BULL. AUSTRAL. MATH. SOC. 06A05, 06A55
VOL. 15 (1976), 13-32.

Groups of automorphisms of linearly ordered sets

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I show that a group of order-automorphisms of a linearly ordered set can be expressed as an unrestricted direct product in which each factor is either the infinite cyclic group or else a group of order-automorphisms of a densely ordered set. From this a couple of simple group embedding theorems can be derived. The technique used to obtain the main result of this paper was motivated by the Erdös-Hajnal inductive classification of scattered sets.

Unless the contrary is either obvious or stated explicitly, throughout this paper we shall take "set" to mean "linearly ordered set". The ordering will always be denoted by "$<$", any ambiguity being resolved by context.

Sets and elements of sets will be denoted by upper and lower case Latin letters respectively, with the exception that "f", "g", "h" will denote functions and "i", "j", "k", "m", "n" integers. Lower case Greek letters will denote order-types, with "ω" always being reserved for the first transfinite ordinal, and the order-type of a set S will sometimes be denoted by "$o(S)$".

Given a set S, we can define a new ordering $<^*$ on S by setting $s <^* t$ whenever $t < s$. The resulting ordered set is called the "converse" of S, and is denoted by "S^*". If $\eta = o(S)$, then we denote $o(S^*)$ by "η^*".

Given two sets R, S, we can form their ordered union as follows. By

Received 11 March 1976.

https://doi.org/10.1017/S000497270003673X Published online by Cambridge University Press

replacing R with $R \times \{R\}$ and S with $S \times \{S\}$ if necessary, we can assume that $R \cap S = \emptyset$. Now put $T = R \cup S$, and order T by setting $x < y$ if either $x \in R$ and $y \in S$ or else $x, y \in R$ or $x, y \in S$ and $x < y$. If $\eta = o(R)$ and $\lambda = o(S)$, then $\eta + \lambda$ is defined to be $o(T)$, and we write "$T = R \dot{\cup} S$" . More generally, if we have an indexed set $\{S_r\}_{r \in R}$ of pairwise disjoint sets with the index set R also ordered, then we can form the ordered union $S = \dot{\bigcup}\{S_r;\ r \in R\}$ by setting $s < s'$ if $s \in S_r$, $s' \in S_{r'}$, and either $r < r'$ or else $r = r'$ and $s < s'$.

We can order the cartesian product $T = R \times S$ of two sets R, S , by setting $(r, s) < (r', s')$ if either $s < s'$ or else $s = s'$ and $r < r'$. We write "$T = R \dot{\times} S$" , and if $\eta = o(R)$ and $\lambda = o(S)$, then we define $\eta\lambda$ to be $o(T)$.

Given two sets S, T , an order-preserving map $f : S \to T$ will be called an "embedding". Clearly an embedding is injective; if $f : S \to T$ is a surjective embedding, then f is called an "isomorphism" and S, T are said to be similar ("$S \simeq T$") . Finally, an isomorphism $f : S \simeq S$ is called an "automorphism". We denote the set of all automorphisms of S by "$A(S)$" : under the operation of composition, $A(S)$ becomes the carrier of a group, which we denote by "$\mathsf{A}(S)$" . Furthermore, we can partially order $A(S)$ by setting $f \leq g$ if $f(s) \leq g(s)$ for all $s \in S$; under this partial ordering, $\mathsf{A}(S)$ is a lattice-ordered group. Holland has shown in [3] that if G is any lattice-ordered group, then for some set S we have a group monomorphism $G \to \mathsf{A}(S)$. Thus in particular any ordered group can be regarded as a subgroup of $\mathsf{A}(S)$ for some set S .

A set S is said to be dense if $S \neq \emptyset$ and for all $x, y \in S$ with $x < y$ there exists $z \in S$ such that $x < z < y$. The obvious example of a dense set is that of the set Q of rational numbers (under the usual ordering), and it is well-known that if S is dense, then there exists an embedding $Q \to S$. A set is said to be scattered if it contains no subset that is a dense set: thus we have that S is scattered if and only if there is no embedding $Q \to S$.

Let S be any nonempty set. Erdös and Hajnal have shown in [2] that there exists an indexed set $\{S_r\}_{r \in R}$ of pairwise disjoint sets such that

(a) $S = \dot{\bigcup}\{S_r;\ r \in R\}$;

(b) for each $r \in R$, S_r is nonempty and scattered;

(c) either $|R| = 1$ or else R is dense.

In (c) of course we are using "$|\ |$" to denote cardinality. The proof of this result that is given in [2] assumes that S is countable; as the authors observe, however, very little extra effort is required for the proof of the general result.

As was indicated in a previous paragraph, we shall denote groups by upper case script letters and - where possible - their carriers by the corresponding upper case Latin letters. Since we shall need to speak of group embeddings and group isomorphisms, we shall attach a subscript "G" to distinguish these from the embeddings and isomorphisms introduced above. If $\{G_x\}_{x \in X}$ is an indexed set of groups (with the index set X not necessarily ordered), then we shall denote the unrestricted direct product of the G_x by "$\times\{G_x;\ x \in X\}$" . We shall also denote the infinite cyclic group by "Z" , and the trivial group by "O" .

Our main result states that for any set S and any subgroup H of $A(S)$, there is an indexed set $\{H_x\}_{x \in X}$ of groups such that

(a) $H \simeq_G \times\{H_x;\ x \in X\}$;

(b) for each $x \in X$, either $H_x = Z$ or $H_x \leq A(R)$ for some dense set R .

A subset R of a set S is called a "segment" if for all $x, y, z \in S$ such that $x \leq y \leq z$, we have $x, z \in R \Rightarrow y \in R$. Let T be any subset of S ; we define the convex hull $\mathrm{Ch}(T)$ of T to be $\bigcap\{R;\ T \subseteq R\ \&\ R \text{ a segment of } S\}$. Obviously $\mathrm{Ch}(T)$ is a segment of S , and $\mathrm{Ch}(R) = R$ for each segment R of S .

