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Range Spaces of Co-Analytic Toeplitz Operators

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Abstract. In this paper we discuss the range of a co-analytic Toeplitz operator. These range spaces are closely related to de Branges–Rovnyak spaces (in some cases they are equal as sets). In order to understand its structure, we explore when the range space decomposes into the range of an associated analytic Toeplitz operator and an identifiable orthogonal complement. For certain cases, we compute this orthogonal complement in terms of the kernel of a certain Toeplitz operator on the Hardy space, where we focus on when this kernel is a model space (backward shift invariant subspace). In the spirit of Ahern–Clark, we also discuss the non-tangential boundary behavior in these range spaces. These results give us further insight into the description of the range of a co-analytic Toeplitz operator as well as its orthogonal decomposition. Our Ahern–Clark type results, which are stated in a general abstract setting, will also have applications to related sub-Hardy Hilbert spaces of analytic functions such as the de Branges–Rovnyak spaces and the harmonically weighted Dirichlet spaces.

1 Introduction

In this paper we consider the range space $\mathscr{M}(\overline{a}) := T_{\overline{a}}H^2$ of the co-analytic Toeplitz operator $T_{\overline{a}}$ on the classical Hardy space H^2. In the above, a is a bounded analytic function on the open unit disk and $\mathscr{M}(\overline{a})$ is endowed with the range norm (3.1), making it a Hilbert space. These spaces are closely related to, and in certain cases equal to, a de Branges–Rovnyak space $\mathscr{H}(b)$. We explore various aspects of $\mathscr{M}(\overline{a})$ and focus on several key questions.

Our first set of questions deals with the actual contents of $\mathscr{M}(\overline{a})$. One can show (Proposition 3.5) that $aH^2 \subset \mathscr{M}(\overline{a})$ with contractive inclusion. We first ask when is aH^2 closed in $\mathscr{M}(\overline{a})$ in the range topology? Of course, this question makes no sense in the H^2-norm. Next, when this is true, what is the corresponding orthogonal complement of aH^2 in $\mathscr{M}(\overline{a})$? We resolve our first question in Proposition 4.2 where we show that aH^2 is closed in $\mathscr{M}(\overline{a})$ if and only if the Toeplitz operator $T_{\overline{a}/a}$ is surjective on H^2. This last condition can be rephrased in terms of a certain Muckenhoupt condition (Theorem 4.3). For our second question, we argue in Section 4 that

$$\mathscr{M}(\overline{a}) = aH^2 \oplus_{\overline{a}} T_{\overline{a}} \operatorname{Ker} T_{\overline{a}/a}, \tag{1.1}$$

where $\oplus_{\overline{a}}$ denotes the orthogonal direct sum in the Hilbert space $\mathscr{M}(\overline{a})$. In Theorem 4.5 we obtain a formula for the orthogonal projection of $\mathscr{M}(\overline{a})$ onto $T_{\overline{a}} \operatorname{Ker} T_{\overline{a}/a}$.

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A description of $T_{\overline{a}} \operatorname{Ker} T_{\overline{a}/a}$ can be complicated. Recall that kernels of Toeplitz operators are so-called nearly invariant subspaces (see Section 2). In Proposition 4.6 we describe those outer a for which $\operatorname{Ker} T_{\overline{a}/a}$ is a model space $K_I = H^2 \ominus IH^2$ (I is some suitable inner function), and in this case, Lemma 4.9 describes when $T_{\overline{a}} K_I = K_I$. Based on these results, we discuss the decomposition in (1.1) (see Section 5) when the outer function a is equal to

$$\begin{aligned} a &= (1+I)^m, \quad m \in \mathbb{N}, I \text{ inner},\\ a &= \prod_{1 \leqslant j \leqslant n} (z - e^{i\theta_j})^{m_j}, \quad m_j \in \mathbb{N},\\ a &= \prod_{1 \leqslant j \leqslant n} (e^{i\theta_j} - I_j)^{m_j}, \quad m_j \in \mathbb{N}, I_j \text{ inner}. \end{aligned}$$

We will show that the orthogonal complement $T_{\overline{a}} \operatorname{Ker} T_{\overline{a}/a}$ is equal to a model space in the first two examples, while in the last example, the situation can be more complicated.

Next we focus on the boundary behavior of functions in $\mathscr{M}(\overline{a})$. Functions, along with their derivatives, in the so-called sub-Hardy Hilbert spaces can have more regularity at particular $\zeta_0 \in \mathbb{T} = \{|z| = 1\}$ than generic functions in H^2. Broadly speaking, these types of results say that when certain conditions are satisfied, then *every* function in a given sub-Hardy Hilbert space has a non-tangential limit at $\zeta_0 \in \mathbb{T}$.

In Theorem 6.3 and Corollary 6.6, we will show, for fixed $\zeta_0 \in \mathbb{T}$ and $N \in \mathbb{N}_0$, that every $f \in \mathscr{M}(\overline{a})$, along with $f', f'', \dots, f^{(N)}$, has a finite non-tangential limit at ζ_0 if and only if

$$\int_0^{2\pi} \frac{|a(e^{i\theta})|^2}{|e^{i\theta} - \zeta_0|^{2N+2}} \, d\theta < \infty. \tag{1.2}$$

Condition (1.2) is in the same spirit as the well-known Ahern–Clark condition for existence of non-tangential boundary values in model spaces [1] and its extensions to other sub-Hardy Hilbert spaces [2, 11, 12, 33].

Observe how the integral in (1.2) depends on the strength of the zero of a at ζ_0. We will use this observation to show (Proposition 6.8) that there is no point $\zeta_0 \in \mathbb{T}$ for which every function in $\mathscr{M}(\overline{a})$ has an analytic continuation to an open neighborhood of ζ_0. This is in contrast to the model spaces K_J where, under certain circumstances, every function in K_J has an analytic continuation across a portion of $\mathbb{T}$ [6]. Such a phenomenon also takes place in certain de Branges–Rovnyak spaces [11]. We point out that our boundary behavior results for $\mathscr{M}(\overline{a})$ make connections to analogous results for the range spaces $T_{\overline{a}} K_J$ [20].

The technique originally used by Ahern and Clark, and extended by others, to discover conditions ensuring existence of non-tangential limits, was to control the norm of the reproducing kernels as one approached the boundary point $\zeta_0 \in \mathbb{T}$. Indeed, if $k_\lambda(z) = (1 - \overline{\lambda} z)^{-1}$ is the standard reproducing kernel for the Hardy space, then

$$(T_{\overline{a}} f)(\lambda) = \langle T_{\overline{a}} f, k_\lambda \rangle_{H^2} = \langle f, a k_\lambda \rangle_{H^2} = \int_0^{2\pi} f(e^{i\theta}) \frac{\overline{a(e^{i\theta})}}{1 - e^{-i\theta}\lambda} \frac{d\theta}{2\pi},$$

and one can see how the condition in (1.2) plays a role in determining the existence of the non-tangential limits of $T_{\overline{a}} f$ at ζ_0. However, as already pointed out by Ahern and Clark, passing to the limit involves a more delicate analysis. We will explore this Ahern–Clark technique in a broader setting to not only capture the boundary behavior of functions in the range spaces $\mathscr{M}(\overline{a})$, the primary focus of this paper, but also the de Branges–Rovnyak spaces $\mathscr{H}(b)$ and the harmonically weighted Dirichlet spaces $\mathscr{D}(\mu)$.

Finally, as an application of our results on decomposition and boundary behavior, we generalize the results from [10, 26] to decompose the de Branges–Rovnyak spaces $\mathscr{H}(b)$ for certain b (see Theorem 6.13).

2 Some Reminders

Let H^2 denote the classical Hardy space of the unit disk $\mathbb{D}$ [7, 18] endowed, via radial boundary values, with the standard L^2 inner product $\langle f, g\rangle_{H^2} := \int_{\mathbb{T}} f\overline{g}\,dm$, where m is normalized Lebesgue measure on $\mathbb{T}$. For an inner function I, we let $K_I = H^2 \ominus IH^2$ denote the model space [14, 27] corresponding to I.

Recall that H^2 is a reproducing kernel Hilbert space with kernel

(2.1)
$$k_\lambda(z) := \frac{1}{1-\overline{\lambda}z}, \quad \lambda, z \in \mathbb{D}.$$

If $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, $\lambda \in \mathbb{D}$, and

(2.2)
$$k_{\lambda,n}(z) := \frac{n! z^n}{(1-\overline{\lambda}z)^{n+1}},$$

then $k_{\lambda,n}$ is the reproducing kernel for the n-th derivative at λ in that

(2.3)
$$f^{(n)}(\lambda) = \langle f, k_{\lambda,n}\rangle_{H^2}, \quad f \in H^2.$$

For a symbol $\varphi \in L^\infty$ define the standard *Toeplitz operator* T_φ on H^2 by $T_\varphi f := P_+(\varphi f)$, where $P_+ : L^2 \to H^2$ is the Riesz projection. When $\varphi \in H^\infty$, T_φ is called an *analytic Toeplitz operator* and is given by the simple formula $T_\varphi f = \varphi f$, while $T_\varphi^* = T_{\overline{\varphi}}$ is called a *co-analytic Toeplitz operator*. Note that $T_{\overline{z}} f = (f - f(0))/z$ is the backward shift operator.

Below are some useful facts about Toeplitz operators from [13, 27, 28].

Proposition 2.1 *Let $\varphi, \psi \in L^\infty$.*

(i) *If $\varphi \in H^\infty$, then $T_{\overline{\varphi}} k_\lambda = \overline{\varphi(\lambda)} k_\lambda$ for every $\lambda \in \mathbb{D}$.*
(ii) *If $\varphi \in H^\infty$ and outer, then the Toeplitz operators T_φ, $T_{\overline{\varphi}}$, and $T_{\varphi/\overline{\varphi}}$ are injective.*
(iii) *If at least one of φ, ψ belongs to H^∞, then $T_{\overline{\psi}} T_\varphi = T_{\overline{\psi}\varphi}$.*
(iv) *If $\varphi \in H^\infty$ and I is the inner factor of φ, then* $\operatorname{Ker} T_{\overline{\varphi}} = K_I$, *where $K_I = (IH^2)^\perp = H^2 \ominus IH^2$.*
(v) *If $\varphi \in H^\infty$ and J is inner, then $T_{\overline{\varphi}} K_J \subset K_J$.*

The kernel $\operatorname{Ker} T_\varphi$ of a Toeplitz operator is well studied and will play an important role in our orthogonal decomposition of $\mathscr{M}(\overline{a})$. Let us recall a few results in this area. A subspace M of H^2 is said to be *nearly invariant* if $f \in M$, $f(0) = 0 \Rightarrow \frac{f}{z} \in M$. It is easy to check that $\operatorname{Ker} T_\varphi$ is nearly invariant for any $\varphi \in L^\infty$. The following is a description of all of the nearly invariant subspaces of H^2.

Theorem 2.2 (Hitt [25], Sarason [32]) *Let M be a non-trivial nearly invariant subspace of H^2. If γ is the unique solution to the extremal problem*

$$\sup\left\{\Re g(0) : g \in M, \|g\|_{H^2} \leqslant 1\right\}, \tag{2.4}$$

then there is an inner function I with $I(0) = 0$ such that $M = \gamma K_I$. Furthermore, $\gamma(0) \neq 0$, γ is an isometric multiplier from K_I onto γK_I and can be written as $\gamma = \alpha/(1-\beta_0 I)$, where $\alpha, \beta_0 \in H^\infty$ and $|\alpha|^2 + |\beta_0|^2 = 1$ almost everywhere on $\mathbb{T}$.

