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GENERALIZED HUGHES PLANES

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1. Introduction. The projective planes discovered in 1957 by Hughes [**3**] were originally described by means of a nearfield F satisfying the following conditions:

(a) F is finite,

(b) the centre and kernel of F coincide,

(c) F is of rank 2 over its kernel.

(The definitions of these terms will be given in § 2; the terminology used throughout the paper is that of [**1**].)

Rosati [**5**] showed in 1960 that condition (a) is not necessary, thus constructing the first "infinite Hughes planes". Condition (b), however, plays an essential part also in Rosati's work.

The aim in this paper is to show that condition (b) is superfluous as well. For the finite case, this has been remarked by Ostrom [**4**] without proof; here we shall show that a "generalized Hughes plane" can be constructed over any nearfield satisfying condition (c) only. (This includes, in particular, the seven irregular nearfields of Zassenhaus [**7**].)

The principal new tool is a representation theorem for nearfields satisfying (c); this will be given in § 2. The actual construction will be carried out in §§ 3, 4, and we shall organize our arguments in such a way that condition (c) will be used only in § 4. It is surprising how much can be done without (c), and although (c) is clearly indispensable for the final steps of the construction, I feel that there may be many presently undiscovered planes for whose construction the results of § 3 are essential.

Finally, in § 5, we shall prove some properties of the collineation groups of the generalized Hughes planes. (These properties were known before in case (b) holds; cf. [**6**].) We show that there is a collineation group transitive on incident pairs of points and lines not belonging to the distinguished Baer subplane, so that the planes may be represented within the group $\mathrm{GL}_3(\mathfrak{K}(F))$; cf. [**1**, p. 15]. This fact was used in my forthcoming paper [**2**].

2. A theorem on nearfields. A *nearfield* is an algebraic structure F with an addition and a multiplication such that all axioms for an associative (but not necessarily commutative) division ring are satisfied except possibly the distributive law

$$\text{(D)} \qquad k(x + y) = kx + ky.$$

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The set of all $k \in F$ such that (D) holds for all $x, y \in F$ is the *kernel* of F, denoted by $\mathfrak{K}(F)$. Clearly, $\mathfrak{K}(F)$ is a division ring, and for any sub-division ring K of $\mathfrak{K}(F)$, we may regard F as a left vector space over K. The dimension of this vector space is the *rank* of F over K; in the most important case, $K = \mathfrak{K}(F)$, we speak simply of the rank of F.

The *centre* of the nearfield F is the set $\mathfrak{Z}(F)$ of all those $z \in F$ for which

$$\text{(C)} \qquad xz = zx$$

holds, for all $x \in F$. The centre need not be closed with respect to addition; hence it is in general not a subfield of F. This is the main reason why the centre will play only a minor role in the remainder of the paper.

2.1. *For any nearfield F, we have $\mathfrak{Z}(F) \subseteq \mathfrak{Z}(\mathfrak{K}(F)) \subseteq \mathfrak{K}(F)$.*

This follows easily from the definitions.

The aim in this section is to prove a theorem on the representation of nearfields of rank $\leqq 2$. For this, we consider an arbitrary division ring K. A pair (u, v) of mappings u, v from $K \times K$ into K will be called a pair of *nearfield functions* on K if the following conditions are satisfied:

$$\text{(1)} \qquad (x, y) \to (u(x, y), v(x, y)) \text{ is a permutation of } K \times K,$$

$$\text{(2)} \qquad u(1, 0) = 0,\ v(1, 0) = 1, \text{ and } u(0, 0) = 0,$$

$$\text{(3)} \quad u(x_1x_2 + y_1u(x_2, y_2), x_1y_2 + y_1v(x_2, y_2)) = u(x_1, y_1)x_2 + v(x_1,y_1)\,u(x_2, y_2),$$

$$\text{(4)} \quad v(x_1x_2 + y_1u(x_2, y_2), x_1y_2 + y_1v(x_2, y_2)) = u(x_1, y_1)y_2 + v(x_1, y_1)v(x_2, y_2).$$

After this preparation, we can state the following result.