We find it convenient to define a binary relation ∇ between subsets of a given set S as follows. For $R, T \subseteq S$ we put $R \nabla T$ if either $R = T$ or $R \cap T = \emptyset$.

Let S and $H \leq A(S)$ be given. A segment R of S is called an "H-block" if for every $f \in H$ we have $R \nabla f''R$, where of course

$f''R = \{f(r);\ r \in R\}$. Clearly if $H \leq K \leq A(S)$, then every K-block is an H-block. We refer to $A(S)$-blocks simply as "blocks".

Let L be a set of pairwise disjoint segments of S . We can define an ordering on L as follows. For $B, C \in L$, we set $B < C$ if we have $s < t$ for some $s \in B$ and some $t \in C$. It is readily seen that this ordering is well-defined.

One final piece of notation is required. Let S be given, and take $H \leq A(S)$. For each $x \in S$ we put $H(x) = \{f(x);\ f \in H\}$, $H^+(x) = \{y \in H(x);\ y > x\}$, and $H^-(x) = \{y \in H(x);\ y < x\}$. If $H = A(S)$, then we write "$A(x)$" , "$A^+(x)$" , "$A^-(x)$" for $H(x), H^+(x)$, and $H^-(x)$ respectively.

We use the same notation for blocks. Thus if D is an H-block, then $H(D) = \{f''D;\ f \in H\}$, and so on.

THEOREM 1. *Let S be a set, and take $x \in S$ and $H \leq A(S)$.*

(1) *$H^+(x)$ has a first element x^+ if and only if $H^-(x)$ has a last element x^- .*

(2) *If $H^+(x)$ is nonempty and has no first element, then $H(x)$ is a dense set.*

(3) *If $H^+(x)$ has a first element x^+ , then the segment $R = \{z \in S;\ x \leq z < x^+\}$ of S is an H-block.*

Proof. Suppose that $H^+(x)$ has a first element x^+ , choose $f \in H$ such that $x^+ = f(x)$, and put $x^- = f^{-1}(x)$: we claim that x^- is the last element of $H^-(x)$.

Clearly $x^- \in H^-(x)$; suppose that $y > x^-$ for some $y \in H^-(x)$, and let $g \in H$ be such that $y = g(x)$. Put $h = fg \in H$; since $x^- < y < x$, we must have $x < f(y) < x^+$, and since $h(x) = f(y)$, this is a contradiction.

In an exactly similar manner we can show that if $H^-(x)$ has a last element, then $H^+(x)$ has a first element.

(2) Suppose that $H^+(x)$ is nonempty and has no first element; we shall show firstly that $H^+(x)$ is a dense set.

Take $u, v \in H^+(x)$ with $u < v$, and choose $f_u, f_v \in H$ such that

$u = f_u(x)$ and $v = f_v(x)$. Put $y = f_u^{-1}(v)$; it is easily shown that $y \in H^+(x)$, and so there exists $z \in H^+(x)$ such that $z < y$. Put $w = f_u(z)$; then $w \in H^+(x)$, and $u < w < v$. Therefore $H^+(x)$ is dense.

By (1), $H^-(x)$ has no last element, and in similar vein we can show that $H^-(x)$ is dense. But $H(x) = H^-(x) \dot{\cup} \{x\} \dot{\cup} H^+(x)$, whence it follows easily that $H(x)$ is dense.

(3) Suppose that $H^+(x)$ has a first element x^+ , and let R be the segment $\{z \in S;\ x \leq z < x^+\}$. Suppose that for some $f \in H$ we do not have $R \nabla f''R$; by replacing f with its inverse if necessary, we may assume that $R < \{y\}$ for some $y \in f''R$. Since $R \cap f''R \neq \emptyset$, it follows that $x^+ \in f''R$; put $u = f^{-1}(x^+)$. Then $u \in R$; that is, $x \leq u < x^+$. But $x < u \Rightarrow u \in H^+(x)$, which contradicts the definition of x^+ . Therefore we must have $x = u$, from which it follows that $R < \{y\}$ for every $y \in f''R$, contradicting our assumption that $\sim (R \nabla f''R)$.

Thus R is an H-block of S .

COROLLARY. *Let S be a scattered set, and take $H \leq A(S)$. Then for each $x \in S$, either $H(x) = \{x\}$ or else every element of $H(x)$ is an immediate predecessor and an immediate successor in H .*

Proof. Since S is scattered, there is no $x \in S$ such that $H(x)$ is dense. Therefore for each $x \in S$, either $H(x) = \{x\}$ or x has both an immediate predecessor and an immediate successor in H .

Take $x \in S$ such that $H(x) \neq \{x\}$, and let y be an element of $H(x)$ with $y \neq x$. Since there exists $g \in H$ with $g(x) = y$, we have $x \in H(y)$, from which we conclude easily that $H(y) = H(x)$.

The corollary now follows.

We retain the convention introduced in the preceding theorem of denoting the first element of $H^+(x)$ - when it exists - by "x^+" and the last element of $H^-(x)$ by "x^-" . We also extend this notation to blocks, with the following justification.

LEMMA 1. *Let S be a set, and let B be an H-block of S , where $H \leq A(S)$ is given. Then the elements of $H(B)$ are pairwise disjoint.*

https://doi.org/10.1017/S000497270003673X Published online by Cambridge University Press

Proof. Take $C, D \in H(B)$, with of course $C \neq D$. There exist $f, g \in H$ such that $f''B = C$ and $g''B = D$. Suppose that $C \cap D \neq \emptyset$. Put $h = f^{-1}g \in H$: then $B \cap h''B \neq \emptyset$, and since B is an H-block we must have $B = h''B$. But then $C = f''B = g''B = D$, a contradiction.

We are going to partition a given set into blocks of a certain type, and then express the automorphisms of the set as products of the automorphisms of these blocks. The key to this procedure is the definition of a certain operation I on the collection of blocks of the given set. This definition we shall present shortly, but first of all we need a result justifying the definition.

