tau^\alpha e^{-m^2\tau}\,, \\
\mathcal{L}\left[\frac{1}{(Q^2)^\alpha}\ln\frac{Q^2}{\nu^2}\right] &= \frac{1}{\Gamma(\alpha)}\tau^\alpha[-\ln\tau\nu^2+\psi(\alpha)]\,, \\
\mathcal{L}\left[\frac{1}{(Q^2)^\alpha}\ln^2\frac{Q^2}{\nu^2}\right] &= \frac{1}{\Gamma(\alpha)}\tau^\alpha[\ln^2\tau\nu^2-2\psi(\alpha)\ln\tau\nu^2+\psi^2(\alpha)-\psi'(\alpha)]\,, \\
\mathcal{L}\left[\frac{1}{(Q^2)^\alpha}\ln^3\frac{Q^2}{\nu^2}\right] &= \frac{1}{\Gamma(\alpha)}\tau^\alpha[-\ln^3\tau\nu^2+3\psi(\alpha)\ln^2\tau\nu^2 \\
&\quad -(3\psi^2(\alpha)-\psi'(\alpha))\ln\tau\nu^2+\psi^3(\alpha)-3\psi(\alpha)\psi'(\alpha)+\psi''(\alpha)]\,, \\
\mathcal{L}\left[\frac{1}{x^\alpha}\frac{1}{(\ln x)^\beta}\right] &= y\,\mu(y,\beta-1,\alpha-1), \\
&\underset{y\to 0}{\simeq} \frac{1}{\Gamma(\alpha)}y^\alpha\frac{1}{(-\ln y)^\beta}\left[1+(\beta)\psi(\alpha)\frac{1}{\ln y}+\mathcal{O}\left(\frac{1}{\ln^2 y}\right)\right]\,, \\
\mathcal{L}\left[\frac{\ln\ln x}{x^\alpha(\ln x)^\beta}\right] &\underset{y\to 0}{\simeq} \frac{1}{\Gamma(\alpha)}y^\alpha\frac{\ln\ln y}{(-\ln y)^\beta}\left[1+\beta\psi(\alpha)\frac{1}{\ln y}+\mathcal{O}\left(\frac{1}{\ln^2 y}\right)\right]\,,
\end{aligned} \tag{G.2}$$

where:

$$\begin{aligned}
\mu(y,\beta,\alpha) &= \int_0^\infty dx\,\frac{x^\beta}{\Gamma(\beta+1)}\frac{y^{\alpha+x}}{\Gamma(\alpha+x+1)}\,, \\
\mu(y,-m,\alpha) &= (-1)^{m-1}\frac{d^{m-1}}{(dx)^{m-1}}\left(\frac{y^{\alpha-x}}{\Gamma(\alpha+x+1)}\right)_{x=0} \quad m=1,2,\ldots,
\end{aligned} \tag{G.3}$$

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with the properties:

$$\mu(y,-1,\alpha) = \frac{y^\alpha}{\Gamma(\alpha+1)} ,$$
$$\mu(y,-2,\alpha) = \frac{y^\alpha}{\Gamma(\alpha+1)}[-\ln y + \psi(\alpha+1)] ,$$
$$\mu(y,-3,\alpha) = \frac{y^\alpha}{\Gamma(\alpha+1)}[\ln^2 y - 2\psi(\alpha+1)\ln y + \psi^2(\alpha+1) - \psi'(\alpha+1)] . \quad (G.4)$$

For the treatment of the QCD continuum, we need the integral:

$$\int_0^{t_c} dt\, t^n\, e^{-t\tau} = (n-1)!\tau^{-n}\,(1-\rho_n) , \quad (G.5)$$

where:

$$\rho_n = e^{-t_c\tau}\left(1 + t_c\tau + \cdots + \frac{(t_c\tau)^n}{n!}\right) . \quad (G.6)$$

G.2 Finite energy sum rule

For the FESR, the integral:

$$\int_0^{t_c} dt\, t^n \ln\frac{t}{\nu^2} , \quad (G.7)$$

induces the extra-term:

$$\frac{t_c^{n+1}}{n+1}\left(-\frac{1}{n}\right) , \quad (G.8)$$

after a renormalization group improvement of the QCD series.