2.2. THEOREM. *If (u, v) is a pair of nearfield functions on the division ring K, then the additive group $F = K \oplus K$ is a nearfield with respect to the following multiplication:*

$$\text{(5)} \qquad (x_1, y_1)(x_2, y_2) = (x_1x_2 + y_1u(x_2, y_2), x_1y_2 + y_1v(x_2, y_2)).$$

The rank of F is 1 or 2 according as the permutation (1) is, or is not, an additive automorphism of F. If F has rank 2, then $\mathfrak{K}(F) = \{(x, 0) | x \in K\} \cong K$.

Conversely, every nearfield of rank 2 over K can be represented in this fashion.

Proof. That (5) turns $F = K \oplus K$ into a nearfield is only a matter of straightforward verification. In particular, (1) guarantees that $F - \{(0, 0)\}$ is a quasi-group with respect to the multiplication (5); the identity element is $(1, 0)$ because of (2), and (3) and (4) yield associativity of multiplication.

The equation

$$(x_1, y_1)[(x_2, y_2) + (x_2', y_2')] = (x_1, y_1)(x_2, y_2) + (x_1, y_1)(x_2', y_2'),$$

which is equivalent to (D) above, degenerates with (5) into the equations

$$\text{(6)} \qquad y_1[u(x_2 + x_2', y_2 + y_2') - u(x_2, y_2) - u(x_2', y_2')] = 0$$

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and

$$(7) \qquad y_1[v(x_2 + x_2', y_2 + y_2') - v(x_2, y_2) - v(x_2', y_2')] = 0.$$

Since these are certainly satisfied if $y_1 = 0$, the set $\{(x, 0) | x \in K\}$ is always in $\mathfrak{K}(F)$. If it is a proper subset of the kernel, then there must exist a $y_1 \neq 0$ such that (6) and (7) hold for all x_2, x_2', y_2, y_2'. This means that the mappings $(x, y) \to u(x, y)$ and $(x, y) \to v(x, y)$ are additive homomorphisms of F into K, whence it follows from (1) that $(x, y) \to (u(x, y), v(x, y))$ is an automorphism of F. On the other hand, if (1) is an automorphism, then (6) and (7) are always satisfied, and F is of rank 1.

Now assume conversely that F is any nearfield of rank 2 over K. (Note that F may be of rank 1, i.e. a division ring, since it is not assumed that K is the kernel.) Then the identity element 1 together with an arbitrary element $e \notin K$ of F are a basis of F over K, so that each element of F can be written in one and only one way as $x + ye$, with $x, y \in K$. We define two functions u and v from F into K by

$$(8) \qquad e(x + ye) = u(x, y) + v(x, y)e.$$

Then, since $x + ye \to e(x + ye)$ is a permutation of F, we obtain (1); on the other hand, $e1 = e$ implies (2).

Next, using the fact that the other (the right) distributive law is valid and that K is in the kernel, we obtain

$$\begin{aligned}(x_1 + y_1e)(x_2 + y_2e) &= x_1(x_2 + y_2e) + y_1e(x_2 + y_2e)\\ &= x_1x_2 + x_1y_2e + y_1[u(x_2, y_2) + v(x_2, y_2)e]\\ &= x_1x_2 + y_1u(x_2, y_2) + [x_1y_2 + y_1v(x_2, y_2)]e.\end{aligned}$$

This means, of course, that multiplication in F obeys the rule (5).

It remains to verify (3) and (4) for the functions u and v defined by (8). This is done in a straightforward fashion by computing the product $e(x_1 + y_1e)(x_2 + y_2e)$ in two different ways, using (5) and the associativity of multiplication in F.

3. The general construction. Let F be a nearfield and K a sub-division ring of $\mathfrak{K}(F)$. We make no assumptions on the rank of F over K here; in particular, we do not exclude $F = \mathfrak{K}(F) = K$. We denote by W the direct sum of three copies of the additive group of F, i.e. the set of all triples $\mathbf{x} = (x_1, x_2, x_3)$, with $x_i \in F$ $(i = 1, 2, 3)$ which is an additive group with respect to componentwise addition. The subset of all those $\mathbf{x} \in W$ for which each x_i is in K $(i = 1, 2, 3)$ will be denoted by V.

We introduce a right scalar multiplication of the elements of W by elements in F by

$$(9) \qquad (x_1, x_2, x_3)f = (x_1f, x_2f, x_3f) \qquad \text{for all } x_i, f \in F.$$

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It is clear that this scalar multiplication satisfies

(10) $\mathbf{x}1 = \mathbf{x}, \quad \mathbf{x}(fg) = (\mathbf{x}f)g, \quad (\mathbf{x}+\mathbf{y})f = \mathbf{x}f + \mathbf{y}f,$

for all $\mathbf{x}, \mathbf{y} \in W$ and $1, f, g \in F$. Furthermore, we have

(11) $\mathbf{v}(f+g) = \mathbf{v}f + \mathbf{v}g \quad$ for $\mathbf{v} \in V$ and $f, g \in F$.