LEMMA 2. *Let* S *be a set, let* $H \leq A(S)$ *be given, and let* B *be an* H*-block of* S. *Assume that* B^+ *exists, let* f, g *be any two elements of* H *such that* $f''B = g''B = B^+$, *and define the subsets* R_0, R_1 *of* S *by*

$$R_0 = \{f^n(x) \in S;\ x \in B \ \&\ n \in Z\}, \quad R_1 = \{g^n(x) \in S\ \ x \in B \ \&\ n \in Z\},$$

where Z *is the set of integers. Then* $\mathrm{Ch}(R_0) = \mathrm{Ch}(R_1)$.

Proof. We shall show that $R_1 \subseteq \mathrm{Ch}(R_0)$, whence we obtain $\mathrm{Ch}(R_1) \subseteq \mathrm{Ch}(R_0)$. Since the argument will be symmetric, we shall also be able to infer $\mathrm{Ch}(R_0) \subseteq \mathrm{Ch}(R_1)$, and so obtain $\mathrm{Ch}(R_0) = \mathrm{Ch}(R_1)$.

Suppose that for some $x \in B$ and some $n \in Z$ we have $g^n(x) \notin \mathrm{Ch}(R_0)$. Then we have either $\{g^n(x)\} < \mathrm{Ch}(R_0)$ or $\mathrm{Ch}(R_0) < \{g^n(x)\}$, and without loss of generality we may assume the latter. Since for each $y \in B$ we have $g^i(y) < g^j(y)$ for all $i, j \in Z$ with $i < j$, it follows that $\mathrm{Ch}(R_0) < \{g^n(x)\}$ for some $x \in B$ and some $n > 0$. However, $g''B = f''B$, and so $g^1(y) \in \mathrm{Ch}(R_0)$ for all $y \in B$. Hence there is a least positive integer n° for which $\mathrm{Ch}(R_0) < \{g^{n^\circ}(x)\}$ for some $x \in B$, and so $g^{n^\circ-1}(y) \in \mathrm{Ch}(R_0)$ for every $y \in B$. It follows that $g^{n^\circ-1}{}''B < f^{m}{}''B$ for some $m > 0$, and since either $f''B = g^{n^\circ-1}{}''B$ or

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$f''B < g^{n^\circ-1}{}''B$, we may assume that m is the least positive integer with this property.

We claim that in fact $g^{n^\circ-1}{}''B = f^{m-1}{}''B$. For if not, then by Lemma 1, $f^{m-1}{}''B < g^{n^\circ-1}{}''B < f^{m}{}''B$. Put $h = f^{1-m}g^{n^\circ-1} \in H$. Then $B < h''B < f''B = B^+$, which is a contradiction. Hence we have $g^{n^\circ-1}{}''B = f^{m-1}{}''B$. However, $\mathrm{Ch}(R_0) < \{g^{n^\circ}(x)\}$ for some $x \in B$, and by Lemma 1 this certainly implies $f^{m}{}''B < g^{n^\circ}{}''B$. We now put $h_0 = g^{1-n^\circ}f^m \in H$ and obtain $B < h_0''B < g''B = B^+$, another contradiction.

Thus our lemma is proved.

DEFINITION 1. Let S be a set, let $H \le A(S)$ be given, and let B be an H-block of S . Define a segment $I(B)$ of S as follows.

(1) If $H(B) = \{B\}$, put $I(B) = B$.

(2) If $H^+(B)$ is nonempty and has no first element, put $I(B) = \mathrm{Ch}(\{f(x) \in S;\ x \in B \ \&\ f \in H\})$.

(3) If $H^+(B)$ has a first element, take any $f \in H$ with $f''B = B^+$, and put $I(B) = \mathrm{Ch}(\{f^n(x) \in S;\ x \in B \ \&\ n \in Z\})$.

Lemma 2 shows that in the case of clause (3), $I(B)$ is independent of the particular choice of $f \in H$; hence our definition of the operator is valid.

LEMMA 3. *Let S be a set, let $H \le A(S)$ be given, let B be an H-block of S, and assume that B^+ exists. For each $n \ge 0$ define $B^{(n)}$ by $B^{(0)} = B$ and $B^{(n+1)} = B^{(n)+}$, and similarly for each $n \le 0$ define $B^{(n)}$ by $B^{(0)} = B$ and $B^{(n-1)} = B^{(n)-}$. Then for every $m \in Z$ and every $f \in H$ such that $f''B = B^+$, we have $f^{m}{}''B = B^{(m)}$.*

Proof. We prove the result for $m \ge 0$ by induction on m , the proo for $m \le 0$ being exactly similar. By assumption $f^{1}{}''B = B^{(1)}$.

Assume that $f^{m}{}''B = B^{(m)}$ and put $f^{m+1}{}''B = D$. Then $D = f''B^{(m)}$, and since it is clear that $B^{(m)} < D$, we must have either $D = B^{(m)+}$ or

$D > B^{(m)+}$. But if the latter holds, then we have

$$B < f^{-m}"B^{(m)+} < f"B = B^{+} ,$$

a contradiction. Therefore we must have $D = B^{(m)+} = B^{(m+1)}$

Again we retain the notation introduced in the preceding result, and note that if B^{+} exists, then $I(B)$ is the smallest segment of S that contains $B^{(m)}$ for each $m \in Z$.

THEOREM 2. *Let S be a set, let $H \leq A(S)$ be given, and let B be an H-block of S .*

(1) $f"I(B) = I(f"B)$ *for every* $f \in H$.

(2) $I(B)$ *is an H-block, and* $B \subseteq I(B)$.

Proof. (1) Take $f \in H$. If $I(B) = B$, then $f"B = B$ and the result is trivial. If $H^{+}(B)$ is nonempty but has no first element, then $I(B) = \mathrm{Ch}\big(\bigcup\{D;\ D \in H(B)\}\big)$, and once again the result is clear. Thus we are left with the case in which B^{+} exists.