Conversely, every space of the form $M = \gamma K_I$, $\gamma(0) \neq 0$, with $\gamma = \alpha/(1 - I\beta_0)$, $\alpha, \beta_0 \in H^\infty$, $|\alpha|^2 + |\beta_0|^2 = 1$ almost everywhere on $\mathbb{T}$, and I inner with $I(0) = 0$, is nearly invariant with the associated extremal function γ.

The parameters γ and $\beta = I\beta_0$ are related by the following formula from [32]:

$$\frac{1+\beta(z)}{1-\beta(z)} = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} |\gamma(\zeta)|^2 dm(\zeta), \quad z \in \mathbb{D}. \tag{2.5}$$

Hayashi [23] identified those nearly invariant subspaces which are kernels of Toeplitz operators. With the notation from Theorem 2.2, set $\gamma_0 := \alpha/(1-\beta_0)$. Using Herglotz's Theorem and the theory of Poisson integrals, it can be shown that γ_0^2 belongs to the Hardy space H^1.

Theorem 2.3 (Hayashi [23]) *A non trivial nearly invariant subspace M is the kernel of a Toeplitz operator if and only if γ_0^2 is rigid in H^1.*

Recall that the H^1 function γ_0^2 is said to be *rigid* if the only H^1 functions having the same argument as γ_0^2 almost everywhere on $\mathbb{T}$ are $\{c\gamma_0^2 : c > 0\}$. One can show that if g and $1/g$ both belong to H^1, then g is rigid. The converse is not always true.

Observe that the extremal function for the kernel of a Toeplitz operator is necessarily outer. In particular, one sees that α is always outer.

If γ is the extremal function for $\operatorname{Ker} T_\varphi$ from (2.4) with associated inner function I, then

$$\operatorname{Ker} T_\varphi = \gamma K_I = \operatorname{Ker} T_{\overline{I\gamma}/\gamma}. \tag{2.6}$$

Note that when γ_0^2 is rigid, then $T_{\overline{\gamma_0}/\gamma_0}$ is injective [33, Theorem X-2]. In this paper we will also need a stronger property, namely the invertibility of $T_{\overline{\gamma_0}/\gamma_0}$. This is characterized in [21] by the well-known Muckenhoupt (A_2)-condition, which is itself connected to the surjectivity of T_φ.

Theorem 2.4 (Hartmann–Sarason–Seip [21]) *With the notation above, suppose that $\operatorname{Ker} T_\varphi \neq \{0\}$. Then the Toeplitz operator T_φ is surjective if and only if $|\gamma_0|^2$ is an (A_2) weight, meaning*

$$\sup_J \left(\frac{1}{m(J)} \int_J |\gamma_0|^2 \, dm\right)\left(\frac{1}{m(J)} \int_J |\gamma_0|^{-2} \, dm\right) < \infty, \tag{2.7}$$

where the supremum above is taken over all arcs $J \subset \mathbb{T}$.

3 Range Spaces

For a bounded linear operator $A: H^2 \to H^2$, define the *range space*

$$\mathscr{M}(A) := AH^2 = \operatorname{Rng}(A)$$

and endow it with the *range norm*

$$\|Af\|_{\mathscr{M}(A)} := \|f\|_{H^2}, \quad f \in H^2 \ominus \operatorname{Ker} A = (\operatorname{Ker} A)^{\perp}. \tag{3.1}$$

The induced inner product

$$\langle Af, Ag\rangle_{\mathscr{M}(A)} := \langle f, g\rangle_{H^2}, \quad f, g \in H^2 \ominus \operatorname{Ker} A, \tag{3.2}$$

makes $\mathscr{M}(A)$ a Hilbert space and A a partial isometry with initial space $H^2 \ominus \operatorname{Ker} A$ and final space AH^2. Moreover, one can show that

$$\langle f, AA^* g\rangle_{\mathscr{M}(A)} = \langle f, g\rangle_{H^2}, \quad f \in \mathscr{M}(A), g \in H^2. \tag{3.3}$$

These range spaces $\mathscr{M}(A)$, as well as their complementary spaces, were formally introduced by Sarason [33], though they appeared earlier in the work of de Branges and Rovnyak [4, 5]. We will discuss this connection in a moment.

Since $\mathscr{M}(A)$ is boundedly contained in H^2, we see that for fixed $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{D}$, the linear functional $f \mapsto f^{(n)}(\lambda)$ is continuous on $\mathscr{M}(A)$. By the Riesz representation theorem, this functional is given by a reproducing kernel $k_{\lambda,n}^{\mathscr{M}(A)} \in \mathscr{M}(A)$, that is to say, $f^{(n)}(\lambda) = \langle f, k_{\lambda,n}^{\mathscr{M}(A)}\rangle_{\mathscr{M}(A)}, f \in \mathscr{M}(A)$.

Recall the definition of the (Cauchy) kernel $k_{\lambda,n}$ from (2.2).

Proposition 3.1 *For fixed $\lambda \in \mathbb{D}$ and $n \in \mathbb{N}_0$, we have $k_{\lambda,n}^{\mathscr{M}(A)} = AA^* k_{\lambda,n}$.*

Proof For any $f \in \mathscr{M}(A)$, use (3.3) along with (2.3) to get

$$\langle f, AA^* k_{\lambda,n}\rangle_{\mathscr{M}(A)} = \langle f, k_{\lambda,n}\rangle_{H^2} = f^{(n)}(\lambda). \qquad \blacksquare$$

When $A = T_{\overline{a}}$ $(a \in H^\infty)$, we obtain a nice formula for the kernel function.

Corollary 3.2 *For each $\lambda \in \mathbb{D}$ and $n \in \mathbb{N}_0$, we have $k_{\lambda,n}^{\mathscr{M}(T_{\overline{a}})} = T_{\overline{a}} a k_{\lambda,n} = T_{|a|^2} k_{\lambda,n}$.*

Proof Note that $T_{\overline{a}}^* = T_a$ and apply Propositions 3.1 and 2.1 (iii). $\blacksquare$

Remark 3.3 Since $\mathscr{M}(T_{\overline{a}})$ is the primary focus on this paper, we will use the following less cumbersome notation:

$$\mathscr{M}(a) := \mathscr{M}(T_a), \qquad \mathscr{M}(\overline{a}) := \mathscr{M}(T_{\overline{a}}), \qquad \langle \cdot, \cdot \rangle_{\overline{a}} := \langle \cdot, \cdot \rangle_{\mathscr{M}(T_{\overline{a}})},$$
$$k_{\lambda,n}^{\overline{a}} := k_{\lambda,n}^{\mathscr{M}(T_{\overline{a}})}, \qquad k_{\lambda}^{\overline{a}} := k_{\lambda,0}^{\overline{a}}.$$

Let us mention a few more structural details concerning $\mathscr{M}(\overline{a})$. For any $a \in H^\infty$, let a_0 be the outer factor of a [7, Chapter 2].

Proposition 3.4 ([13, Lemma 17.3]) *$\mathscr{M}(\overline{a}) = \mathscr{M}(\overline{a_0})$ as Hilbert spaces.*

When a is outer, the Toeplitz operator $T_{\overline{a}}$ is injective, and hence the corresponding inner product $\langle\cdot,\cdot\rangle_{\overline{a}}$ on $\mathscr{M}(\overline{a})$ from (3.2) becomes

$$\langle T_{\overline{a}}f, T_{\overline{a}}g\rangle_{\overline{a}} = \langle f,g\rangle_{H^2}, \quad f,g\in H^2. \tag{3.4}$$

Use the identity $T_a = T_{\overline{a}}T_{a/\overline{a}}$ (Proposition 2.1) to see the following.

Proposition 3.5 ([13, 33]) *For $a\in H^\infty$ we have $\mathscr{M}(a)\subset\mathscr{M}(\overline{a})$ with contractive inclusion.*

To connect the results of this paper with those of [10, 26], let us briefly recall some facts about the de Branges–Rovnyak spaces [13, 33]. For $b\in H_1^\infty$, the closed unit ball in H^∞, observe that T_b is contractive, which allows us to define $A := (I - T_bT_{\overline{b}})^{1/2}$. The *de Branges–Rovnyak space* $\mathscr{H}(b)$ is defined to be

$$\mathscr{H}(b) := \mathscr{M}(A), \tag{3.5}$$

endowed with the corresponding range norm from (3.1). Analogously, we have $\mathscr{H}(\overline{b}) := \mathscr{M}(A)$ when $A := (I - T_{\overline{b}}T_b)^{1/2}$.

Remark 3.6 In a similar vein to Remark 3.3, we set $\langle\cdot,\cdot\rangle_b := \langle\cdot,\cdot\rangle_{\mathscr{M}(A)}$, $k_{\lambda,n}^b := k_{\lambda,n}^{\mathscr{M}(A)}$, $k_\lambda^b := k_{\lambda,0}^b$, when $A = (I - T_bT_{\overline{b}})^{1/2}$ and $n\in\mathbb{N}_0$.

When $\|b\|_\infty < 1$, we have $\mathscr{H}(b) = H^2$ with an equivalent norm. If $b = I$ is an inner function, then $\mathscr{H}(I) = K_I := H^2\ominus IH^2$ is a model space endowed with the H^2 norm.

Suppose $a\in H_1^\infty$ is outer and satisfies $\log(1-|a|)\in L^1 = L^1(\mathbb{T},m)$. This log integrability condition is equivalent to the fact that a is a *non-extreme* point of H_1^∞. Then there exists an outer function b, unique if we require the additional condition that $b(0) > 0$ that satisfies $|a|^2 + |b|^2 = 1$ almost everywhere on $\mathbb{T}$. We call b, which is necessarily in H_1^∞, the *Pythagorean mate* for a. If $\mathscr{H}(b)$ is the associated de Branges–Rovnyak space from (3.5), it is known [33, p. 24] that $\mathscr{M}(a)\subset\mathscr{M}(\overline{a})\subset\mathscr{H}(b)$, though neither $\mathscr{M}(a)$ nor $\mathscr{M}(\overline{a})$ is, in general, closed in $\mathscr{H}(b)$. Still, $\mathscr{M}(\overline{a})$ is always dense in $\mathscr{H}(b)$. Furthermore, when (a,b) is a *corona pair*, that is to say,

$$\inf\big\{|a(z)| + |b(z)| : z\in\mathbb{D}\big\} > 0, \tag{3.6}$$

then $\mathscr{H}(b) = \mathscr{M}(\overline{a})$ [13, Theorem 28.7] or [33]. The equality $\mathscr{M}(\overline{a}) = \mathscr{H}(b)$ is a set equality but the norms, though equivalent by the closed graph theorem, need not be equal.

4 An Orthogonal Decomposition

The goal of this section is to discuss the following questions concerning $\mathscr{M}(\overline{a})$, endowed with its range norm. When is $\mathscr{M}(a)$ a closed subspace of $\mathscr{M}(\overline{a})$? What is the orthogonal complement of $\mathscr{M}(a)$ in $\mathscr{M}(\overline{a})$?

To avoid trivialities, we point out the following.

Proposition 4.1 *Let $a\in H^\infty$ be outer.*

(i) *If $T_{\overline{a}}$ is surjective, then $\mathscr{M}(a) = \mathscr{M}(\overline{a}) = H^2$.*

(ii) *$\mathscr{M}(a) = \mathscr{M}(\overline{a})$ if and only if $T_{a/\overline{a}}$ is surjective.*

Proof (i) From Proposition 2.1 (ii) we know that $T_{\overline{a}}$ is injective. Thus if $T_{\overline{a}}$ were surjective it would also be invertible (as would T_a). Hence $\mathscr{M}(a) = \mathscr{M}(\overline{a}) = H^2$.