Next, consider the group $\mathrm{GL}_3(K)$ of all non-singular (3, 3)-matrices with entries in K. For each $A = (a_{ij}) \in \mathrm{GL}_3(K)$, we define a permutation $\gamma(A)$ of W by

(12) $$\mathbf{x}^{\gamma(A)} = A\mathbf{x} = \left(\sum_{j=1}^{3} a_{1j}x_j, \sum_{j=1}^{3} a_{2j}x_j, \sum_{j=1}^{3} a_{3j}x_j \right).$$

Although we write the elements of W as row vectors (for typographical convenience), it becomes apparent here that they should be considered as *column vectors*. We could have indicated this by the usual transposition superscript, but that would have caused much more cumbersome notation while adding practically nothing to the clarity.

3.1. *Let* Γ *denote the set of all* $\gamma(A)$, *with* $A \in \mathrm{GL}_3(K)$. *Then*
(i) *each* $\gamma \in \Gamma$ *satisfies*

(13) $(\mathbf{x}+\mathbf{y})^\gamma = \mathbf{x}^\gamma + \mathbf{y}^\gamma$ *and* $(\mathbf{x}f)^\gamma = \mathbf{x}^\gamma f$ *for all* $\mathbf{x}, \mathbf{y} \in W, f \in F$,

which we express by saying that γ *is an "F-automorphism" of* W, *and*
(ii) Γ *is a group isomorphic to* $\mathrm{GL}_3(K)$.

The proof of this is straightforward and can be left to the reader.

Now we define the *points* of W as the subsets

$$\mathbf{x}F = \{\mathbf{x}f \mid f \in F\}, \text{ with } \mathbf{o} \neq \mathbf{x} \in W.$$

Here and later, $\mathbf{o}$ denotes the zero element of W. Clearly $\mathbf{x} \in \mathbf{x}F$, and if $\mathbf{o} \neq \mathbf{y} \in \mathbf{x}F$, then $\mathbf{y}F = \mathbf{x}F$. This shows the following.

3.2. *The intersection of two distinct points consists of* $\mathbf{o}$ *alone.*

Also, the second equation of (13) yields the following.

3.3. *Each* $\gamma \in \Gamma$ *maps points onto points.*

Next, we define the lines. For this we associate with each $f \in F$ a mapping f^* from W into F by

(14) $$f^*(x_1, x_2, x_3) = x_1 + fx_2 + x_3$$

and consider the subsets

$$L(f) = \{\mathbf{x} \in W \mid f^*(\mathbf{x}) = 0\} \quad \text{for all } f \in F.$$

The *lines* of W are then defined as the images of these sets under the F-automorphisms in the group Γ of 3.1, i.e. as the subsets

$$L(f)^{\gamma(A)} = \{A\mathbf{x} \mid \mathbf{x} \in L(f)\} \quad \text{with } A \in \mathrm{GL}_3(K), f \in F.$$

Since (14) implies that $f^*(\mathbf{x}g) = f^*(\mathbf{x})g$ for all $\mathbf{x} \in W$ and $g \in F$, we have $f^*(\mathbf{x}F) = 0$ if and only if $f^*(\mathbf{x}) = 0$. Therefore we have the following.

3.4. *Each line is a union of points.*

This suggests a natural definition of *incidence* between points and lines, namely that by set-theoretical inclusion. We denote the incidence structure obtained in this way by $\mathbf{H}(F, K)$. We show in this section that $\mathbf{H}(F, K)$ is always a partial plane (cf. [**1**, p. 9] for definition) in which any two lines intersect in a point. In the next section it will then be proved that if condition (c) of the Introduction is satisfied, $\mathbf{H}(F, K)$ is even a projective plane.

The following is clear from the definition of lines.

3.5. *Each* $\gamma \in \Gamma$ *maps lines onto lines.*

Thus Γ acts (not necessarily faithfully, see 5.1 below) as a collineation group on $\mathbf{H}(F, K)$.