We claim first of all that $f"B^{+} = (f"B)^{+}$. For if this is not so, then there exists $g \in H$ such that $f"B < g"(f"B) < f"B^{+}$, whence we obtain the contradiction $B < \big(f^{-1}gf\big)"B < B^{+}$.

Thus $f"B^{+} = (f"B)^{+}$, and by induction we can show that $f"B^{(m)} = (f"B)^{(m)}$ for each $m \in Z$. Therefore $f"B^{(m)} \subseteq I(f"B)$ for each such m , from which it follows that $f"I(B) \subseteq I(f"B)$. On the other hand, $(f"B)^{(m)} = f"B^{(m)} \subseteq f"I(B)$ for each $m \in Z$, and by the remark above we see that $I(f"B) \subseteq f"I(B)$.

Hence $f"I(B) = I(f"B)$.

(2) Obviously $B \subseteq I(B)$, and so it suffices to show that $I(B)$ is an H-block. This is clear if $I(B)$ is defined by clauses (1) or (2) of Definition 1, for in the former case we have $I(B) = B$, whilst in the latter case $f"I(B) = I(B)$ for every $f \in H$. Thus we are left with the case in which $I(B)$ is defined by clause (3).

Take $f \in H$ and suppose that $I(B) \cap f"I(B) \neq \emptyset$. By part (1) we have $I(B) \cap I(f"B) \neq \emptyset$. Put $C = f"B$; we are assuming that B^{+}

exists, and so we know from the proof of part (1) that C^+ exists; furthermore, we can show as in the proof of Lemma 2 that $C^{(i)} = B^{(j)}$ for some $i, j \in Z$, since it follows from our assumption $I(B) \cap I(C) \neq \emptyset$ that $C^{(k)} \subseteq I(B)$ for some $k \in Z$. It is clear, however, that $I\left(C^{(i)}\right) = I(C)$ and $I\left(B^{(j)}\right) = I(B)$. Thus $I(B) = I(C) = f''I(B)$, and so $I(B)$ is an H-block.

THEOREM 3. *Let* S *be a set, let* $H \leq A(S)$ *be given, and let* B, C *be* H*-blocks of* S *such that* $B \subseteq C$. *Then* $I(B) \subseteq I(C)$, *or* $I(C) \subseteq I(B)$.

Proof. We consider three cases.

(1) Suppose that $I(B) = B$. Then $I(B) \subseteq C \subseteq I(C)$.

(2) Suppose that B^+ exists, and let $f \in H$ be such that $f''B = B^+$. If $B^{(m)} \subseteq C$ for each $m \in Z$, then $I(B) \subseteq C \subseteq I(C)$, and so we may without loss of generality assume that $C < B^{(n)}$ for some $n > 0$. Since $B^{(0)} = B \subseteq C$, there is a least $n^\circ > 0$ for which $B^{(n^\circ)} > C$, and we claim that C^+ exists and that $g''C = C^+$, where $g = f^{n^\circ} \in H$.

Certainly $B^{(n^\circ)} \subseteq g''C$, and thus $C < g''C$. Suppose that we have $C < h''C < g''C$ for some $h \in H$. Then $B < h''B$, and so $h''B = B^{(i)}$ for some i with $0 < i < n^\circ$. However, $B^{(j)} \subseteq C$ for every j with $0 < j < n^\circ-1$, and certainly $B^{(n^\circ-1)} \cap C \neq \emptyset$. Thus we have a contradiction, and hence C^+ exists and $g''C = C^+$.

A routine induction argument now shows that for each $m > 0$ there exists $n > 0$ such that $C^{(n)} > B^{(m)}$. Similarly we can show that for each $m < 0$ there exists $n < 0$ such that $C^{(n)} < B^{(m)}$. Thus $I(B) \subseteq I(C)$.

(3) Suppose that $H^+(B)$ is nonempty but has no first element. Thus $H(B)$ is dense and $I(B) = \mathrm{Ch}\left(\bigcup\{D;\ D \in H(B)\}\right)$. Now if $I(C) = C$, then obviously $D \subseteq C$ for every $D \in H(B)$, whence we obtain $I(B) \subseteq C = I(C)$. Hence we may assume that $I(C) \neq C$, and it follows from Definition 1 that for each $E \in H(C)$ with $E \subseteq I(C)$, there exist $E_0, E_1 \in H(C)$ such that

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$E_0, E_1 \subseteq I(C)$ and $E_0 < E < E_1$. But clearly for each $E \in H(C)$ there exists $D \in H(B)$ with $D \subseteq E$. Therefore we have

$$I(C) = \mathrm{Ch}\left(\bigcup\{D \in H(B);\ \exists E \in H(C)\left(D \subseteq E \subseteq I(C)\right)\}\right),$$

whence it follows at once that $I(C) \subseteq I(B)$.

With reference to the above theorem: a slightly deeper examination of Case (3) shows that $I(C) \subseteq I(B)$ and $I(C) \neq I(B)$ only if C^+ exists and $H(B)$ is dense.

LEMMA 4. *Let* S *be a set, let* $H \leq A(S)$ *be given, and let* B *be an* H*-block of* S . *Suppose that* B^+ *exists, and let* D *be such that* $B \dot{\cup} D \dot{\cup} B^+$ *is a segment. Then* D *is an* H*-block, and if* $D \neq \emptyset$ *then* $I(D) = I(B)$.

Proof. Take $f \in H$ and suppose that $D \cap f''D \neq \emptyset$. Obviously we cannot have $B \cap f''D \neq \emptyset$, since this would lead to $D \cap f^{-1}{}''B \neq \emptyset$. Similarly we cannot have $B^+ \cap f''D \neq \emptyset$. Thus $f''D \subseteq D$. But if $f''D \neq D$, then either $B \cap f^{-1}{}''D \neq \emptyset$ or $B^+ \cap f^{-1}{}''D \neq \emptyset$. Therefore $f''D = D$, and so D is an H-block.