(ii) As we have already seen,

$$\mathscr{M}(a) = T_{\overline{a}} T_{a/\overline{a}} H^2, \tag{4.1}$$

and since $T_{\overline{a}}$ is injective, we get that $\mathscr{M}(a) = \mathscr{M}(\overline{a}) \Leftrightarrow T_{a/\overline{a}}H^2 = H^2$. ∎

Henceforth, we will assume that $T_{a/\overline{a}}$ is not surjective. This next result determines when $\mathscr{M}(a)$ is closed in $\mathscr{M}(\overline{a})$.

Proposition 4.2 *For $a \in H^\infty$ and outer, the following are equivalent.*

(i) *$\mathscr{M}(a)$ is a closed subspace of $\mathscr{M}(\overline{a})$.*
(ii) *$T_{a/\overline{a}}H^2$ is a closed subspace of H^2.*
(iii) *$T_{a/\overline{a}}$ is left invertible.*
(iv) *$T_{\overline{a}/a}$ is surjective.*

Proof Using (4.1) and the fact that $T_{\overline{a}}$ is an isometry from H^2 onto $\mathscr{M}(\overline{a})$, we see that $\mathscr{M}(a)$ is a closed subspace of $\mathscr{M}(\overline{a})$ if and only if $T_{a/\overline{a}}H^2$ is a closed subspace of H^2. This proves (i) $\Leftrightarrow$ (ii). The remaining implications follow from standard ideas using the observation that $T_{a/\overline{a}}$ is injective (see Proposition 2.1 (ii)). ∎

Thus, when $\mathscr{M}(a)$ is a closed subspace of $\mathscr{M}(\overline{a})$, the operator $T_{a/\overline{a}}$ has closed range. Hence,

$$H^2 = T_{a/\overline{a}}H^2 \oplus_{H^2} (H^2 \ominus_{H^2} T_{a/\overline{a}}H^2) = T_{a/\overline{a}}H^2 \oplus_{H^2} \operatorname{Ker} T_{\overline{a}/a}. \tag{4.2}$$

Since a is outer, then $T_{\overline{a}}$ is injective (Proposition 2.1 (ii)) and so by (3.4), $T_{\overline{a}}$ is an isometry from H^2 onto $\mathscr{M}(\overline{a})$. Applying $T_{\overline{a}}$ to both sides of (4.2) and using the earlier mentioned operator identity $T_{\overline{a}}T_{a/\overline{a}} = T_a$ (Proposition 2.1 (iii)), we obtain

$$\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} T_{\overline{a}} \operatorname{Ker} T_{\overline{a}/a}. \tag{4.3}$$

This brings us to some of the subtleties of $\operatorname{Ker} T_{\overline{a}/a}$ discussed earlier. Note that $\operatorname{Ker} T_{\overline{a}/a} \neq \{0\}$, since $T_{a/\overline{a}}$ is not surjective but left invertible. Thus from Theorem 2.2 and the discussion thereafter, we have $\operatorname{Ker} T_{\overline{a}/a} = \gamma K_I$, where $\gamma = \frac{\alpha}{1-\beta_0 I}$, $\alpha \in H_1^\infty$ is outer, β_0 is a Pythagorean mate, and I is an inner function with $I(0) = 0$. As a consequence of Proposition 4.2 and Theorem 2.4, we see that $T_{a/\overline{a}}$ has closed range if and only if $|\gamma_0|^2$ is an (A_2) weight, where $\gamma_0 = \frac{\alpha}{1-\beta_0}$. Thus $\mathscr{M}(a)$ is a closed non-trivial subspace of $\mathscr{M}(\overline{a})$ if and only if $|\gamma_0|^2$ is an (A_2) weight. We summarize this discussion with the following.

Theorem 4.3 *Let $a \in H^\infty$ be outer.*

(i) *Then $\mathscr{M}(a)$ is a closed subspace of $\mathscr{M}(\overline{a})$ if and only if $|\gamma_0|^2$ is an (A_2) weight.*
(ii) *If γ and I are the associated functions as above, then $\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} T_{\overline{a}}(\gamma K_I)$.*

Although Theorem 4.3 seems implicit, it actually yields a recipe to construct non-trivial examples of decompositions of $\mathscr{M}(\overline{a})$. For example, choose an outer $\alpha \in H_1^\infty$

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such that its Pythagorean mate β_0 satisfies the property that $|\gamma_0|^2$ is an (A_2) weight. We will see a specific example of this in a moment. As mentioned earlier, the (A_2) condition implies that γ_0^2 is a rigid function. Let I be any inner function with $I(0) = 0$ and set $\gamma = \frac{\alpha}{1-I\beta_0}$. From (2.6) we have $\gamma K_I = \operatorname{Ker} T_{\overline{I\gamma}/\gamma}$. With $a = (1+I)\gamma$ we have $\frac{\overline{a}}{a} = \frac{\overline{I\gamma}}{\gamma}$ almost everywhere on $\mathbb{T}$ and so $\operatorname{Ker} T_{\overline{a}/a} = \gamma K_I$, whence

$$\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} T_{\overline{a}}(\gamma K_I). \tag{4.4}$$

Here is an example that uses this recipe.

Example 4.4 Let $\varepsilon \in (0, \frac{1}{2})$ and define the outer function $\alpha \in H_1^\infty$ by

$$\alpha(z) = \Big(\frac{1-z}{2}\Big)^\varepsilon.$$

With β_0 the outer Pythagorean mate for α, an estimate from [22, p. 359–360] yields

$$|1-\beta_0(\zeta)| \asymp |1-\zeta|^{2\varepsilon}, \quad \zeta \in \mathbb{T}.$$

From this we see that the function $\gamma_0 = \alpha/(1-\beta_0)$ satisfies $|\gamma_0(\zeta)| \asymp |1-\zeta|^{-\varepsilon}$ for all $\zeta \in \mathbb{T}$. A routine estimate will show that (2.7) holds and so $|\gamma_0|^2$ is an (A_2) weight. For any inner I with $I(0) = 0$, define $\gamma = \alpha/(1-I\beta_0)$ and $a = \gamma(1+I)$ and follow the above argument to obtain the decomposition in (4.4).

It is also possible to start from $\gamma_0(z) = (1-z)^\varepsilon$. Then β_0 can be expressed using the integral representation (2.5) and $\alpha = \gamma_0(1-\beta_0)$.

Before discussing several interesting special cases where we can obtain more precise information on the orthogonal complement $T_{\overline{a}}(\gamma K_I)$, we would like to give a formula for the orthogonal projection of $\mathscr{M}(\overline{a})$ onto $T_{\overline{a}}(\gamma K_I)$.

Theorem 4.5 *In the notation from Theorem 4.3, let P_I denote the orthogonal projection of H^2 onto K_I. Then*

$$P = T_{\overline{a}}\gamma P_I \overline{\gamma} T_{1/\overline{a}} \tag{4.5}$$

is the orthogonal projection of $\mathscr{M}(\overline{a})$ onto $T_{\overline{a}}(\gamma K_I)$.

In the formula for the projection P from (4.5), we need to be more precise about the meaning of the operator $T_{1/\overline{a}}$ since, in general, $1/\overline{a} \notin L^\infty$ (a necessary and sufficient condition for a Toeplitz operator to be bounded). To do this, we use the results of [34] on unbounded Toeplitz operators with symbols in the Smirnov class

$$N^+ := \{f/g : f, g \in H^\infty, g \text{ outer}\}.$$

See [7, Chapter 2] for more on N^+. Note that $\varphi := 1/a \in N^+$. It was shown in [34, Proposition 3.1] that we can uniquely decompose φ as $\varphi = \frac{b_0}{a_0}$, where $a_0, b_0 \in H_1^\infty$, a_0 is an outer function, $a_0(0) > 0$, and $|a_0|^2 + |b_0|^2 = 1$ almost everywhere on $\mathbb{T}$. Moreover, the unbounded Toeplitz operator $T_\varphi : \mathcal{D}(T_\varphi) \to H^2$, $T_\varphi f = \varphi f$, $f \in \mathcal{D}(T_\varphi)$, is closed and densely defined with domain $\mathcal{D}(T_\varphi)$ equal to $a_0 H^2$. Hence, its adjoint T_φ^* is also densely defined and closed. Using [34, Proposition 5.4], we have $\mathcal{D}(T_\varphi^*) = \mathscr{H}(b_0)$, where $\mathscr{H}(b_0)$ denotes the de Branges–Rovnyak space from (3.5). We now define $T_{\overline{\varphi}}$ to be T_φ^*. Using the facts that $a_0 = ab_0$ and $|a_0|^2 + |b_0|^2 = 1$ almost everywhere on $\mathbb{T}$, it is easy to see that $|a_0|^2 = |a|^2/(1+|a|^2)$, almost everywhere on $\mathbb{T}$, whence $|a|^2/2 \leqslant |a_0|^2 \leqslant |a|^2$ almost everywhere on $\mathbb{T}$. This implies that a/a_0 and a_0/a belong to L^∞. Using [10, Proposition 3.6], we see that $\mathscr{M}(\overline{a}) = \mathscr{M}(\overline{a_0})$ which yields the containment

$$\mathscr{M}(\overline{a}) \subset \mathscr{H}(b_0) = \mathcal{D}(T_{\overline{\varphi}}).$$

Now using [34, Proposition 6.5], we have

$$T_{1/\overline{a}} T_{\overline{a}} f = f, \quad f \in H^2. \tag{4.6}$$

This last fact is important for the proof of Theorem 4.5.

Proof of Theorem 4.5 From Theorem 2.2 we know that γ is an isometric multiplier of K_I. The operator $P_0 := \gamma P_I \overline{\gamma}$ is the orthogonal projection from H^2 onto γK_I. Indeed, it is clear that its range is γK_I. From Theorem 2.2 we deduce that $P_0(\gamma f) = \gamma f$ when $f \in K_I$. Finally, it is straightforward to see that $P_0 f = 0$ whenever $f \perp \gamma K_I$. Using (4.6), the result now follows by composition. ∎

The expression of the orthogonal complement $T_{\overline{a}}(\gamma K_I)$ in Theorem 4.3 is not quite satisfactory, since it is rather implicit. The next result aims at characterizing the situation where $\gamma = 1$ (equivalently when $a/\overline{a}$ is inner, see Proposition 4.6), and, when this is the case, whether or not $T_{\overline{a}} K_I = K_I$.

Before proving the following result which answers the first question, we need a so-called Frostman shift of the inner function I. This is given by

$$J = \frac{I - I(0)}{1 - \overline{I(0)} I}. \tag{4.7}$$

Observe that J is inner with $J(0) = 0$. Moreover, $K_I = (1 - \overline{I(0)} I) K_J = k_0^I K_J$, where $k_0^I = 1 - \overline{I(0)} I$ is the reproducing kernel for K_I at 0 (see [3]).

In particular, taking into account that $T_{1/\overline{k_0^I}}$ is invertible,

$$\operatorname{Ker} T_{\overline{a}/a} = K_I \Longleftrightarrow \operatorname{Ker} T_{\overline{a}/(a/k_0^I)} = K_J \Longleftrightarrow \operatorname{Ker} T_{\overline{(a/k_0^I)}/(a/k_0^I)} = K_J.$$

Proposition 4.6 *For a bounded outer function a and an inner function I, the following are equivalent.*

(i) $\operatorname{Ker} T_{\overline{a}/a} = K_I$.

