1 Basic Game Theory

Game theory is the study of mathematical modeling of strategic interactions among agents. Agents, or players in the game, are usually considered rational (i.e., seeking their own maximum benefit from the game's outcome). The outcomes of the game, or the consequences of the actions of agents, depend on their (strategic) interactions and (nonstrategic) environmental factors. Thus, a rational player should observe, analyze, and predict the actions of other players in the game in order to select the appropriate action that leads to the most desired outcome. In game theory, the main goal is to predict the outcome of the game given the rationality of players and defined environments.

A game is composed of at least the following elements: players, actions, outcomes, and utility functions. Players, which are denoted as $\mathcal{N} = \{1, 2, 3, \ldots, N\}$, are the agents who would rationally maximize their utilities in the game. The way for Player i to maximize the corresponding utility is to choose the action a_i from the action set $\mathcal{A}_i$. An action profile $\mathbf{a} = \{a_1, a_2, \ldots, a_N\}$ represents the actions that each player may select in the game. When the action profile $\mathbf{a}$ is determined, the outcome of the game can then be derived by the outcome function $O(\mathbf{a})$. Given the outcome, each player may receive a utility according to the utility function $U_i(O(\mathbf{a}))$. In sum, we may describe a game as a tuple $\mathcal{G} = \{\mathcal{N}, \mathcal{A}, \mathcal{U}, O\}$.

1.1 Strategic-Form Games and Nash Equilibrium

The strategic-form game is one of the most basic game structures, where the relation between outcome and action can be represented in a matrix form. A well-known strategic-form game is the prisoner's dilemma. Two players are questioned by the prosecutor regarding the evidence of a crime. They may choose to stay silent or betray the other for a shorter prison sentence. The prisoner's dilemma can be explained with the payoff matrix in Table 1.1. In the matrix, each cell represents the utility received by Players A and B, respectively, if they choose the action profile.

In a strategic-form game, we seek the Nash equilibrium, which represents the expected action profile (i.e., the outcome of the game) if the players are rational.

https://doi.org/10.1017/9781108859783.002 Published online by Cambridge University Press

Table 1.1. Prisoner's dilemma: a strategic-form game

(A B)	Stay Silent	Betray
Stay Silent	(−1,−1)	(−8,0)
Betray	(0,−8)	(−5,−5)

DEFINITION 1.1 (Nash equilibrium) Nash equilibrium is the action profile $\mathbf{a}^* = \{a_1^*, a_2^*, \ldots, a_N^*\} \in \mathcal{A}$ that

$$U_i(O(a_i^*, \mathbf{a}_{-i}^*)) \geq U_i(O(a_i, \mathbf{a}_{-i}^*)), \forall i \in \mathcal{N}, a_i \in \mathcal{A}_i,$$

where a_i is the action of Player i and $\mathbf{a}_{-i}$ is the action profile of all players except Player i.

The concept of the Nash equilibrium states that every player is satisfied with the selected action in the profile given the actions selected by other players in the profile. In other words, no player can receive higher utility by changing their own action. Therefore, rational players have no incentive to change their actions when the game falls into Nash equilibria. Notice that the Nash equilibrium or pure-strategy Nash equilibrium we defined in Definition 1.1 may not always be unique or even may not exist. It is necessary to analyze the existence and uniqueness of Nash equilibria in the defined game model.

In the prisoner's dilemma game illustrated in Table 1.1, readers may observe that, given any action selected by one player, the other player will have higher utility if they choose to betray. Therefore, $\{Betray, Betray\}$ is the unique Nash equilibrium. Given that a Nash equilibrium exists and is unique in the prisoner's dilemma, we could predict that the final outcome of the game, if all players are rational, will be $\{Betray, Betray\}$.

One may notice that the best choice for the players, if they cooperate, would be the $\{StaySilent, StaySilent\}$, for the utility of $(-1, -1)$. Nevertheless, this better outcome is impossible in the prisoner's dilemma, since both players have the incentive to betray for a shorter prison sentence. This rational choice eventually leads them to a worse outcome, which explains why this is a "dilemma." This example suggests that rational decisions in the game may lead to suboptimal outcomes from the perspective of overall system efficiency.

1.2 Extensive-Form Games and Subgame-Perfect Nash Equilibrium

In a strategic-form game, players do not have knowledge of the actions selected by the other players. Such games are suitable for problems involving simultaneous decision-making or if the decisions are privately made. For scenarios in which the actions of players can be observed, either fully or partially, by other players, extensive-form games would be more suitable.