We call a point *interior* if it can be written in the form $\mathbf{v}F$, with $\mathbf{o} \neq \mathbf{v} \in V$. Clearly, V is essentially the three-dimensional right vector space over the division ring K, and the interior points are in a natural one-to-one correspondence with the one-dimensional subspaces of this vector space. A line is called *interior* if it is of the form $L(k)^{\gamma(A)}$ with $k \in K$, i.e. if it corresponds to a two-dimensional subspace of the vector space V. From these definitions we have, by standard arguments, the following.

3.6. *The substructure* $\mathbf{H}_0 = \mathbf{H}_0(F, K)$ *of the interior points and interior lines of* $\mathbf{H}(F, K)$ *is the Desarguesian projective plane over* K. *The group* Γ *leaves this substructure invariant and acts as the projective group* $\mathrm{PGL}_3(K)$ *on it.*

Points and lines which are not interior will be called *exterior*.

So far, things were easy; from now on we have to work a little harder. An arbitrary line can be characterized as the set of all elements $\mathbf{x} \in W$ whose coordinates x_i satisfy an equation of the form

$$(15) \qquad a_1x_1 + a_2x_2 + a_3x_3 + f(b_1x_1 + b_2x_2 + b_3x_3) = 0,$$

with $f \in F$ and $a_i, b_i \in K$ $(i = 1, 2, 3)$. These equations are in general too complicated to be of much help, and so we shall use another way to represent lines; see 3.7 below. In special cases, however, (15) reduces to a very simple, and hence useful, relationship; we give two such examples for later reference:

$$(16) \qquad L(0)^{\gamma(R)} = \{(x_1, x_2, x_3) \in W \mid x_2 = 0\}, \qquad \text{where } R = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

and

$$(17) \qquad L(f)^{\gamma(S)} = \{(x_1, x_2, x_3) \in W \mid x_3 = fx_1\}, \qquad \text{where } S = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

For the general case, we have the following.

3.7. *If* $f \in F$ *and* $A = (a_{ij}) \in \mathrm{GL}_3(K)$, *define*

$$b_i = a_{i1} - a_{i3} \quad \text{and} \quad c_i = c_i(f) = a_{i2} - a_{i3}f \qquad (i = 1, 2, 3).$$

Then the line $L(f)^{\gamma(A)}$ *consists precisely of those elements of* W *which are of the form*

$$\mathbf{p}_{A,f}(x, y) = (b_1x + c_1y, b_2x + c_2y, b_3x + c_3y), \qquad x, y \in F.$$

The proof is a straightforward application of (12) and will be omitted.

For the following, we consider the interior point

(18) $$Q = \mathbf{q}F, \quad \text{with } \mathbf{q} = (1, 0, -1) \in V.$$

We determine the lines containing Q. One such line is clearly that given by (16); this particular line will henceforth be denoted by $L(\infty)$. We first show the following.

3.8. *If* $f \neq g$, *then* $L(f) \cap L(g) = Q = L(f) \cap L(\infty)$.

Proof. We have just remarked that Q is on $L(\infty)$; that Q is on each $L(f)$ follows from $f^*(\mathbf{q}) = 0$; cf. (14) and (18). Suppose, conversely, that $\mathbf{o} \neq \mathbf{x} = (x_1, x_2, x_3) \in L(f) \cap L(g)$. Then (14) yields

$$x_1 + fx_2 + x_3 = x_1 + gx_2 + x_3 = 0,$$

whence $x_2 = 0$ (since $f \neq g$) and $x_3 = -x_1$, i.e. $x = (1, 0, -1)x_1 \in Q$. Hence $\mathbf{x}F = Q$, and similarly, if $\mathbf{o} \neq \mathbf{x} \in L(f) \cap L(\infty)$, then $\mathbf{x}F = Q$.

The converse is more difficult.

3.9. *Any line containing* $Q = \mathbf{q}F$ *is either* $L(\infty)$ *or* $L(f)$ *for some* $f \in F$.

Proof. Suppose that $Q \subset L(f)^{\gamma(A)}$. Then, by 3.7, there exist $u, v \in F$ such that $\mathbf{q} = \mathbf{p}_{A,f}(u, v)$ or, more explicitly,

(19) $$b_1u + c_1v = 1, \qquad b_2u + c_2v = 0, \qquad b_3u + c_3v = -1,$$

with $b_i \in K$ and $c_i = c_i(f)$ as defined in 3.7.