The remainder of the lemma now follows easily.

THEOREM 4. *Let* S *be a set, let* $H \leq A(S)$ *be given, and let* B, C *be* H*-blocks of* S *such that* $B \cap C = \emptyset$. *Assume that* $I(B) \cap I(C) \neq \emptyset$. *Then* $I(B) \subseteq I(C)$ *or* $I(C) \subseteq I(B)$.

Proof. Since $I(B) \cap I(C) \neq \emptyset$, it is obvious that we cannot have both $I(B) = B$ and $I(C) = C$. Hence we may assume without loss of generality that $I(B) \neq B$ and that $B < C$.

From $I(B) \cap I(C) \neq \emptyset$ it follows that $I(B) \cap E \neq \emptyset$ for some $E \in H(C)$ with $E \subseteq I(C)$. Since $I(E) = I(C)$, we may as well assume that $I(B) \cap C \neq \emptyset$. But $I(B) \neq B$, and so for each $D \in H(B)$ with $D \subseteq I(B)$, there exists $D^\circ \in H(B)$ such that $D^\circ \subseteq I(B)$ and $D < D^\circ$. But then, since $C > B$ and $I(B) \cap C \neq \emptyset$, it follows that there are two possibilities.

(1) $D \subseteq C$ for some $D \in H(B)$ with $D \subseteq I(B)$. From Theorem 3 we have either $I(D) \subseteq I(C)$ or $I(C) \subseteq I(D)$. Since $I(D) = I(B)$, the result follows.

(2) $C \subseteq I(B)$ and $C \cap D \neq \emptyset$ for at most one $D \in H^+(B)$ with $D \subseteq I(B)$. Let us deal firstly with the case in which there is no such D. Then if $H(B)$ is dense, we must have $D_0 < I(C) < D_1$ for some $D_0, D_1 \in H^+(B)$, and so $I(C) \subseteq I(B)$. On the other hand, if B^+ exists, then we must have $B^{(m)} < C < B^{(m+1)}$ for some $m \geq 0$, and the result follows by Theorem 3 and Lemma 4.

This leaves us with the case in which $C \cap D \neq \emptyset$ for exactly one $D \in H^+(B)$ with $D \subseteq I(B)$. Again if $H(B)$ is dense it is easy to show that $E \subseteq I(B)$ for each $E \in H(C)$, and hence that $I(C) \subseteq I(B)$.

Thus assume that B^+ exists, let $f \in H$ be such that $f''B = B^+$, and let m be the unique positive integer such that $C \cap B^{(m)} \neq \emptyset$. Then $f''C \cap B^{(m+1)} \neq \emptyset$, and it is easy to see that $f''C = C^+$. By induction we obtain $B^{(n+m-1)} < C^{(n)} < B^{(n+m+1)}$ for each $n \in Z$, whence it follows that $I(C) \subseteq I(B)$.

This proves our theorem.

DEFINITION 2. Let S be a set, let $H \leq A(S)$ be given, let B be an H-block of S, and let α be a nonzero ordinal. Define a segment $I^\alpha(B)$ of S as follows.

(1) $I^1(B) = I(B)$.

(2) $I^{\beta+1}(B) = I(I^\beta(B))$.

(3) If α is a limit ordinal, then $I^\alpha(B) = \bigcup\{I^\gamma(B);\ \gamma < \alpha\}$.

THEOREM 5. *Let S be a set, let $H \leq A(S)$ be given, let B be an H-block of S, and let α be a nonzero ordinal.*

(1) *For every $f \in H$, $f''I^\alpha(B) = I^\alpha(f''B)$.*

(2) *$I^\alpha(B)$ is an H-block of S.*

Proof. (1) In view of Theorem 2, it suffices to prove this when α is a limit ordinal. We have $f''I^\omega(B) = \bigcup\{f''I^n(B);\ n < \omega\}$, and so $f''I^\omega(B) = \bigcup\{I^n(f''B);\ n < \omega\} = I^\omega(f''B)$. Now take $\alpha > \omega$, and assume that

$f''I^\gamma(B) = I^\gamma(f''B)$ for each $\gamma < \alpha$. Then

$$f''I^\alpha(B) = \bigcup\{f''I^\gamma(B);\ \gamma < \alpha\} = \bigcup\{I^\gamma(f''B);\ \gamma < \alpha\} = I^\alpha(f''B)$$

(2) Again it suffices to prove this when α is a limit ordinal, and we assume that $I^\gamma(B)$ is an H-block for each $\gamma < \alpha$. Take $f \in H$ and suppose that $I^\alpha(B) \cap f''I^\alpha(B) \neq \emptyset$. Then for some $\beta, \gamma < \alpha$ we have $I^\beta(B) \cap f''I^\gamma(B) \neq \emptyset$.

Put $\delta = \max\{\beta, \gamma\}$; then $\delta < \alpha$ and $I^\delta(B) \cap f''I^\delta(B) \neq \emptyset$, by part (1). But by assumption, $I^\delta(B)$ is an H-block, and so $I^\delta(B) = f''I^\delta(B)$. Using part (1) again, we see that $I^\xi(B) = f''I^\xi(B)$ for every ξ with $\delta \leq \xi < \alpha$, from which it follows that $I^\alpha(B) = f''I^\alpha(B)$. Thus $I^\alpha(B)$ is an H-block.

THEOREM 6. *Let S be a set, let $H \leq A(S)$ be given, and let B, C be H-blocks of S such that $B \cap C = \emptyset$. Let α be a nonzero ordinal, and assume that $I^\alpha(B) \cap I^\alpha(C) \neq \emptyset$. Then $I^\alpha(B) \subseteq I^\alpha(C)$ or $I^\alpha(C) \subseteq I^\alpha(B)$.*

Proof. Once more, in view of previous results, it suffices to consider the case in which α is a nonzero limit ordinal.