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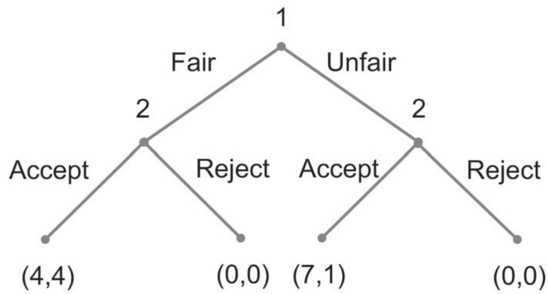

Figure 1.1 Ultimatum game

An extensive-form game can be represented by a game tree, in which the terminal nodes are the outcomes of the game with payoffs to the players, and the rest of the nodes are the decision timings when the players (and/or the nature) may select the actions (and/or external influence) to direct the game toward certain outcomes.

The ultimatum game, which is illustrated in Figure 1.1, is a famous example of an extensive-form game. In the ultimatum game, Player 1 is requested to propose an offer for sharing a cake, while Player 2 can choose to accept or reject the offer. If Player 2 accepts the offer, the cake will be allocated as is. Nevertheless, if Player 2 chooses to reject the offer, neither player will receive the cake.

In this game, we may identify three Nash equilibria according to Definition 1.1:

(1) Player 1 offers to fairly share the cake and Player 2 only accepts when the offer is fair, or $\{Fair, Fair|Accept, Unfair|Reject\}$.
(2) Player 1 offers to unfairly share the cake and Player 2 only accepts when the offer is unfair, or $\{Unfair, Fair|Reject, Unfair|Accept\}$.
(3) Player 1 offers to unfairly share the cake and Player 2 accepts any offer, or $\{Unfair, Fair|Accept, Unfair|Accept\}$.

The first one is the offer that we sometimes see when people are facing the situation similar to Player 2; that is, they will make an ultimatum or threat that they will damage both of them if they are treated unfairly. This is why the game is called the ultimatum game. Nevertheless, although such a claim can be supported by a Nash equilibrium, it may not be a suitable solution concept for the extensive-form game, as it does not consider the fact that Player 2 already knows which action Player 1 has selected at the time of their decision.

The subgame-perfect Nash equilibrium is a refined solution concept for the extensive-form game. A subgame is a subtree of the game tree starting from any node and including all branches following the starting node. A subgame-perfect Nash equilibrium is defined as follows:

DEFINITION 1.2 (Subgame-perfect Nash equilibrium) A Nash equilibrium is a subgame-perfect Nash equilibrium if it is a Nash equilibrium in every subgame.

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The concept of the subgame-perfect Nash equilibrium captures the rationality of players when they know the actions of other players beforehand. In the ultimatum game, for instance, Player 2 already knows whether Player 1 has offered a fair share when they choose to accept or not. In both subgames, after Player 1 proposes the offer, the only Nash equilibrium in the subgames would be Player 2 choosing to accept the offer, since rejecting will give Player 2 zero utility. With this refinement, the only subgame-perfect Nash equilibrium would be $\{Unfair, Fair|Accept, Unfair|Accept\}$.

1.3 Incomplete Information: Signal and Bayesian Equilibrium

In some problems, the outcome of the game involves uncertainty. The uncertainty may come from the external factors that could not be observed directly or the private preferences and actions of the players. The exact outcome of the game therefore may not be known at the time of the decision-making. In such a game, rational players must estimate the influence of this uncertainty. Instead of seeking utility maximization, they aim to maximize their expected utility.

Let us consider a game with player set $\mathcal{N}$ and action set $\mathcal{A}$. The game is in a state $\theta \in \Theta$, which is unknown to some or all players. The utility of Player i is given by $U_i(\mathbf{a}, \theta)$, $\mathbf{a} \in \mathcal{C}$, which depends on the action selected by Player i, the actions selected by the other players, and the state θ.

The uncertainty in the state can be estimated if the distribution of the state is known. It can either be known in advance (signaling game) or learned about from the received information (social learning). Let us assume that the probability of the state $\mathbf{p_i}$ (or belief), on state θ over state space Θ is known or derived after strategic thinking. Players then may maximize their expected utilities based on the belief as follows:

$$\mathbf{p_i}(\mathbf{I_i}) = \{p_{i,\theta} | \theta \in \Theta\}, \sum_{\theta \in \Theta} p_{i,\theta}(I_i) = 1,$$

where I_i is the information received by Player i in the game.

$$p_{i,\theta}(I_i) = \frac{Prob(I_i|\theta)}{\sum_{\theta' \in \Theta} Prob(I_i|\theta')}$$

Given this new objective of the players, we may extend the equilibrium concept to a Bayesian Nash equilibrium.