Case 1. $v = 0$. Then $u \neq 0$ by (19), and it follows that $b_2 = 0$ and $b_3 = -b_1 = -u^{-1} \neq 0$. We can assume that $c_2 \neq 0$ since otherwise $L(f)^{\gamma(A)} = L(\infty)$; then $g = -(c_1 + c_3)c_2^{-1}$ is well-defined. A simple calculation shows that

$$g^*(\mathbf{p}_{A,f}(x, y)) = 0,$$

so that, by 3.7, $L(f)^{\gamma(A)} \subseteq L(g)$.

Case 2. $v \neq 0$. Here (19) yields

$$c_1 = (1 - b_1u)v^{-1}, \qquad c_2 = -b_2uv^{-1}, \qquad c_3 = -(1 + b_3u)v^{-1}.$$

If $b_2 = 0$, we also obtain $c_2 = 0$, and again $L(f)^{\gamma(A)} = L(\infty)$. Hence we may assume that $b_2 \neq 0$, so that $g = -(b_1 + b_3)b_2^{-1}$ is well-defined. Since the b_i

are, by definition, in K, so is g, and therefore g and the b_i are in the kernel of F. Using this, we obtain for all $x, y \in F$:

$$\begin{aligned} g^*(\mathbf{p}_{A,f}(x, y)) &= b_1x + v^{-1}y - b_1uv^{-1}y + g(b_2x - b_2uv^{-1}y) \\ &\qquad\qquad + b_3x - v^{-1}y - b_3uv^{-1}y \\ &= (b_1 + b_3)x - (b_1 + b_3)uv^{-1}y - (b_1 + b_3)b_2^{-1}b_2(x - uv^{-1}y) \\ &= (b_1 + b_3)(x - uv^{-1}y - x + uv^{-1}y) \\ &= 0, \end{aligned}$$

whence, again by 3.7, $L(f)^{\gamma(A)} \subseteq L(g)$.

It remains to show that $L(f)^{\gamma(A)} \subseteq L(g)$ implies $L(f)^{\gamma(A)} = L(g)$. We prove a little more.

3.10. *No line is properly contained in another line.*

For if $L(f)^\gamma \subseteq L(g)^\delta$, then $L(f) \subseteq L(g)^{\delta\gamma^{-1}}$ and therefore $Q = \mathbf{q}F \subset L(g)^{\delta\gamma^{-1}}$. This implies, by what we have already proved of 3.9, that $L(g)^{\delta\gamma^{-1}}$ is contained either in $L(\infty)$ or in some $L(h)$. The first alternative is impossible since $L(f)$ is not contained in $L(\infty)$; hence we are left with

$$L(f) \subseteq L(g)^{\delta\gamma^{-1}} \subseteq L(h),$$

which clearly implies that $f = h$, and therefore $L(f)^\gamma = L(g)^\delta$. This completes the proofs of 3.9 and 3.10.

The following is the main result of this section.

3.11. *The intersection of any two distinct lines is a unique point.*

Proof. Without loss of generality, one of the two lines in question may be taken of the form $L(g)$, $g \in F$. Let the other be $L(f)^\gamma$, with $\gamma = \gamma(A) \in \Gamma$. If $L(f)^\gamma = L(h)$ or $L(\infty)$, the unique intersection is $Q = \mathbf{q}F$ by 3.8. Thus we may assume that $L(f)^\gamma$ is not of this form or, by 3.9, that $\mathbf{q} \notin L(f)^\gamma$. This means, because of 3.7, that

$$\mathbf{p}_{A,f}(x, y) \neq \mathbf{q}z = (z, 0, -z) \quad \text{for all } x, y, z \in F. \tag{20}$$

Now $\mathbf{p}_{A,f}(x, y) \in L(f)^\gamma$ is in $L(g)$ if and only if $g^*(\mathbf{p}_{A,f}(x, y)) = 0$; cf. (14). Letting $b = b_1 + b_3$ and $c = c_1 + c_3$ for brevity (cf. 3.7), this becomes

$$bx + cy + g(b_2x + c_2y) = 0. \tag{21}$$

We have to determine the solutions x, y of (21).