G.3 Coordinate space integrals

In some applications, one works in the x-space instead of the usual momentum one. Using the Fourier transform:

$$f(x) = \int \frac{d^4q}{(2\pi)^4}\, e^{iqx} f(q) , \quad (G.9)$$

one has the correspondence ($Q^2 \equiv -q^2 > 0$) for $x \to 0$ [394]:

G.4 Cauchy contour integrals

We shall be concerned with the integral entering e.g. into the τ-like decay processes (see Section 25.5), and which can be evaluated using the Cauchy contour integral along the circle of radius M_τ.

$$I_{ij} = \oint_{|s|=M_\tau^2} dt(-t)^i \left(\ln\frac{\nu^2}{-t}\right)^j . \quad (G.10)$$

Results are given for some particular values of i and j [878]. $L \equiv \ln \nu^2/M_\tau^2$.

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Table G.1. *Some useful Fourier transforms*

Q-space	x-space
$Q^2 \ln Q^2$	$\frac{8}{\pi^2}\frac{1}{x^6}$
$\ln Q^2$	$-\frac{1}{\pi^2}\frac{1}{x^4}$
$\frac{1}{Q^2}$	$\frac{1}{4\pi^2}\frac{1}{x^2}$
$\frac{1}{Q^2}\ln Q^2$	$-\frac{1}{4\pi^2}\frac{1}{x^2}\ln^2 x^2$
$\frac{1}{Q^4}$	$-\frac{1}{16\pi^2}\ln x^2$
$\frac{1}{Q^4}\ln Q^2$	$\frac{1}{64\pi^2}\ln^2 x^2$
$\frac{1}{Q^6}$	$\frac{1}{8\times 16\pi^2}x^2 \ln x^2$
$\frac{1}{Q^6}\ln Q^2$	$-\frac{1}{496\pi^2}x^2 \ln^2 x^2$
$\frac{1}{Q^8}$	$-\frac{1}{8\times 16\times 24\pi^2}x^4 \ln x^2$
$\frac{1}{Q^8}\ln Q^2$	$\frac{1}{496\times 24\pi^2}x^4 \ln^2 x^2$

Table G.2. *Some useful Cauchy integrals*

i	j	$I_{ij}/2i\pi$	i	j	$I_{ij}/2i\pi$
-3	0	0	-2	0	0
	1	$-\frac{1}{2}$		1	1
	2	$\frac{1}{2} - L$		2	$-2 + 2L$
	3	$-\frac{3}{4} + \frac{3}{2}L - \frac{3}{2}\pi L^2 + \frac{1}{2}\pi^2$		3	$6 - 6L + 3L^2 - \pi^2$
-1	0	-1	0	0	0
	1	$-L$		1	-1
	2	$-L^2 + \frac{\pi^2}{3}$		2	$-2 - 2L$
	3	$-L^3 + \pi^2 L$		3	$-6 - 6L - 3L^2 + \pi^2$
1	0	0	2	0	0
	1	$\frac{1}{2}$		1	$-\frac{1}{3}$
	2	$\frac{1}{2} + L$		2	$-\frac{2}{9} - \frac{2}{3}L$
	3	$\frac{3}{4} + \frac{3}{2}L + \frac{3}{2}L^2 - \frac{\pi^2}{2}$		3	$-\frac{2}{9} - \frac{2}{3}L - L^2 + \frac{\pi^2}{3}$
3	0	0			
	1	$\frac{1}{4}$			
	2	$\frac{1}{8} + \frac{1}{2}L$			
	3	$\frac{3}{32} + \frac{3}{8}L + \frac{3}{4}L^2 - \frac{\pi^2}{4}$			