Since $I^\alpha(B) \cap I^\alpha(C) \neq \emptyset$, there must exist $\beta, \gamma < \alpha$ such that $I^\beta(B) \cap I^\gamma(C) \neq \emptyset$; putting $\delta = \max\{\beta, \gamma\} < \alpha$, we see that $I^\delta(B) \cap I^\delta(C) \neq \emptyset$, whence it follows that $I^\xi(B) \cap I^\xi(C) \neq \emptyset$ for every ξ with $\delta \leq \xi < \alpha$. By induction therefore, we may assume that for each such ξ, either $I^\xi(B) \subseteq I^\xi(C)$ or $I^\xi(C) \subseteq I^\xi(B)$.

Suppose that it is not the case that $I^\alpha(B) \subseteq I^\alpha(C)$. Then for some $x \in S$, we must have $x \in I^\alpha(B) - I^\alpha(C)$, whence $x \in I^\beta(B)$ for some $\rho < \alpha$, but $x \in I^\tau(C)$ for no $\tau < \alpha$. Let θ be the least ordinal ρ for which $x \in I^\rho(B)$; then for each ζ with $\theta \leq \zeta < \alpha$, it cannot be the case that $I^\zeta(B) \subseteq I^\zeta(C)$, and so for each ζ with

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$\max\{\theta, \delta\} \le \zeta < \alpha$, we must have $I^{\zeta}(C) \subseteq I^{\zeta}(B)$.

Put $\sigma = \max\{\theta, \delta\}$; then the sequence $(I^{\psi}(C))_{\psi<\sigma}$ is nondecreasing, and so $\cup\{I^{\psi}(C);\ \psi < \sigma\} \subseteq I^{\sigma}(B)$. Thus we have $I^{\alpha}(C) \subseteq I^{\alpha}(B)$.

LEMMA 5. *Let* S *be a set, and let* $H \le A(S)$ *be given. For any* H*-block* B *, there is an ordinal* $\alpha > 0$ *such that* $I^{\alpha}(B) = I^{\alpha+1}(B)$.

Proof. For any H-block B and any ordinals β, γ with $0 < \beta < \gamma$, we have $I^{\beta}(B) \subseteq I^{\gamma}(B)$. Thus if α is any ordinal such that $|\alpha| = |S|$, then we must have $I^{\alpha}(B) = I^{\alpha+1}(B)$.

DEFINITION 3. Let S be a set, and let $H \le A(S)$ be given. For each ordinal α define a set L_{α} of H-blocks of S as follows.

(1) $L_0 = \{\{x\};\ x \in S\}$.

(2) For $\alpha > 0$, $L_{\alpha} = \left\{I^{\alpha}(B);\ B \in L_0\right\}$.

The elements of L_{α} will be called the "α-lines" of S (with respect to H).

LEMMA 6. *Let* S *be a set, and let* $H \le A(S)$ *be given. There is an ordinal* α *such that* $L_{\alpha} = L_{\alpha+1}$.

Proof. By Lemma 5, for each $x \in S$ there exists an ordinal $\beta = \beta_x$ such that $I^{\beta}(\{x\}) = I^{\beta+1}(\{x\})$. Put $\alpha = \sup\{\beta_x;\ x \in S\}$. Then $I^{\alpha}(\{x\}) = I^{\alpha+1}(\{x\})$ for each $x \in S$, and so $L_{\alpha} = L_{\alpha+1}$.

DEFINITION 4. Let S be a set, and let $H \le A(S)$ be given. The order $\mathrm{ord}_H(S)$ (with respect to H) of S is defined to be the least ordinal α such that $L_{\alpha} = L_{\alpha+1}$.

THEOREM 7. *Let* S *be a set, let* $H \le A(S)$ *be given, and let* α *be a fixed ordinal. There exists a unique set* K *of pairwise disjoint* H*-blocks of* S *having the following properties.*

(1) $S = \dot{\cup}\{B;\ B \in K\}$ *, where the ordering on* K *is that induced*

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by the ordering on S .

(2) *For each* $B \in K$, *either*

(i) $B \in L_\alpha$, *and for any* $C \in L_\alpha$, $C \supseteq B \Rightarrow C = B$; *or*

(ii) $B = \bigcup\{C_\beta;\ \beta < \gamma\}$ *for some increasing sequence* $(C_\beta)_{\beta<\gamma}$ *of* α*-lines, and there is no* $C \in L_\alpha$ *with* $C \supseteq B$.

Proof. Take any $C \in L_\alpha$. Either there exists a maximal $C^\circ \in L_\alpha$ with $C^\circ \supseteq C$, or else there is an increasing sequence $(C_\beta)_{\beta<\gamma}$ of α-lines with $C = C_0$ and such that $D \supseteq \bigcup\{C_\beta;\ \beta < \gamma\}$ for no $D \in L_\alpha$. In the first case we put $C^\# = C^\circ$, and in the second case we put $C^\# = \bigcup\{C_\beta;\ \beta < \gamma\}$. We now define K by $K = \left\{C^\#;\ C \in L_\alpha\right\}$.

We must show first of all that each $B \in K$ is an H-block. If B is of the first type given above, then this is obvious. Therefore we assume that B is of the second type, and put $B = \bigcup\{C_\beta;\ \beta < \gamma\}$ for some increasing sequence $(C_\beta)_{\beta<\gamma}$ of α-lines. Take $f \in H$ and suppose that $B \cap f''B \neq \emptyset$. Then we have $C_\delta \cap f''C_\theta \neq \emptyset$ for some $\delta, \theta < \gamma$, and hence $C_\xi \cap f''C_\xi \neq \emptyset$ for all ξ with $\max\{\delta, \theta\} < \xi < \gamma$. Therefore $C_\xi = f''C_\xi$ for all such ξ , from which it follows very easily that $B = f''B$. Thus B is an H-block.