Case 1. $b_2 = 0$. Then $b \neq 0$, for otherwise $b_3 = -b_1$ and $\mathbf{p}_{A,f}(1, 0) = \mathbf{q}b_1$, contradicting (20). Hence (21) is equivalent to

$$x = -b^{-1}(c + gc_2)y,$$

and it follows that $L(f)^\gamma \cap L(g)$ is the unique point $\mathbf{p}_{A,f}(-b^{-1}(c + gc_2), 1)F$.

Case 2. $b_2 \neq 0$. Then we can rearrange (21) as follows:

$$\begin{aligned} 0 &= b(b_2^{-1}b_2x + b_2^{-1}c_2y - b_2^{-1}c_2y) + cy + g(b_2x + c_2y) \\ &= bb_2^{-1}(b_2x + c_2y) - bb_2^{-1}c_2y + cy + g(b_2x + c_2y) \\ &= (bb_2^{-1} + g)(b_2x + c_2y) - (bb_2^{-1}c_2 - c)y. \end{aligned}$$

Here we have used the fact that b and b_2 are in K and therefore in the kernel of F. We write this briefly as

$$(22) \qquad u(b_2x + c_2y) = vy, \quad \text{with } u = bb_2^{-1} + g \text{ and } v = bb_2^{-1}c_2 - c.$$

Assume that $u = v = 0$. Then $g = -bb_2^{-1} \in K$ and $bb_2^{-1}c_2 = c$. Now $c_2 \neq 0$, since otherwise $c = c_1 + c_3 = 0$, i.e. $c_3 = -c_1$ and $\mathbf{p}_{A,f}(0, 1) = \mathbf{q}c_1$, contradicting (20). Therefore

$$g = -(b_1 + b_3)b_2^{-1} = -(c_1 + c_3)c_2^{-1} \in K \subseteq \mathfrak{K}(F).$$

This, however, implies, for any $\mathbf{p}_{A,f}(x, y) \subset L(f)^\gamma$, that

$$\begin{aligned} g^*(\mathbf{p}_{A,f}(x, y)) &= b_1x + c_1y + g(b_2x + c_2y) + b_3x + c_3y \\ &= (b_1 - (b_1 + b_3)b_2^{-1}b_2 + b_3)x + (c_1 - (c_1 + c_3)c_2^{-1}c_2 + c_3)y \\ &= 0, \end{aligned}$$

whence $L(f)^\gamma = L(g)$ by 3.7 and 3.10, contrary to hypothesis.

Therefore, if $u = 0$ in (22), we must have $v \neq 0$. Hence all solutions have $y = 0$ in this case, and it follows that $L(f)^\gamma \cap L(g) = \mathbf{p}_{A,f}(1, 0)F$.

Finally, if $u \neq 0$, (22) becomes $b_2x + c_2y = u^{-1}vy$, or

$$x = b_2^{-1}(u^{-1}v - c_2)y.$$

In this case, $L(f)^\gamma \cap L(g) = \mathbf{p}_{A,f}(b_2^{-1}(u^{-1}v - c_2), 1)F$. This completes the proof of 3.11.

The dual of 3.11, that any two distinct points are joined by a unique line, is not true without additional hypotheses, as may be seen from the example $K = \mathrm{GF}(q)$, $F = \mathrm{GF}(q^3)$. We can, however, prove the following weaker statement.

3.12. *If one of two distinct points is interior, then there is a unique line joining these points.*

Proof. By 3.6, the collineation group Γ is transitive on the interior points; hence there is no loss of generality in assuming that the interior point in question is the point Q of (18). Let $\mathbf{x}F$, with $\mathbf{x} = (x_1, x_2, x_3) \in W$, be any point $\neq Q$. If $x_2 = 0$, then $L(\infty)$ is a line joining Q and $\mathbf{x}F$. If $x_2 \neq 0$, then put $f = -(x_1 + x_3)x_2^{-1}$; it follows that

$$f^*(\mathbf{x}) = x_1 + fx_2 + x_3 = x_1 - (x_1 + x_3)x_2^{-1}x_2 + x_3 = 0,$$

i.e. $\mathbf{x}F \subset L(f)$. Since $L(f)$ also contains Q by 3.8, we have again found a line joining Q and $\mathbf{x}F$. Finally, 3.11 ensures the uniqueness of the line joining Q and $\mathbf{x}F$, and our proof is complete.

We sum up the results of this section as follows.