(ii) $a = i\frac{\Theta_1 - i\Theta_2}{\Theta_1 + i\Theta_2}(1 + \xi J) k_0^I$, *where $\xi \in \mathbb{T}$, Θ_1, and Θ_2 are inner functions, J is the Frostman shift of I from (4.7), and $\Theta_1 + i\Theta_2$ is outer.*

Proof Suppose that $a = i\frac{\Theta_1 - i\Theta_2}{\Theta_1 + i\Theta_2}(1 + \xi J) k_0^I$. Then

$$\frac{\overline{a}}{a} = -\overline{\Big(\frac{\Theta_1 - i\Theta_2}{\Theta_1 + i\Theta_2}\Big)} \frac{\Theta_1 + i\Theta_2}{\Theta_1 - i\Theta_2} \frac{\overline{1 + \xi J}}{1 + \xi J} \frac{\overline{k_0^I}}{k_0^I} = -\frac{\overline{\Theta}_1 + i\overline{\Theta}_2}{\overline{\Theta}_1 - i\overline{\Theta}_2} \frac{\Theta_1 + i\Theta_2}{\Theta_1 - i\Theta_2} \overline{\xi J} \frac{1 - I(0)\overline{I}}{1 - \overline{I(0)} I} = \overline{\xi} \overline{I},$$

and hence $\operatorname{Ker} T_{\overline{a}/a} = K_I$.

Conversely, suppose $\operatorname{Ker} T_{\overline{a}/a} = K_I$ for some inner function I. By our introductory remark, this is equivalent to

$$\operatorname{Ker} T_{\overline{(a/k_0^I)}/(a/k_0^I)} = K_J,$$

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where J is the Frostman shift of I from (4.7). Let us set $a_1 = a/k_0^I$. Then since $J(0) = 0$, we get $T_{\overline{a_1}/a_1}1 = 0$ which means that $\frac{\overline{a_1}}{a_1} = \overline{\psi}$, for some $\psi \in H^2$ with $\psi(0) = 0$. In particular, $|\psi| = 1$ almost everywhere on $\mathbb{T}$, which implies that ψ is inner. Hence, almost everywhere on $\mathbb{T}$,

$$\frac{\overline{a_1}}{a_1} = \overline{\psi} = \frac{\overline{1+\psi}}{1+\psi}, \qquad \frac{\overline{a_1}}{\overline{1+\psi}} = \frac{a_1}{1+\psi} =: F,$$

and $F \in N^+$. Thus $F \in N^+$ with $F(e^{i\theta}) \in \mathbb{R}$ for almost every θ. By a result of Helson [24], F can be written as $F = i\frac{\Theta_1 - i\Theta_2}{\Theta_1 + i\Theta_2}$, where Θ_1, Θ_2 are inner and $\Theta_1 + i\Theta_2$ is outer, and thus

$$a_1 = i\frac{\Theta_1 - i\Theta_2}{\Theta_1 + i\Theta_2}(1+\psi).$$

Since $\overline{a_1}/a_1 = \overline{\psi}$, we have $K_\psi = \operatorname{Ker} T_{\overline{\psi}} = \operatorname{Ker} T_{\overline{a_1}/a_1} = K_J$, from which we deduce that $\psi = \xi J$ for some $\xi \in \mathbb{T}$. Finally $a = a_1 k_0^I = i\frac{\Theta_1 - i\Theta_2}{\Theta_1 + i\Theta_2}(1+\xi J)k_0^I$. ∎

Remark 4.7 • It is clear from the proof that $a/\overline{a}$ is inner if and only if a takes the form in Proposition 4.6.

• Notice that whenever $I = \prod_{k=1}^n I_k$ is a factorization of the inner function I into inner functions I_k, we can replace $(1+I)$ by $\prod_{k=1}^n (1+I_k)$ without changing $\overline{a}/a$. This only changes the inner functions Θ_1 and Θ_2.

Example 4.8 One might ask whether it is possible to find non-trivial inner functions Θ_1, Θ_2 satisfying the conditions of Proposition 4.6, all the while ensuring that the corresponding function a is bounded. Garcia and Sarason [17] gave a method of constructing real outer functions (see also [15]). The idea is rather simple: in order that a function f in the Smirnov class be real on $\mathbb{T}$, its argument must be $\pi\chi_E$, where E is a measurable subset of $\mathbb{T}$. Hence we are interested in

$$f_E = \exp(\pi(i\chi_E - \widetilde{\chi}_E)),$$

where $\widetilde{\chi}_E$ denotes the harmonic conjugate of χ_E. The simplest example is when $E = [e^{it_1}, e^{it_2}]$ is an arc of the circle. In this case, its conjugate function is nicely behaved outside the endpoints of the arc. Close to the endpoint, it essentially behaves like a logarithm, and so $|f_E(e^{it})| \simeq |t - t_k|^{-\alpha}$ for some power $\alpha > 0$. Now pick a nicely behaved inner function I that is analytic near e^{it_k} and which takes the value -1 at these points. Then for an integer $N \geqslant \alpha$, we can take $a = f_E(1+I)^N$, which is a bounded outer function.

In Proposition 4.6, we discussed when the extremal function γ of $\operatorname{Ker} T_{\overline{a}/a}$ is a constant. The next natural question is then to ask when $T_{\overline{a}}K_I = K_I$ (we always have $T_{\overline{a}}K_I \subset K_I$). This situation is characterized by the following result.

Lemma 4.9 *For $a \in H^\infty$ outer and I inner, the following are equivalent.*

(i) $T_{\overline{a}}K_I = K_I$.
(ii) *There exists a function ψ in H^∞ such that $a\psi - 1 \in IH^\infty$.*
(iii) *There exists a constant $\delta > 0$ such that $|a| + |I| \geqslant \delta$ on $\mathbb{D}$.*

In other words, $T_{\overline{a}}K_I = K_I$ if and only if (a, I) is a corona pair.

Proof The equivalence (ii) $\Leftrightarrow$ (iii) is just an application of the corona theorem [18].

That (ii) $\Rightarrow$ (i) is rather simple. Indeed, when $a\psi - 1 \in IH^\infty$, we have

$$T_{\overline{a}}T_{\overline{\psi}}K_I = T_{1+\overline{Ih}}K_I = K_I$$

for some $h \in H^\infty$. The last equality follows from the identity $T_{\overline{Ih}}K_I = T_{\overline{h}}T_{\overline{I}}K_I = \{0\}$ (Proposition 2.1 (iv)). This shows that $T_{\overline{a}}K_I = K_I$.

Let us discuss (i) $\Leftrightarrow$ (ii). Observe that condition (ii) is a kind of interpolation condition, and our argument is a simple version of generalized interpolation based on the commutant lifting theorem [29, Chapter C3].

As we have already seen, since a is outer, $T_{\overline{a}}$ is injective on H^2 and hence on K_I. By (i), $T_{\overline{a}}K_I = K_I$, so that the operator $T_{\overline{a}}$ must be invertible on K_I. This is equivalent to saying that the compression of the analytic Toeplitz operator T_a to K_I (a truncated Toeplitz operator [16]), *i.e.*, $a(S_I) := P_I T_a|_{K_I}$ is invertible on K_I. Here P_I is the orthogonal projection of L^2 onto K_I, $S_I := P_I T_z|_{K_I}$ is the compression of the shift T_z, and $a(S_I)$ is defined via the functional calculus.

If $a(S_I)$ is invertible, then its inverse commutes with S_I [28, p. 231]. By the commutant lifting theorem, there is a $\psi \in H^\infty$ such that $(a(S_I))^{-1} = \psi(S_I)$ and thus for every $f \in K_I$, $P_I(a\psi f) = f$, or equivalently, $(a\psi - 1)f \in IH^2$. This translates to the condition $a\psi - 1 \in IH^\infty$ (pick, for instance, $f = 1 - \overline{I(0)}I$, which is outer with bounded reciprocal). ∎

We summarize the above discussion with the following result.

Corollary 4.10 *Let $a \in H^\infty$ be outer and assume that* Ker $T_{\overline{a}/a} = K_I$. *If*

$$\inf_{z\in\mathbb{D}}(|a(z)| + |I(z)|) > 0,$$

then $\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} K_I$.

When (a, I) is not necessarily a corona pair, we can decompose $\mathscr{M}(\overline{a})$ using the following proposition.

Proposition 4.11 *Let $a \in H^\infty$ and outer and such that* Ker $T_{\overline{a}/a} = K_I$ *for some inner function I. Then $\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} (\mathscr{M}(\overline{a}) \cap K_I)$.*

Proof By (4.3) we have

$$\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} T_{\overline{a}}K_I. \tag{4.8}$$

First observe that the H^2-closure of $T_{\overline{a}}K_I$ is equal to K_I (easy to conclude from the definitions of $T_{\overline{a}}$ and K_I). Second, we observe that $T_{\overline{a}}K_I$ is a closed subspace of $\mathscr{M}(\overline{a})$ which is invariant with respect to the backward shift operator. Finally, we use the proof in [13, Theorem 17.22] to see that $T_{\overline{a}}K_I = K_I \cap \mathscr{M}(\overline{a})$. Combine this with (4.8) to complete the proof. ∎

5 Examples

We now discuss some examples where Proposition 4.6 and Lemma 4.9 hold, as well as an example where Lemma 4.9 does not apply. Let us begin with the case when

$$\frac{\overline{a}}{a} = \overline{I}, \quad I(0) = 0, \quad (a, I) \text{ is a corona pair.} \tag{5.1}$$

In this situation we will always have

$$\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} K_I. \tag{5.2}$$

Here are some examples of when (5.1) holds.

(1) Let I be any inner function with $I(0) = 0$ and $a = 1 + I$. Then it is clear that $\frac{\overline{a}}{a} = \overline{I}$ and $|a| + |I| = |1 + I| + |I| \geqslant 1$.
(2) The exact same argument works also when $a = (1 + I)^m$, $m \in \mathbb{N}$.
(3) For the outer function $a(z) := \prod_{1 \leqslant j \leqslant n}(z - \zeta_j)^{m_j}$, where $\zeta_j \in \mathbb{T}$, $m_j \in \mathbb{N}$, we have $\overline{a}/a = c\overline{z^N}$ on $\mathbb{T}$ where $N = \sum_{j=1}^n m_j$. Thus, putting $I(z) = z^N$, it is clear that

$$|a(z)| + |I(z)| = \prod_{1 \leqslant j \leqslant n} |z - \zeta_j|^{m_j} + |z|^N \geqslant \delta > 0,$$

which yields (5.2). In this situation $K_I = \mathcal{P}_{N-1}$ are the polynomials of degree at most $N - 1$. See Corollary 6.12 for another description of the orthogonal decomposition.