Now take $B, D \in K$, and suppose that $B \cap D \neq \emptyset$. We must show that $B = D$. If B, D are both of the first type, then we have $B = I^\alpha(\{x\})$ and $D = I^\alpha(\{y\})$ for some $x, y \in S$, and so by Theorem 6 we have either $B \subseteq D$ or $D \subseteq B$. However, B, D are both maximal, and so $B = D$. Now suppose that B is of the first and D of the second type, and put $D = \bigcup\{D_\beta;\ \beta < \gamma\}$. Then $B \cap D_\beta \neq \emptyset$ for some $\beta < \gamma$, and so $B \cap D_\delta \neq \emptyset$ for every δ with $\beta \leq \delta < \gamma$, whence either $B \subseteq D_\delta$ or $D_\delta \subseteq B$ for every such δ . But there is no $E \in L_\alpha$ with $E \supseteq D_\beta$ for all $\beta < \gamma$, from which it follows that $B \subseteq D$. Since B is maximal, this gives $B = D$.

Finally, suppose that B, D are both of the second type, and put $B = \bigcup\{B_\psi;\ \psi < \theta\}$ with D as before. Then $B_\tau \cap D_\delta \neq \emptyset$ for some $\tau < \theta$ and some $\delta < \gamma$, and so $B_\tau \cap D_\beta \neq \emptyset$ for every β with $\delta \leq \beta < \gamma$, which tells us that either $B_\tau \subseteq D_\beta$ or $D_\beta \subseteq B_\tau$ for every such β. As above we can show from this that $B_\tau \subseteq D$, and in a similar manner we obtain $B_\psi \subseteq D$ for every ψ with $\tau \leq \psi < \theta$. Thus we conclude that $B \subseteq D$. By symmetry, $D \subseteq B$. Hence $B = D$.

Therefore the elements of K are pairwise disjoint, and as for each $x \in S$ we have $x \in I^\alpha(\{x\}) \subseteq I^\alpha(\{x\})^\# \in K$, it is clear that K satisfies the conditions. It remains to prove uniqueness.

Suppose that K' is another set of pairwise disjoint H-blocks of S satisfying the two conditions. Take $B \in K$, $B' \in K'$, and suppose that $B \cap B' \neq \emptyset$. Then exactly the same arguments as above show that either $B \subseteq B'$ or $B' \subseteq B$, whence maximality tells us that $B = B'$. Thus $K = K'$ and our theorem is proved.

DEFINITION 5. Let H be a group. H is called an "A-group" if there exists an indexed set $\{H_x\}_{x \in X}$ of groups such that

(1) $H \simeq_G \times\{H_x;\ x \in X\}$;

(2) For each $x \in X$, either $H_x = Z^Y$ for some set Y, or else $H_x \leq A(R)$ for some dense set R.

THEOREM 8. *A group* H *is an* A*-group if and only if* $H \leq A(S)$ *for some set* S.

Proof. We assume firstly that $H \leq A(S)$ for some set S, and show by induction on $\mathrm{ord}_H(S)$ that H is an A-group.

Now if $\mathrm{ord}_H(S) = 0$, then we have $f(x) = x$ for every $x \in S$ and $f \in H$. Thus $H \simeq_G 0$, and obviously 0 is an A-group.

Let α be a fixed positive ordinal. We make the following induction assumption: that for every set R and every $K \leq A(R)$, if $\mathrm{ord}_K(R) < \alpha$, then K is an A-group. Now let S be a set, let $H \leq A(S)$ be given, and assume that $\mathrm{ord}_H(S) = \alpha$.

With respect to α , let K be the set of H-blocks of S whose existence was proved in Theorem 7. Since $\alpha = \mathrm{ord}_H(S)$, we have $f''B = B$ for every $f \in H$ and $B \in L_\alpha$, whence it follows easily that $f''B = B$ for every $f \in H$ and $B \in K$. Thus we have $H \simeq_G \times\{H_B;\ B \in K\}$, where for each $B \in K$, H_B is the subgroup of H whose carrier H_B is defined by $H_B = \{f \in H;\ \forall x \in S-B(f(x) = x)\}$. Since it is clear that the unrestricted direct product of a set of A-groups is again an A-group, it suffices to show that each H_B is an A-group.

Take $B \in K$ and suppose that $B \notin L_\alpha$. Then $B = \bigcup\{C_\beta;\ \beta < \gamma\}$ for some increasing sequence $(C_\beta)_{\beta<\gamma}$ of α-lines, and there is no $D \in L_\alpha$ with $D \supseteq B$. Thus γ is a limit ordinal. Now take $f \in H_B$ and $\xi, \zeta < \gamma$ with $\xi < \zeta$. Then either $C_\xi = f''C_\xi$ or $C_\xi \cap f''C_\xi = \emptyset$, and we conclude via Theorem 6 that either $C_\zeta \supseteq f''C_\zeta$ or $C_\zeta \cap f''C_\xi = \emptyset$. It follows that $H_B \simeq_G \times\{H_{C_\beta};\ \beta < \gamma\}$, where H_{C_β} is defined in the obvious manner.

We have therefore reduced the problem of showing H to be an A-group to that of showing each H_D , $D \in L_\alpha$, to be an A-group. Thus we may assume without loss of generality that $S = I^\alpha(\{x\})$ for some $x \in S$; for typographical convenience we put $D_\psi = I^\psi(\{x\})$ for each $\psi < \alpha$.

Suppose firstly that α is a limit ordinal. Then we have $S = \bigcup\{D_\psi;\ \psi < \alpha\}$, and the same argument as above shows that $H \simeq_G \times\{H_{D_\psi};\ \psi < \alpha\}$. In view of our induction assumption therefore, the result in this case will follow once we have shown that $\mathrm{ord}_K(D_\psi) < \alpha$, where for convenience we are setting $K = H_{D_\psi}$.