3.13. THEOREM. *For any nearfield F and any sub-division ring K of its kernel $\mathfrak{K}(F)$, the incidence structure $\mathbf{H}(F, K)$ is a partial plane in which any two lines have a common point. Moreover, each interior point is connected by a line with any other point.*

4. The generalized Hughes planes. Let F, K, V, W, and Γ be as in the preceding section, but from now on with the following additional hypothesis:

(H) $\qquad F$ is of rank 2 over K.

Note that (H) is a little weaker than condition (c) of the Introduction, because we assume only that $K \subseteq \mathfrak{K}(F)$, not $K = \mathfrak{K}(F)$.

We choose an element $e \notin K$ of F which will be kept fixed throughout. Then 1 and e form a (left K-) basis of F over K, i.e. each $f \in F$ can be written in one and only one way as

$$f = x + ye \quad \text{with } x, y \in K. \tag{23}$$

From this we conclude the following.

4.1. *Each element $\mathbf{w} \in W$ can be written in one and only one way as*

$$\mathbf{w} = \mathbf{x} + \mathbf{y}e \quad \textit{with } \mathbf{x}, \mathbf{y} \in V. \tag{24}$$

For if $\mathbf{w} = (f_1, f_2, f_3)$, write $f_i = x_i + y_i e$ $(i = 1, 2, 3)$; then (24) holds with $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$, and the uniqueness of $\mathbf{x}$, $\mathbf{y}$ in (24) follows from that of x, y in (23).

4.2. *If $\mathbf{w} = \mathbf{x} + \mathbf{y}e$ represents an exterior point, then $\mathbf{x}F$ and $\mathbf{y}F$ are distinct interior points.*

Proof. If $\mathbf{x} = \mathbf{o}$ or $\mathbf{y} = \mathbf{o}$, then $\mathbf{w}F = \mathbf{y}F$ or $\mathbf{w}F = \mathbf{x}F$, respectively, and the point represented by $\mathbf{w}$ would be interior. Therefore, $\mathbf{x} \neq \mathbf{o}$ and $\mathbf{y} \neq \mathbf{o}$, so that $\mathbf{x}$ and $\mathbf{y}$ each represents an interior point. If these points were equal, then $\mathbf{x} = \mathbf{y}f$ for some $f \in F$, and we could infer from (11) that

$$\mathbf{w}F = (\mathbf{x} + \mathbf{y}e)F = (\mathbf{y}f + \mathbf{y}e)F = \mathbf{y}(f + e)F = \mathbf{y}F,$$

which is an interior point, contradicting our hypothesis.

4.3. *Γ acts transitively on the exterior points.*

Proof. Let $\mathbf{w} = \mathbf{x} + \mathbf{y}e$ and $\mathbf{w}' = \mathbf{x}' + \mathbf{y}'e$ represent two exterior points. Then 4.2 implies that the vectors $\mathbf{x}$ and $\mathbf{y}$ as well as the vectors $\mathbf{x}'$ and $\mathbf{y}'$ of V are linearly independent. But Γ, which induces the general linear group $\mathrm{GL}_3(K)$ on V, is transitive on ordered pairs of linearly independent vectors of V. Hence there exists $\gamma \in \Gamma$ such that $\mathbf{x}^\gamma = \mathbf{x}'$ and $\mathbf{y}^\gamma = \mathbf{y}'$, and then 3.1 yields

$$(\mathbf{w}F)^\gamma = (\mathbf{x} + \mathbf{y}e)^\gamma F = (\mathbf{x}' + \mathbf{y}'e)F = \mathbf{w}'F.$$

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After these preparations, we can show the following.

4.4. *If one of two distinct points is exterior, then there is a unique line joining these points.*

Proof. We need only prove the existence of such a line; its uniqueness will then follow from 3.11. Because of 4.3, there is no loss of generality in assuming that the exterior point in question is the point

$$E = \mathbf{e}F, \quad \text{with } \mathbf{e} = (1, 0, e) \in W - V. \tag{25}$$

Clearly, E is incident with the line $L(\infty) = L(0)^{\gamma(R)}$; cf. (16). Hence, for the second point we need consider only those $\mathbf{x}F$ for which $x_2 \neq 0$. But any such point may be written in the form $(g, 1, h)F$, with $g, h \in F$. Therefore, it will be sufficient to prove the existence of a matrix $A \in \mathrm{GL}_3(K)$ such that $L(e)^{\gamma(A)}$ contains $(1, 0, e)$ as well as $(g, 1, h)$.