A natural example to consider next would be $a := \prod_{1 \leqslant j \leqslant n}(\zeta_j - I_j)^{m_j}$, where I_j are inner functions, $\zeta_j \in \mathbb{T}$, and $m_j \in \mathbb{N}$. However, in this case, we do not, in general, obtain such a nice decomposition as in (5.2). Note that we obviously still have $\overline{a}/a = \overline{I}$, where $I = \prod_{1 \leqslant j \leqslant n} I_j$. Hence $\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} T_{\overline{a}} K_I$. However, we will show that for appropriate choices of I_j we have $T_{\overline{a}} K_I \subsetneq K_I$. To do so, we will consider the situation when $a := (1 - I_1)(1 - I_2)$ and $I = I_1 I_2$, where I_1 and I_2 are suitable Blaschke products. More precisely, $\lambda_n = 1 - 4^{-n^2}$, $\mu_n = 1 - 4^{-n^2 - n}$, and let I_1 and I_2 be the Blaschke products with these zeros. In order to show that $\inf\{|a(z)| + |I(z)| : z \in \mathbb{D}\} = 0$, it is enough to show that $I_1(\mu_{n_k}) \to 1$ when $k \to \infty$ for some suitable sub-sequence. Clearly $I_1(\mu_n)$ is a real number. Since the zeros of I_1 are simple, I_1 changes sign on $[0, 1)$ at each λ_n. Thus we can assume that for alternating μ_n, we have $I_1(\mu_n) > 0$. Note these μ_n by μ_n^+. Finally, since the sequence is interpolating with increasing pseudohyperbolic distances between successive points, we necessarily have $I_1(\mu_n^+) \to 1$. Hence

$$a(\mu_n^+) = (1 - I_1(\mu_n^+))(1 - I_2(\mu_n^+)) \to 0, \quad n \to \infty,$$

and $I(\mu_n^+) = 0$, which proves the claim.

6 Boundary Behavior in Sub-Hardy Hilbert Spaces

Next we discuss the boundary behavior of functions in $\mathscr{M}(\overline{a})$ spaces. While the $\mathscr{M}(\overline{a})$ spaces are the focus of this paper, we will discuss the boundary behavior of functions in a large class of "admissible" reproducing kernel Hilbert spaces of analytic functions on $\mathbb{D}$. These "admissible" classes also include the de Branges–Rovnyak spaces and the harmonically weighted Dirichlet spaces. In Section 6.3 we will apply some of the results on boundary behavior in $\mathscr{M}(\overline{a})$ to the orthogonal decomposition $\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} T_{\overline{a}}(\gamma K_I)$.

6.1 An Abstract Result

To get started, let $\mathscr{H}$ be a Hilbert space of analytic functions on $\mathbb{D}$ with norm $\|\cdot\|_{\mathscr{H}}$ such that for each $\lambda \in \mathbb{D}$, the evaluation functional $f \mapsto f(\lambda)$ is continuous on $\mathscr{H}$. By the Riesz representation theorem, there is a $k_\lambda^{\mathscr{H}} \in \mathscr{H}$, called the *reproducing kernel* [30], such that $f(\lambda) = \langle f, k_\lambda^{\mathscr{H}}\rangle_{\mathscr{H}}$, $f \in \mathscr{H}$. For each $j \in \mathbb{N}_0$ it follows that the linear functional $f \mapsto f^{(j)}(\lambda)$ is also continuous on $\mathscr{H}$ and thus given by a reproducing kernel $k_{\lambda,j}^{\mathscr{H}} \in \mathscr{H}$, *i.e.*, $f^{(j)}(\lambda) = \langle f, k_{\lambda,j}^{\mathscr{H}}\rangle_{\mathscr{H}}$, where $f \in \mathscr{H}, \lambda \in \mathbb{D}$. A brief argument from [13, Vol. 2, p. 205] will show that

$$k_{\lambda,j}^{\mathscr{H}} = \frac{\partial^j}{\partial \overline{\lambda}^j} k_\lambda^{\mathscr{H}}, \quad f \in \mathscr{H}, \lambda \in \mathbb{D}. \tag{6.1}$$

Define the following linear transformations T and B on $\mathscr{O}(\mathbb{D})$ (the vector space of analytic functions on $\mathbb{D}$) by

$$(Tf)(z) = zf(z), \quad (Bf)(z) = \frac{f(z) - f(0)}{z}.$$

Observe that $S := T|_{H^2}$ is the well-known unilateral shift operator on H^2 and $S^* = B|_{H^2}$ is the equally well-known backward shift. Observe further that, in terms of Toeplitz operators on H^2, we have $S = T_z$ and $S^* = T_{\overline{z}}$.

Definition 6.1 A reproducing kernel Hilbert space $\mathscr{H}$ of analytic functions on $\mathbb{D}$ satisfying

(i) $B\mathscr{H} \subset \mathscr{H}$ and $\|B\|_{\mathscr{H}\to\mathscr{H}} \leqslant 1$,
(ii) $\sigma_p(X_{\mathscr{H}}^*) \subset \mathbb{D}$, where $X_{\mathscr{H}} := B|_{\mathscr{H}}$,

will be called *admissible*. Here, $\sigma_p(X_{\mathscr{H}}^*)$ is the point spectrum of the operator $X_{\mathscr{H}}^*$.

We will discuss some specific examples, such as $\mathscr{M}(\overline{a})$ (range space), $\mathscr{H}(b)$ (de Branges–Rovnyak space), and $\mathscr{D}(\mu)$ (harmonically weighted Dirichlet space) towards the end of this section.

Note that if condition (i) in Definition 6.1 is satisfied, then

$$\sigma_p(X_{\mathscr{H}}^*) \subset \sigma(X_{\mathscr{H}}^*) = \{\overline{z} : z \in \sigma(X_{\mathscr{H}})\} \subset \mathbb{D}^-.$$

Thus condition (ii) in Definition 6.1 is equivalent to saying that no unimodular complex number is an eigenvalue of $X_{\mathscr{H}}^*$.

The following result, valid beyond the setting of admissible spaces (see [13, Vol. 2, p. 205] for an alternate proof given for $\mathscr{H}(b)$ spaces), gives us a useful formula for the reproducing kernels $k_{\lambda,j}^{\mathscr{H}}$.

Lemma 6.2 *Let $\mathscr{H}$ be a reproducing kernel Hilbert space of analytic functions on $\mathbb{D}$ such that $B\mathscr{H} \subset \mathscr{H}$ and $\|B\| \leqslant 1$. Then for each $j \in \mathbb{N}_0$ and $\lambda \in \mathbb{D}$ we have*

$$k_{\lambda,j}^{\mathscr{H}} = j!(I - \overline{\lambda}X_{\mathscr{H}}^*)^{-(j+1)} {X_{\mathscr{H}}^*}^j k_0^{\mathscr{H}}. \tag{6.2}$$

Proof We first establish (6.2) when $j = 0$. Since B is a contraction, the operator $(I - \overline{\lambda}X^*_{\mathscr{H}})$ is invertible when $\lambda \in \mathbb{D}$ and the formula in (6.2), for $j = 0$, is equivalent to the identity $(I - \overline{\lambda}X^*_{\mathscr{H}})k^{\mathscr{H}}_{\lambda} = k^{\mathscr{H}}_0$. Observe how this identity holds if and only if for every $f \in \mathscr{H}$, $\langle f, (I - \overline{\lambda}X^*_{\mathscr{H}})k^{\mathscr{H}}_{\lambda}\rangle_{\mathscr{H}} = \langle f, k^{\mathscr{H}}_0\rangle_{\mathscr{H}} = f(0)$.

To prove this last identity, observe that

$$\begin{aligned}\langle f, (I - \overline{\lambda}X^*_{\mathscr{H}})k^{\mathscr{H}}_{\lambda}\rangle_{\mathscr{H}} &= \langle f, k^{\mathscr{H}}_{\lambda}\rangle_{\mathscr{H}} - \lambda\langle f, X^*_{\mathscr{H}}k^{\mathscr{H}}_{\lambda}\rangle_{\mathscr{H}}\\ &= f(\lambda) - \lambda\langle X_{\mathscr{H}}f, k^{\mathscr{H}}_{\lambda}\rangle_{\mathscr{H}} = f(\lambda) - \lambda\frac{f(\lambda) - f(0)}{\lambda}\\ &= f(0).\end{aligned}$$

This proves (6.2) when $j = 0$.

The formula for $k^{\mathscr{H}}_{\lambda,j}$ now follows by using (6.1) and differentiating the identity

$$k^{\mathscr{H}}_{\lambda} = (I - \overline{\lambda}X^*_{\mathscr{H}})^{-1}k^{\mathscr{H}}_0$$

j times with respect to the variable $\overline{\lambda}$. ∎

We are now ready to state the main result of this section. For fixed $\zeta_0 \in \mathbb{T}$ and $\alpha > 1$ let $\Gamma_\alpha(\zeta_0) := \{z \in \mathbb{D} : |z - \zeta_0| < \alpha(1 - |z|)\}$ be a standard Stolz domain anchored at ζ_0. We say that $f \in \mathscr{O}(\mathbb{D})$ has a finite Γ_α-limit at ζ_0 if $\lim_{z\to\zeta_0, z\in\Gamma_\alpha(\zeta_0)} f(z)$ exists. We say that an $f \in \mathscr{O}(\mathbb{D})$ has a finite *non-tangential limit* at ζ_0 if f has a Γ_α-limit for every $\alpha > 1$ and this limit is the same, denoted by $f(\zeta_0)$, for every $\alpha > 1$. When $\alpha = 1$, $\Gamma_1(\zeta_0)$ degenerates to the radius connecting 0 and ζ_0 and the limit within $\Gamma_1(\zeta_0)$ becomes a radial limit.

The following theorem was inspired by an operator theory result of Ahern and Clark [1], where they discussed non-tangential limits of functions in the classical model spaces $K_I = H^2 \ominus IH^2$.

Theorem 6.3 *Let $\mathscr{H}$ be an admissible space, $\zeta_0 \in \mathbb{T}$, and $N \in \mathbb{N}_0$. Then the following are equivalent.*

(i) *For every $f \in \mathscr{H}$, the functions $f, f', f'', \dots, f^{(N)}$ have finite non-tangential limit at ζ_0.*

(ii) *For each $\alpha \geqslant 1$, $\sup\{\|k^{\mathscr{H}}_{\lambda,N}\|_{\mathscr{H}} : \lambda \in \Gamma_\alpha(\zeta_0)\} < \infty$.*

(iii) *There exists an $\alpha \geqslant 1$ with $\sup\{\|k^{\mathscr{H}}_{\lambda,N}\|_{\mathscr{H}} : \lambda \in \Gamma_\alpha(\zeta_0)\} < \infty$.*

(iv) $(X_{\mathscr{H}})^{*N}k^{\mathscr{H}}_0 \in \operatorname{Rng}(I - \overline{\zeta}_0X^*_{\mathscr{H}})^{N+1}$.

Moreover, if any one of the above equivalent conditions hold, then

$$(I - \overline{\zeta_0}X^*_{\mathscr{H}})^{N+1}k^{\mathscr{H}}_{\zeta_0,N} = N!\,{X^*_{\mathscr{H}}}^{N}k^{\mathscr{H}}_0, \tag{6.3}$$

where $k^{\mathscr{H}}_{\zeta_0,N} \in \mathscr{H}$ satisfies $f^{(N)}(\zeta_0) = \langle f, k^{\mathscr{H}}_{\zeta_0,N}\rangle_{\mathscr{H}}$, $f \in \mathscr{H}$.

We emphasize that the above theorem shows that in admissible spaces, the existence of radial limits (consider $\alpha = 1$ in condition (iii)) implies existence of non-tangential limits.

Proof (i) ⇒ (ii). Since the norms of the reproducing kernels $k^{\mathscr{H}}_{\lambda,N}$ are the norms of the evaluation functionals $f \mapsto f^{(N)}(\lambda)$, we can apply the uniform boundedness principle to see, for fixed $\alpha \geqslant 1$, that if the N-th derivative of every function in $\mathscr{H}$ has a finite limit as $\lambda \to \zeta_0$ with $\lambda \in \Gamma_\alpha(\zeta_0)$, then the norms of the kernels $k^{\mathscr{H}}_{\lambda,N}$ are uniformly bounded for $\lambda \in \Gamma_\alpha(\zeta_0)$.