Now $\mathrm{ord}_K(D_\psi)$ is defined by means of an operator I° , and this operator I° is in turn defined using the group K , whereas the operator

I is defined in terms of the group H. But K is the restriction of the group H to the H-block D_ψ. It follows that if $B \subseteq D_\psi$ is a K-block, then B is also an H-block, and $I^\circ(B) \subseteq I(B)$. Therefore if δ is any ordinal such that $D_\psi \cap I^\delta(B) = D_\psi \cap I^{\delta+1}(B)$, then we also have $I^{\circ\delta}(B) = I^{\circ\delta+1}(B)$. Hence $I^{\circ\psi}(\{y\}) = I^{\circ\psi+1}(\{y\})$ for any $y \in D_\psi$, and so $\mathrm{ord}_K(D_\psi) \le \psi < \alpha$.

Therefore our result is proved for the case in which α is a limit ordinal.

We now assume that $\alpha = \delta + 1$ for some δ, and put $B = D_\delta$. Thus $S = I(B)$, and $H \simeq_G H_B \times K$ where K is the factor group H/H_B. In the same way as above we can show that H_B is an A-group, and so it suffices to show that K is an A-group.

Suppose that B^+ exists. Then we know that for each $f \in H$ we have $f''B = B^{(m)}$ for some $m \in Z$. Let $f^\circ \in H$ be such that $f^{\circ\prime\prime}B = B^+$, and put $g = f^\circ H_B \in K$. Then g has infinite order and generates K; thus $K \simeq_G Z$, and so K is an A-group.

Now assume that B^+ does not exist. Then $H(B)$ is dense, and $C \subseteq I(B)$ for every $C \in H(B)$. It follows that we have an embedding $K \to_B A(H(B))$, and so once more K is an A-group.

We have therefore shown that if $H \le A(S)$ for some set S, then is an A-group. We must now prove the converse.

Let $\{H_x\}_{x\in X}$ be some indexed set of groups such that for each $x \in X$, either $H_x = Z^Y$ for some set Y, or else $H_x = A(R)$ for some dense set R. Put $H = \times\{H_x;\ x \in X\}$; clearly it suffices to show that $H = A(S)$ for some set S.

By "gathering terms", we may assume that there is exactly one $x \in X$ such that $H_x = Z^Y$ for some set Y. Let κ be the smallest ordinal for which $|\kappa| = |Y|$, and let S_0 be any set of order-type $(\omega^*+\omega)\kappa$. Then

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we have $A(S_0) \simeq_G Z^Y$. Now let $(x_\tau)_{\tau<\eta}$ be some well-ordered enumeration of X such that $H_{x_0} = Z^Y$, and for each τ with $0 < \tau < \eta$, let W_τ be a set of order-type $\tau + 1$ and let S_τ be a dense set such that $H_{x_\tau} \simeq_G A(S_\tau)$. Put $S = S_0 \mathbin{\dot\cup} \dot{\bigcup}\{S_\tau \mathbin{\dot\cup} W_\tau ;\ 0 < \tau < \eta\}$. Then it is routine to show that $H \simeq_G A(S)$.

Our theorem is thus proved.

THEOREM 9. *Let S be a scattered set. Then $A(S) = Z^Y$ for some set Y.*

Proof. By examining the proof of Theorem 8, we see that $A(S)$ contains as a factor a group $K \leq A(R)$ for some dense set R only if $A(B)$ is dense for some block B of S. But obviously there is an embedding $A(B) \to S$, and so S cannot be scattered.

Taking the contrapositive, we see that if S is scattered, then $A(S)$ contains no such group K as a factor, and thus by Theorem 8 we must have $A(S) = Z^Y$ for some set Y.

COROLLARY. *If S is a scattered set, then $A(S)$ is an ordered group.*

Proof. By our theorem we know that $A(S)$ is abelian. Cohn in [*1*] has shown, however, that for any set S, $A(S)$ is an ordered group if and only if it is abelian.

Holland has shown in [3] that if H is a lattice-ordered group, then $H \leq A(S)$ for some set S. Since every ordered group is lattice-ordered, the same result holds for ordered groups, and hence from Theorem 8 we can conclude that every ordered group is an A-group.

We follow current custom in calling a group "torsion-free" if every nontrivial element of it has infinite order.

THEOREM 10. *Every torsion-free abelian group is an A-group.*

Proof. Levi has shown in [*4*] that every torsion-free abelian group is an ordered group. The result now follows from Holland's result and Theorem 8.

THEOREM 11. *Let* F *be a free group of rank greater than* 1 . *Then* $F \leq A(S)$ *for some dense set* S .

Proof. Neumann has shown in [5] that every free group is an ordered group. Hence by Holland's result and Theorem 8, F is an A-group. Let $F = \times\{F_x;\ x \in X\}$ be the "A-group representation" of F given by Theorem 8. We claim firstly that $|X| = 1$.

For suppose that $|X| \geq 2$. Then there exist A-groups F_0, F_1 , neither of which is the trivial group, such that $F \simeq_G F_0 \times F_1$; for convenience we assume that $F = F_0 \times F_1$, and represent the elements of F as ordered pairs. The identity element of F_i will be denoted by "e_i" , $i = 0, 1$. Take $f_i \in F_i - \{e_i\}$, $i = 0, 1$, and consider the elements (f_0, e_1), (e_0, f_1) of F . We then have

$$(f_0, e_1)(e_0, f_1) = (f_0, f_1) = (e_0, f_1)(f_0, e_1) ,$$

and since F is free, it follows that for some positive integer n we must have either $(f_0, e_1)^n = (e_0, f_1)$ or else $(f_0, e_1) = (e_0, f_1)^n$. Since this implies $f_i = e_i$ for some $i = 0, 1$, we have a contradiction.

Therefore $|X| = 1$; put $X = \{x\}$. If $F_x = Z^Y$ for some set Y , then F would be abelian, contradicting the fact that F is free with rank greater than 1 . Therefore we must have $F_x \leq A(S)$ for some dense set S .

This proves our theorem.

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