DEFINITION 1.3 (Bayesian Nash equilibrium) A Bayesian Nash equilibrium is the action profile $\mathbf{a}^*$, where

$$\sum_{\theta \in \Theta} p_{i,\theta}(I_i) U_i(a_i^*, \mathbf{a}_{-i}^*, \theta) \geq \sum_{\theta \in \Theta} p_{i,\theta}(I_i) U_i(a_i, \mathbf{a}_{-i}^*, \theta), \forall i \in \mathcal{N}, a_i \in \mathcal{A}_i.$$

The reputation game is a famous example of a game with incomplete information. In the game we have two firms. Firm 1 is in the market and prefers a monopoly. Firm 2 is new and would like to enter the market. There are two possible types of

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Table 1.2. Reputation game

	Stay	Exit
Sane/Prey	(2,5)	(X,0)
Sane/Accommodate	(5,5)	(10,0)
Crazy/Prey	(0,−10)	(0,0)

Firm 1: Sane and Crazy, each with 0.5 probability. Their actions and corresponding utilities are illustrated in the game matrix in Table 1.2.

In such a game, there are two possible Bayesian Nash equilibria, while their existence depends on the value of X.

Pooling equilibrium: When X = 8, both Sane and Crazy Firm 1 will choose to prey. In such a case, Firm 2 has no way to distinguish between these two types. Given the distribution of the type, Firm 2 will choose to exit instead of stay.

Separating equilibrium: When X = 2, Sane Firm 1 will accommodate and Crazy Firm 1 will prey. Firm 2 will stay when seeing accommodate and exit when seeing prey. In this equilibrium, Firm 2 can judge Firm 1's type through the observed action. In other words, Firm 1's action can be treated as a signal to improve the estimation of the unknown types. The rules of signaling in incomplete information will be discussed in detail in Part III.

1.4 Repeated Games and Stochastic Games

The previous game models are based on the assumption that the game will be played only once (i.e., a one-shot game). In practice, agents may face the same problem multiple times, each time with the same or different players. In such a scenario, we may formulate the problem as a repeated game. A repeated game consists of a series of base games in which the players play the same base game sequentially. It can be written as an extensive-form game by expanding the base game in a game tree repeatedly.

There are two kinds of repeated games: finite and infinite repeated games. For finite repeated games, the number of rounds of the base game is finite. This suggests that an ending base game exists, and the game tree in the extensive form is finite. In such a scenario, the game can be analyzed directly within the concept of a subgame-perfect Nash equilibrium. For the other case, where the rounds are infinite, there is no ending game and therefore a subgame-perfect Nash equilibrium cannot be applied directly.

The utility of Player i in a repeated game can be written as follows:

$$U_i = \lim_{T\to\infty} \sum_{t=0}^{T} \delta^t u_i(a_i(t), \mathbf{a}_{-i}(t)), \tag{1.1}$$

where $a_i(t)$ and $\mathbf{a}_{-i}(t)$ are the actions selected by Player i and the other players at round t, respectively, and $0 < \delta < 1$ is the discount factor for evaluating the utility in the future to the players in the present.

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Taking the prisoner's dilemma as an example, we may consider a repeated version of the prisoner's dilemma, which is called the iterated prisoner's dilemma. When the rounds are finite, it can be easily shown that the original $\{Betray, Betray\}$ equilibrium still holds as the unique Nash equilibrium of the game. Nevertheless, when the rounds are infinite, the equilibrium becomes nonunique as cooperation between players becomes possible. For instance, a tit-for-tat strategy (i.e., stay silent in the first round and then choose the action selected by the opponent in the previous round) is also a Nash equilibrium, since any deviation from such a strategy will lead to $\{Betray, Betray\}$ in every round, while continuing to use the strategy will keep them at $\{StaySilent, StaySilent\}$ and eventually lead to higher utility in the long run. This example suggests that cooperation is likely to emerge in a repeated game. Readers will find more related examples in Part I.

In practice, even if the agent is facing the same problem and applying the same action multiple times, the results could be different due to external uncertainty or randomness in the system. We may capture this characteristic with stochastic games, which are repeated games with uncertainty.

The uncertainty is captured through adding a state $\theta \in \Theta$ to the game. The state is observable by the players but it may change in the next round with the probability described by $P(\theta'|\theta, \mathbf{a})$, which depends on the current state and the action profiles selected by the players. The utility of the players depends on not only the action profile, but also the state. Therefore, the utility of Player i in the stochastic game can be written as follows:

$$\begin{aligned}U_i = \lim_{T\to\infty} & u_i(\theta(0), a_i(0), \mathbf{a}_{-i}(0)) \\ & + \sum_{t=1}^{T} P(\theta(t)|\theta(t-1), \mathbf{a}(t-1))\delta^t u_i(\theta(t), a_i(t), \mathbf{a}_{-i}(t)). \end{aligned} \tag{1.2}$$

Given that the state transitions depend on the previous state and the action profile, the system can be formulated as a Markov decision process when the action profile is given, which helps us to derive the expected utility given certain action profiles. Then, a Bayesian Nash equilibrium can be applied to derive the equilibrium of the game. Readers will find more examples in Parts II and III.