(ii) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (iv). Let α be as in (iii). By Lemma 6.2, the vectors

$$(I - \overline{z_n} X^*_{\mathscr{H}})^{-(N+1)} {X^*_{\mathscr{H}}}^N k_0^{\mathscr{H}}$$

are uniformly bounded for any sequence $\{z_n\}_{n\geqslant 1} \subset \Gamma_\alpha(\zeta_0)$ tending to ζ_0. By our assumption $\sigma_p(X^*_{\mathscr{H}}) \subset \mathbb{D}$ (Definition 6.1) we see that the operator $I - \overline{\zeta_0} X^*_{\mathscr{H}}$ is injective. Now apply [13, Corollary 21.22] to conclude that ${X^*_{\mathscr{H}}}^N k_0^{\mathscr{H}} \in \operatorname{Rng}(I - \overline{\zeta}_0 X^*_{\mathscr{H}})^{N+1}$. (Strictly speaking, [13, Corollary 21.22] is stated for non-tangential limits, but its proof is based on [13, Lemma 21.20], which also works for radial limits.)

(iv) $\Rightarrow$ (i). Let $\alpha \geq 1$. Again using [13, Corollary 21.22], we see that

$$(I - \overline{z_n} X^*_{\mathscr{H}})^{-(N+1)} {X^*_{\mathscr{H}}}^N k_0^{\mathscr{H}} \to (I - \overline{\zeta_0} X^*_{\mathscr{H}})^{-(N+1)} {X^*_{\mathscr{H}}}^N k_0^{\mathscr{H}}$$

weakly for any sequence $\{z_n\}_{n\geqslant 1} \subset \Gamma_\alpha(\zeta_0)$ tending to ζ_0. However, Lemma 6.2 says that the left-hand side of the identity above is precisely $\frac{1}{N!} k^{\mathscr{H}}_{z_n,N}$. Hence, for any $f \in \mathscr{H}$, the N-th derivative, $f^{(N)}(z_n)$, has a finite limit as z_n tends to ζ_0 within $\Gamma_\alpha(\zeta_0)$.

To see that the lower order derivatives, $f, f', f'', \dots, f^{(N-1)}$, have finite non-tangential limits at ζ_0, use an inductive argument based on the formula

$$I - (I - \overline{\zeta_0} X^*_{\mathscr{H}})^N = -\sum_{\ell=1}^{N} \binom{N}{\ell} (-\overline{\zeta_0})^\ell {X^*_{\mathscr{H}}}^\ell,$$

which gives

$$\begin{aligned} {X^*_{\mathscr{H}}}^{(N-1)} k_0^{\mathscr{H}} &= (I - \overline{\zeta_0} X^*_{\mathscr{H}})^N {X^*_{\mathscr{H}}}^{(N-1)} k_0^{\mathscr{H}} \\ &\qquad - \sum_{\ell=1}^{N} \binom{N}{\ell} (-\overline{\zeta_0})^\ell {X^*_{\mathscr{H}}}^{\ell-1} {X^*_{\mathscr{H}}}^N k_0^{\mathscr{H}}; \end{aligned}$$

see the proof of Theorem 21.26 in [13].

Finally, the equivalent conditions of the theorem show that the linear functional $f \mapsto f^{(N)}(\zeta_0)$ is continuous on $\mathscr{H}$, and thus, by the Riesz representation theorem, it is induced by a kernel $k^{\mathscr{H}}_{\zeta_0,N} \in \mathscr{H}$ satisfying

$$(I - \overline{\zeta_0} X^*_{\mathscr{H}})^{-(N+1)} {X^*_{\mathscr{H}}}^N k_0^{\mathscr{H}} = \frac{1}{N!} k^{\mathscr{H}}_{\zeta_0,N}.$$

This proves (6.3). ∎

The next result allows us to produce a large class of admissible reproducing kernel Hilbert spaces.

Lemma 6.4 *Let $\mathscr{H}$ be a B-invariant reproducing kernel Hilbert space of analytic functions on $\mathbb{D}$ such that the analytic polynomials are dense in $\mathscr{H}$. Then $\sigma_p(X^*_{\mathscr{H}}) = \varnothing$. In particular, if $X_{\mathscr{H}} = B|_{\mathscr{H}}$ acts as a contraction on $\mathscr{H}$, then $\mathscr{H}$ is an admissible space.*

Proof Suppose $\lambda \in \mathbb{C}$ and $f \in \mathscr{H} \setminus \{0\}$ with $X^*_{\mathscr{H}} f = \lambda f$. On one hand,

$$\langle X^*_{\mathscr{H}} f, z^n \rangle_{\mathscr{H}} = \lambda \langle f, z^n \rangle_{\mathscr{H}},$$

while on the other hand,

$$\langle X^*_{\mathscr{H}} f, z^n \rangle_{\mathscr{H}} = \langle f, X_{\mathscr{H}} z^n \rangle_{\mathscr{H}} = \langle f, z^{n-1} \rangle_{\mathscr{H}}, \quad n \geqslant 1.$$

Combining these two facts yields

$$\lambda \langle f, z^n \rangle_{\mathscr{H}} = \langle f, z^{n-1} \rangle_{\mathscr{H}}, \quad n \geqslant 1. \tag{6.4}$$

When $\lambda = 0$, the previous identity shows that $\langle f, z^k \rangle_{\mathscr{H}} = 0$ for all $k \geqslant 0$. By the density of the polynomials in $\mathscr{H}$ we see that $f = 0$, a contradiction.

When $\lambda \neq 0$, then

$$\lambda \langle f, 1 \rangle_{\mathscr{H}} = \langle X^*_{\mathscr{H}} f, 1 \rangle_{\mathscr{H}} = \langle f, X_{\mathscr{H}} 1 \rangle_{\mathscr{H}} = 0$$

and thus $\langle f, 1 \rangle_{\mathscr{H}} = 0$. Use this last identity and repeatedly apply (6.4) to see that $\langle f, z^k \rangle_{\mathscr{H}} = 0$ for all $k \geqslant 0$. Again, by our assumption that the polynomials are dense in $\mathscr{H}$, we see that $f = 0$. ∎

Remark 6.5 If $\mathscr{H}$ contains all of the Cauchy kernels k_w, $w \in \mathbb{D}$ (see (2.1)), then we can use the fact that $X_{\mathscr{H}} k_w = \overline{w} k_w$ to replace the identity in (6.4) with $\lambda \langle f, k_w \rangle_{\mathscr{H}} = w \langle f, k_w \rangle_{\mathscr{H}}$. Thus the hypothesis "the polynomials are dense in $\mathscr{H}$" in Lemma 6.4 can be replaced with "the linear span of Cauchy kernels is dense in $\mathscr{H}$". We would like to thank Omar El Fallah for some fruitful discussions concerning an earlier version of this result.

6.2 Three Examples of Admissible Spaces

Besides considering the application of Theorem 6.3 to $\mathscr{M}(\overline{a})$-spaces, we discuss two other prominent admissible spaces.

6.2.1 $\mathscr{M}(\overline{a})$-spaces

For $a \in H^\infty$, let us show that the range space $\mathscr{M}(\overline{a})$ is admissible. By Proposition 3.4 we can assume that a is outer. To verify that $\mathscr{M}(\overline{a})$ is admissible, we will check the hypothesis of Lemma 6.4. It is clear that $\mathscr{M}(\overline{a})$ is B-invariant (use the identity $T_{\overline{z}} T_{\overline{a}} = T_{\overline{a}} T_{\overline{z}}$ from Proposition 2.1 (iii)).

To show that $B = T_{\overline{z}}$ is contractive on $\mathscr{M}(\overline{a})$, notice that for any $g \in H^2$ we have

$$\|B T_{\overline{a}} g\|_{\overline{a}} = \|T_{\overline{z}} T_{\overline{a}} g\|_{\overline{a}} = \|T_{\overline{a}} T_{\overline{z}} g\|_{\overline{a}} = \|T_{\overline{z}} g\|_{H^2} \leqslant \|g\|_{H^2} = \|T_{\overline{a}} g\|_{\overline{a}}.$$

Thus $\|B\|_{\mathscr{M}(\overline{a}) \to \mathscr{M}(\overline{a})} \leqslant 1$.

To finish, using Lemma 6.4 and Remark 6.5, we need to show that the Cauchy kernels k_λ belong to $\mathscr{M}(\overline{a})$ and have dense linear span. From Proposition 2.1 (i) we have $k_\lambda = T_{\overline{a}}(k_\lambda / \overline{a(\lambda)}) \in \mathscr{M}(\overline{a})$. Furthermore, since $T_{\overline{a}}$ is a partial isometry from H^2 *onto* $\mathscr{M}(\overline{a})$, it maps a dense subset of H^2 onto a dense subset of $\mathscr{M}(\overline{a})$. Thus the density in $\mathscr{M}(\overline{a})$ of the linear span of k_λ for $\lambda \in \mathbb{D}$ follows from the well-known density of this span in H^2. We remark that one can also obtain the admissibility of $\mathscr{M}(\overline{a})$ by using the density of the polynomials in $\mathscr{M}(\overline{a})$ [13, Vol. 2, p. 47].

Using Theorem 6.3, we obtain the following more explicit characterization of the boundary behavior for functions in $\mathscr{M}(\overline{a})$.

Corollary 6.6 *Let $a \in H^\infty$ be outer, $\zeta_0 \in \mathbb{T}$, and $N \in \mathbb{N}_0$. Then for every $f \in \mathscr{M}(\overline{a})$, the functions $f, f', f'', \dots, f^{(N)}$ have finite non-tangential limits at ζ_0 if and only if*

$$\int_0^{2\pi} \frac{|a(e^{it})|^2}{|e^{it} - \zeta_0|^{2N+2}}\, dt < \infty. \tag{6.5}$$

We will write $\zeta_0 \in (AC)_{\overline{a},N}$ if (6.5) holds. In this case, we have

$$k^{\overline{a}}_{\zeta_0,\ell} = T_{\overline{a}}(ak_{\zeta_0,\ell}), \quad 0 \leqslant \ell \leqslant N,$$

where $ak_{\zeta_0,\ell} = \ell! \frac{z^\ell a}{(1-\overline{\zeta}_0 z)^{\ell+1}}$. Moreover, for each $\alpha > 1$ we have

$$\lim_{\substack{\lambda \to \zeta_0 \\ \lambda \in \Gamma_\alpha(\zeta_0)}} \|k^{\overline{a}}_{\lambda,\ell} - k^{\overline{a}}_{\zeta_0,\ell}\|_{\overline{a}} = 0.$$

Proof Corollary 3.2 gives us

$$\|k^{\overline{a}}_{\lambda,N}\|^2_{\overline{a}} = (N!)^2 \int_0^{2\pi} \frac{|a(e^{it})|^2}{|e^{it} - \lambda|^{2N+2}} \frac{dt}{2\pi}.$$

If λ approaches ζ_0 from within a fixed Stolz domain $\Gamma_\alpha(\zeta_0)$, then

$$\frac{1}{|e^{it} - \lambda|} \leqslant \frac{\alpha + 1}{|e^{it} - \zeta_0|}, \quad t \in [0, 2\pi],$$

and so

$$\frac{|a(e^{it})|^2}{|e^{it} - \lambda|^{2N+2}} \leqslant (\alpha + 1)^{2N+2} \frac{|a(e^{it})|^2}{|e^{it} - \zeta_0|^{2N+2}}. \tag{6.6}$$

If

$$\int_0^{2\pi} \frac{|a(e^{it})|^2}{|e^{it} - \zeta_0|^{2N+2}} \frac{dt}{2\pi} < \infty,$$

we see that

$$\sup\big\{ \|k^{\overline{a}}_{\lambda,N}\|_{\overline{a}} : \lambda \in \Gamma_\alpha(\zeta_0) \big\} < \infty. \tag{6.7}$$

Now apply Theorem 6.3.

Conversely, if for every $f \in \mathscr{M}(\overline{a})$, the functions $f, f', f'', \dots, f^{(N)}$ have non-tangential limits at ζ_0, then Theorem 6.3 implies that for each fixed $\alpha > 1$ (6.7) is satisfied. Thus

$$\sup_{\lambda \in \Gamma_\alpha(\zeta_0)} \int_0^{2\pi} \frac{|a(e^{it})|^2}{|e^{it} - \lambda|^{2N+2}} \frac{dt}{2\pi} < \infty.$$

By Fatou's Lemma

$$\int_0^{2\pi} \frac{|a(e^{it})|^2}{|e^{it} - \zeta_0|^{2N+2}} \frac{dt}{2\pi} \leqslant \liminf_{\substack{\lambda \to \zeta_0 \\ \lambda \in \Gamma_\alpha(\zeta_0)}} \int_0^{2\pi} \frac{|a(e^{it})|^2}{|e^{it} - \lambda|^{2N+2}} \frac{dt}{2\pi} < \infty.$$

Now let $\zeta_0 \in (AC)_{\overline{a},N}$. Then, via (3.3), for any $f = T_{\overline{a}} g \in \mathscr{M}(\bar{a})$ and $0 \leqslant \ell \leqslant N$, we have $\langle f, T_{\overline{a}}(ak_{\zeta_0,\ell}) \rangle_{\overline{a}} = \langle g, ak_{\zeta_0,\ell} \rangle_{H^2}$. Note that $ak_{\lambda,\ell} \to ak_{\zeta_0,\ell}$ in H^2 as $\lambda \to \zeta_0$ within $\Gamma_\alpha(\zeta_0)$. Indeed this is true pointwise and, by using the inequality in (6.6) and

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the dominated convergence theorem, we also have $\|ak_{\lambda,\ell}\|_{H^2} \to \|ak_{\zeta_0,\ell}\|_{H^2}$ as $\lambda \to \zeta_0$ within $\Gamma_\alpha(\zeta_0)$. By a standard Hilbert space argument,

$$\|ak_{\lambda,\ell} - ak_{\zeta_0,\ell}\|_{H^2} \to 0. \tag{6.8}$$

The above analysis says that

$$\langle f, T_{\overline{a}}(ak_{\zeta_0,\ell})\rangle_{\overline{a}} = \lim_{\substack{\lambda\to\zeta_0\\ \lambda\in\Gamma_\alpha(\zeta_0)}} \langle g, ak_{\lambda,\ell}\rangle_{H^2} = \lim_{\substack{\lambda\to\zeta_0\\ \lambda\in\Gamma_\alpha(\zeta_0)}} \langle f, T_{\overline{a}} ak_{\lambda,\ell}\rangle_{\overline{a}}.$$

By Corollary 3.2, $T_{\overline{a}}(ak_{\lambda,\ell}) = k^{\overline{a}}_{\lambda,\ell}$, whence

$$\begin{aligned}\langle f, T_{\overline{a}}(ak_{\zeta_0,\ell})\rangle_{\overline{a}} &= \lim_{\substack{\lambda\to\zeta_0\\ \lambda\in\Gamma_\alpha(\zeta_0)}} \langle f, k^{\overline{a}}_{\lambda,\ell}\rangle_{\overline{a}} = \lim_{\substack{\lambda\to\zeta_0\\ \lambda\in\Gamma_\alpha(\zeta_0)}} f^{(\ell)}(\lambda)\\ &= f^{(\ell)}(\zeta_0) = \langle f, k^{\overline{a}}_{\zeta_0,\ell}\rangle_{\overline{a}},\end{aligned}$$

which proves that $k^{\overline{a}}_{\zeta_0,\ell} = T_{\overline{a}}(ak_{\zeta_0,\ell})$. Finally, from (6.8),

$$\|k^{\overline{a}}_{\lambda,\ell} - k^{\overline{a}}_{\zeta_0,\ell}\|_{\overline{a}} = \|ak_{\lambda,\ell} - ak_{\zeta_0,\ell}\|_{H^2} \to 0, \quad \lambda\to\zeta_0, \quad \text{as } \lambda\in\Gamma_\alpha(\zeta_0). \qquad \blacksquare$$

Remark 6.7 • In a general admissible space $\mathscr{H}$ we see that if

$$\sup\{\|k^{\mathscr{H}}_{\lambda,N}\|_{\mathscr{H}} : \lambda\in\Gamma_\alpha(\zeta_0)\} < \infty$$

for each $\alpha > 1$, then $k^{\mathscr{H}}_{\lambda,N} \to k^{\mathscr{H}}_{\zeta_0,N}$ weakly in $\mathscr{H}$ as $\lambda\to\zeta_0$ non-tangentially. However, it is not immediately clear whether we also have norm convergence of the kernels. Corollary 6.6 shows this is true when $\mathscr{H} = \mathscr{M}(\overline{a})$. See also [13] where this was shown to be true when $\mathscr{H}$ is a de Branges–Rovnyak space $\mathscr{H}(b)$.

• Condition (6.5) yields an estimate of the rate of decrease of the outer function a, along with its derivatives, at the distinguished point ζ_0. Indeed, using the facts that $(\zeta_0 - z)^{N+1}$ is outer, along with (6.5), and Smirnov's theorem [7] (if the boundary function of an outer function belongs to L^2, then the function belongs to H^2), the function $h(z) := \frac{a(z)}{(z-\zeta_0)^{N+1}}$ belongs to H^2. Recall the following standard estimates for the derivatives of $h\in H^2$: $|h^{(\ell)}(r\zeta)| = o((1-r)^{-\ell-\frac{1}{2}})$, as $r\to 1^-$. Thus Leibniz's formula yields

$$a^{(k)}(r\zeta_0) = \sum_{\ell=0}^{k}\binom{k}{\ell} h^{(\ell)}(r\zeta_0)\frac{d^{k-\ell}}{dz^{k-\ell}}(z-\zeta_0)^{N+1}\Big|_{z=r\zeta_0} = o((1-r)^{N+\frac{1}{2}-\ell}), \quad \text{as } r\to 1^-.$$

In particular, we see that the functions $a, a', \dots, a^{(N)}$ have radial (and even non-tangential) limits $a^{(\ell)}(\zeta_0)$ that vanish for each $0\leqslant\ell\leqslant N$.

Corollary 6.6 yields the following interesting observation, which shows a sharp difference between $\mathscr{M}(\overline{a})$ spaces and the model, or more generally, de Branges–Rovnyak spaces $\mathscr{H}(b)$. More precisely, when $\log(1-|b|)\notin L^1$, it is sometimes the case that every function in $\mathscr{H}(b)$ can be analytically continued to an open neighborhood of a point $\zeta_0\in\mathbb{T}$. For example, if b is an inner function and $\zeta_0\in\mathbb{T}$ with

$$\liminf_{\lambda\to\zeta_0}|b(\lambda)| > 0,$$

then every $f \in \mathscr{H}(b)$ (which turns out to be a model space K_b) can be analytically continued to some open neighborhood Ω_{ζ_0} of ζ_0 (see [6, Corollary 3.1.8] for details). This phenomenon never happens in $\mathscr{M}(\overline{a})$.

Proposition 6.8 *For any $a \in H^\infty \setminus \{0\}$, there is no point $\zeta_0 \in \mathbb{T}$ such that every $f \in \mathscr{M}(\overline{a})$ can be analytically continued to some open neighborhood of ζ_0.*

Proof Suppose there exists such a $\zeta_0 \in \mathbb{T}$ where every function in $\mathscr{M}(\overline{a})$ has an analytic continuation to an open neighborhood Ω_{ζ_0} of ζ_0. Then the function $a \in \mathscr{M}(a) \subset \mathscr{M}(\overline{a})$ would have an analytic continuation to Ω_{ζ_0} and thus could be expanded in a power series around ζ_0. If every function in $\mathscr{M}(\overline{a})$ had an analytic continuation to Ω_{ζ_0}, then every function in $\mathscr{M}(\overline{a})$, and its derivatives of all orders, would have finite non-tangential limits at ζ_0. In particular, (6.5) would hold for every $N \in \mathbb{N}$ at ζ_0. By Remark 6.7, this would imply that all of the Taylor coefficients of a at ζ_0 would vanish, implying $a \equiv 0$ on $\mathbb{D}$, a contradiction. ∎

6.2.2 $\mathscr{H}(b)$-spaces

We have seen that $\mathscr{H}(b)$ spaces are special cases of $\mathscr{M}(A)$-spaces. It turns out that they are admissible. Indeed, they are B-invariant reproducing kernel Hilbert spaces contained in H^2 with $\|B\|_{\mathscr{H}(b)\to\mathscr{H}(b)} \leqslant 1$ [13, Theorem 18.13]. Furthermore, by [13, Theorem 18.26], $\sigma_p(X^*_{\mathscr{H}}) \subset \mathbb{D}$. Thus Theorem 6.3 applies, allowing us to reproduce some of the results of [11]. In particular, the condition that every $f \in \mathscr{H}(b)$, along with $f', \dots, f^{(N)}$, has a non-tangential limit at ζ_0 is equivalent to the condition that the norms of the reproducing kernels for $\mathscr{H}(b)$ are uniformly bounded in every Stolz domain anchored at ζ_0. The difficult part of [11] is to prove that the boundedness of the kernels is equivalent to the condition that

$$\sum_{n\geqslant 1} \frac{1-|a_n|}{|\zeta_0 - a_n|^{2N+2}} + \int_{\mathbb{T}} \frac{d\mu(\xi)}{|\zeta_0 - \xi|^{2N+2}} + \int_{\mathbb{T}} \frac{|\log|b(\xi)||}{|\zeta_0 - \xi|^{2N+2}}\, dm(\xi) \tag{6.9}$$

is finite, where $b = BS_\mu b_0$, B is a Blaschke product with zeros $\{a_n\}_{n\geqslant 1} \subset \mathbb{D}$, S_μ is a singular inner function with corresponding positive measure μ on $\mathbb{T}$ with $\mu \perp m$, and b_0 is the outer factor of b. See also [33, Chapter VII] for an equivalent condition in terms of the Aleksandrov–Clark measure associated with b as well as [2] for a condition in terms of a Schwarz-Pick matrix.

Remark 6.9 As already mentioned in Section 3, if $a \in H^\infty_1$ is such that $\log(1-|a|) \in L^1$ and b is its (outer) Pythagorean mate, then we have $\mathscr{M}(\overline{a}) \subset \mathscr{H}(b)$. If $N \in \mathbb{N}_0$ and $\zeta_0 \in \mathbb{T}$ are such that for every $f \in \mathscr{H}(b)$, the functions $f, f', \dots, f^{(N)}$ admit a finite non-tangential limit at ζ_0, then this is also true for every function $f \in \mathscr{M}(\overline{a})$. What is more surprising here is that the converse is true. This is a byproduct of Corollary 6.6 and [11, Theorem 3.2]. Indeed, since $|b|^2 = 1 - |a|^2$ almost everywhere on $\mathbb{T}$, we see (remembering b is outer) that condition (6.9) implies

$$\int_{\mathbb{T}} \frac{|\log(1-|a(\zeta)|^2)|}{|\zeta - \zeta_0|^{2N+2}}\, dm(\zeta) < \infty,$$

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which is equivalent to

$$\int_0^{2\pi} \frac{|a(e^{it})|^2}{|e^{it} - \zeta_0|^{2N+2}} \, dt < \infty.$$

Thus the conditions in (6.9) and (6.5) are equivalent which shows that the existence of boundary derivatives for functions in $\mathscr{H}(b)$ and $\mathscr{M}(\overline{a})$ (in the case when b is outer) are equivalent.

6.2.3 $\mathscr{D}(\mu)$-spaces

For a finite positive Borel measure μ on $\mathbb{T}$, let

$$\varphi_\mu(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\xi - z|^2} d\mu(\xi), \quad z \in \mathbb{D},$$

denote the Poisson integral of μ. The *harmonically weighted Dirichlet space* $\mathscr{D}(\mu)$ [9, 31] is the set of all $f \in \mathscr{O}(\mathbb{D})$ for which $\int_{\mathbb{D}} |f'|^2 \varphi_\mu dA < \infty$, where $dA = dxdy/\pi$ is normalized planar measure on $\mathbb{D}$. Notice that when $\mu = m$ is a Lebesgue measure on $\mathbb{T}$, then $\varphi_\mu \equiv 1$ and $\mathscr{D}(\mu)$ becomes the classical Dirichlet space [9]. One can show that $\mathscr{D}(\mu) \subset H^2$ [31, Lemma 3.1] and the norm $\|\cdot\|_{\mathscr{D}(\mu)}$ satisfying

$$\|f\|^2_{\mathscr{D}(\mu)} := \|f\|^2_{H^2} + \int_{\mathbb{D}} |f'|^2 \varphi_\mu \, dA$$

makes $\mathscr{D}(\mu)$ into a reproducing kernel Hilbert space of analytic functions on $\mathbb{D}$. It is known that both the polynomials and the linear span of the Cauchy kernels form a dense subset of $\mathscr{D}(\mu)$ [31, Corollary 3.8].

The backward shift B is a well-defined contraction on $\mathscr{D}(\mu)$. Indeed, we have

$$\|zf\|_{\mathscr{D}(\mu)} \geqslant \|f\|_{\mathscr{D}(\mu)}, \quad f \in \mathscr{D}(\mu),$$

and the constant function 1 is orthogonal to $z\mathscr{D}(\mu)$ [31, Theorem 3.6]. Thus

$$\|f\|^2_{\mathscr{D}(\mu)} = \|f(0) + zBf\|^2_{\mathscr{D}(\mu)} = |f(0)|^2 + \|zBf\|^2_{\mathscr{D}(\mu)} \geqslant \|Bf\|^2_{\mathscr{D}(\mu)}.$$

We thank Stefan Richter for showing us this elegant argument. From Lemma 6.4 we see that $\mathscr{D}(\mu)$ is an admissible space. From here, one can apply Theorem 6.3 to discuss the existence of non-tangential limits at a particular boundary point in terms of the boundedness of the norms of the reproducing kernels $k_\lambda^{\mathscr{D}(\mu)}$ for $\mathscr{D}(\mu)$. Using a kernel function estimate from [8], one can show that if $\mu = \sum_{1 \leqslant j \leqslant n} c_j \delta_{\zeta_j}$ with $c_j > 0, \zeta_j \in \mathbb{T}$, then each of the kernels $k_{r\zeta_j}^{\mathscr{D}(\mu)}, 1 \leqslant j \leqslant n$, remains norm-bounded as $r \to 1^-$. Thus the radial limits of every function from $\mathscr{D}(\mu)$ exist at each of the ζ_j. Other radial limit results along these lines can be stated in terms of an associated capacity for $\mathscr{D}(\mu)$ [8, 19]. Indeed, we can combine Theorem 6.3, which yields the equivalence of radial and non-tangential limits with the boundedness of the norms of the kernels, together with [8, Theorem 2], to obtain the following.

Theorem 6.10 *For a finite positive Borel measure μ on $\mathbb{T}$ and $\zeta_0 \in \mathbb{T}$, the following are equivalent.*

(i) *Every $f \in \mathscr{D}(\mu)$ has a finite non-tangential limit at ζ_0.*
(ii) $c_\mu(\zeta_0) > 0$.

(iii) $\int_0^1 \frac{dt}{(1-t)\varphi_\mu(t\zeta_0)+(1-t)^2} < \infty$.

Here, c_μ is the capacity in the sense of Beuring–Deny [8]. It would be interesting to obtain the analog of this result for the derivatives of functions from $\mathscr{D}(\mu)$. By Theorem 6.3, this is equivalent to estimating the norms of the kernel functions for the derivatives.

6.3 An Application to the Orthogonal Decomposition of $\mathscr{M}(\overline{a})$

In Theorem 4.3 we discovered the orthogonal decomposition

$$\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} T_{\overline{a}}(\gamma K_I).$$

We will use our boundary behavior results in order to better understand some of the contents of $T_{\overline{a}}(\gamma K_I)$. Recall the definition of the set $(AC)_{\overline{a},N}$ from (6.5). We begin with the following.

Proposition 6.11 *If $\zeta_0 \in (AC)_{\overline{a},N}$, then $k^{\overline{a}}_{\zeta_0,\ell} \in T_{\overline{a}}(\gamma K_I)$, $0 \leqslant \ell \leqslant N$.*

Proof Notice that $\zeta_0 \in (AC)_{\overline{a},\ell} \Rightarrow \zeta_0 \in (AC)_{\overline{a},\ell'}$, $0 \leqslant \ell' \leqslant \ell$, and so it suffices to prove the result when $\ell = N$. By Theorem 4.3, it suffices to show that $k^{\overline{a}}_{\zeta_0,N} \perp_{\overline{a}} aH^2$. To prove this last fact, set $f = ah$, where $h \in H^2$. By Leibniz's formula,

$$\langle f, k^{\overline{a}}_{r\zeta_0,N}\rangle_{\overline{a}} = f^{(N)}(r\zeta_0) = \sum_{0\leqslant k\leqslant N} \binom{N}{k} a^{(k)}(r\zeta_0)h^{(N-k)}(r\zeta_0).$$

By Remark 6.7 we have $|a^{(k)}(r\zeta_0)h^{(N-k)}(r\zeta_0)| = o((1-r)^{N+\frac{1}{2}-k}(1-r)^{k-N-\frac{1}{2}}) = o(1)$. Thus $\lim_{r\to 1}\langle f, k^{\overline{a}}_{r\zeta_0,N}\rangle_{\overline{a}} = 0$, and, using Corollary 6.6, yields $\langle f, k^{\overline{a}}_{\zeta_0,N}\rangle_{\overline{a}} = 0$. ∎

Using Proposition 6.11, we can revisit Example (3) from Section 5 and give an alternate description of the orthogonal complement of $\mathscr{M}(a)$ in $\mathscr{M}(\overline{a})$ when $a = \prod_{1\leqslant j\leqslant n} (z-\zeta_j)^{m_j}$. Indeed, since a is a polynomial, it is clear that $\zeta_j \in (AC)_{\overline{a},m_j-1}$ and so

$$k^{\overline{a}}_{\zeta_j,\ell} \in T_{\overline{a}}(\gamma K_I) = \mathcal{P}_{N-1}, \quad 1 \leqslant j \leqslant n, 0 \leqslant \ell \leqslant m_j - 1,$$

where we recall that $\mathcal{P}_{N-1}$ are the polynomials of degree at most $N-1$ and $N = \sum_{j=1}^n m_j$. Since the functions $\{k^{\overline{a}}_{\zeta_j,\ell} : j = 1,\dots,n, \ell = 0,\dots,m_j - 1\}$ are linearly independent, we obtain $\mathcal{P}_{N-1} = \bigvee\{k^{\overline{a}}_{\zeta_j,\ell} : j = 1,\dots,n, \ell = 0,\dots,m_j = 1\}$.

Corollary 6.12 *If $a = \prod_{j=1}^n (z-\zeta_j)^{m_j}$, then*

$$\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} \bigvee\{k^{\overline{a}}_{\zeta_j,\ell} : j = 1,\dots,n, \ell = 0,\dots,m_i = 1\}.$$

Our techniques yield the following generalization of results from [10, 26].

Theorem 6.13 *For an inner function I with $I(0) = 0$, set $a = \frac{1-I}{2}$ and $b = \frac{1+I}{2}$. Then $\mathscr{H}(b) = \mathscr{M}(a) \oplus_b K_I$.*

Proof Since $|a|+|b|\geqslant|a|^2+|b|^2=\frac{1}{2}(1+|I|^2)\geqslant\frac{1}{2}$, (a,b) forms a corona pair (see (3.6)) and hence we know from [33] that $\mathscr{H}(b)=\mathscr{M}(\overline{a})$. As in example (1) from Section 5, we can decompose $\mathscr{H}(b)$ as the orthogonal sum of $\mathscr{M}(a)$ and K_I with respect to $\langle\cdot,\cdot\rangle_{\overline{a}}$, the inner product in $\mathscr{M}(\overline{a})$. It remains to prove that $\mathscr{M}(a)$ and K_I are also orthogonal with respect to $\langle\cdot,\cdot\rangle_b$, the inner product in $\mathscr{H}(b)$. In other words, we need to check that given any $f\in H^2$ and any $g\in K_I$, we have $\langle af,g\rangle_b=0$.

We remind the reader of the well-known formula [33] for the inner product in $\mathscr{H}(b)$

$$\langle f_1,f_2\rangle_b=\langle f_1,f_2\rangle_{H^2}+\langle f_1^+,f_2^+\rangle_{H^2}, \tag{6.10}$$

where f_j^+ is the unique function from H^2 satisfying $T_{\overline{b}}f_j=T_{\overline{a}}f_j^+$. When $(b/a)f_j\in L^2$, then f_j^+ is given by

$$f_j^+=P_+((\overline{b/a})f_j). \tag{6.11}$$

We will use (6.10) and (6.11) with $f_1=af$ so that obviously $f_1^+=P_+((\overline{b/a})af)$ and $f_2=g\in K_I$. Observe that, since $\operatorname{Ker}T_{\overline{I}}=K_I$ and by the special form of a and b, we have $T_{\overline{b}}g=T_{\overline{a}}g$, so that automatically $f_2^+=g^+=g$. We thus get $\langle af,g\rangle_b=\langle af,g\rangle_{H^2}+\langle P_+((\overline{b/a})af),g\rangle_{H^2}$. Also observe that $a+b=1$ (by definition). Hence

$$\begin{aligned}\langle af,g\rangle_b&=\langle af,g\rangle_{H^2}+\langle P_+((\overline{b/a})af),g\rangle_{H^2}=\langle af+P_+((\overline{b/a})af),g\rangle_{H^2}\\&=\langle P_+((1+\overline{b/a})af),g\rangle_{H^2}=\langle P_+(\frac{a}{\overline{a}}f),g\rangle_{H^2}=\langle T_{a/\overline{a}}f,g\rangle_{H^2}.\end{aligned}$$

Since $T_{a/\overline{a}}f=-If\perp K_I$, we see that $\langle T_{a/\overline{a}}f,g\rangle_{H^2}=0$, which completes the proof. ∎

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