In order to construct such a matrix we have to use a pair of nearfield functions u, v describing F over K, i.e. a pair of mappings from $K \times K$ into K satisfying conditions (1)–(4). (The existence of such u, v is ensured by 2.2.) The multiplication in F will then satisfy

$$e(x + ye) = u(x, y) + v(x, y)e, \quad \text{where } x, y \in K; \tag{26}$$

cf. (8).

We decompose the given $g, h \in F$ as in (23):

$$g = g_1 + g_2 e \quad \text{and} \quad h = h_1 + h_2 e, \quad \text{with } g_i, h_i \in K. \tag{27}$$

By condition (1), there exists $k \in K$ such that

$$v(k, g_2) = h_2. \tag{28}$$

With this k, put $a_{11} = g_1 - k$ and $a_{31} = h_1 - u(k, g_2) - 1$; then consider the matrix

$$A = \begin{pmatrix} a_{11} & 1 & 0 \\ 1 & 0 & 0 \\ a_{31} & 0 & -1 \end{pmatrix} \in \mathrm{GL}_3(K).$$

The b_i and $c_i = c_i(e)$ of 3.7 $(i = 1, 2, 3)$ become

$$b_1 = a_{11}, \quad b_2 = 1, \quad b_3 = a_{31} + 1, \qquad c_1 = 1, \quad c_2 = 0, \quad c_3 = e,$$

and therefore we have

$$\mathbf{p}_{A,e}(0, 1) = (1, 0, e) = \mathbf{e}$$

and, because of (26)–(28),

$$\begin{aligned} \mathbf{p}_{A,e}(1, k + g_2 e) &= (a_{11} + k + g_2 e, 1, a_{31} + 1 + e(k + g_2 e)) \\ &= (g_1 + g_2 e, 1, h_1 + h_2 e) \\ &= (g, 1, h). \end{aligned}$$

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In view of 3.7, this proves our contention that $(1, 0, e)$ and $(g, 1, h)$ are on $L(e)^{\gamma(A)}$.

We can now prove the main result of this paper.

4.5. THEOREM. *If the nearfield F is of rank 2 over the sub-division ring K of its kernel $\mathfrak{K}(F)$, then the incidence structure $\mathbf{H}(F, K)$ is a projective plane. Moreover, the Desarguesian subplane $\mathbf{H}_0(F, K)$ of 3.6 is in this case a Baer sub-plane.*

That $\mathbf{H}(F, K)$ is a projective plane follows immediately from 3.11, 3.12, and 4.4. It remains to show that (i) each exterior line carries an interior point and, dually, that (ii) each exterior point is on an interior line.

(i) By 3.8, each line of the special form $L(f), f \in F$, contains the interior point Q. It follows that $L(f)^{\gamma(A)}$ contains $Q^{\gamma(A)}$ which, by 3.6, is also interior.

(ii) Here it suffices, because of 4.3, to find one exterior point which is on an interior line. But the point E of (25) is exterior and on the interior line $L(\infty)$.

5. Collineations. The group Γ of 3.1 always acts as a collineation group on the partial plane $\mathbf{H}(F, K)$, but this action need not be faithful. We call Γ^* the collineation group induced by Γ, and we shall now determine the structure of Γ^*. For this, we do not need condition (H).

For any multiplicative subgroup G of K, we denote by $S(G)$ the group of all scalar matrices gI, $g \in G$. (I denotes the identity matrix in $\mathrm{GL}_3(K)$.) Also, K^* denotes, as usual, the set of all elements $\neq 0$ in K, i.e. the full multiplicative group of K. Finally, $\mathfrak{Z}(F)$ is, as in § 2, the centre of F.

5.1. Γ^* *is isomorphic to* $\mathrm{GL}_3(K)/S(K^* \cap \mathfrak{Z}(F))$.

Proof. If $0 \neq z \in K^* \cap \mathfrak{Z}(F)$, then, for any $\mathbf{x} = (x_1, x_2, x_3) \in W$,

$$\mathbf{x}^{\gamma(zI)} = (zI)(x_1, x_2, x_3) = (zx_1, zx_2, zx_3) = \mathbf{x}z,$$

and therefore $(\mathbf{x}F)^{\gamma(zI)} = \mathbf{x}zF = \mathbf{x}F$. Hence $S(K^* \cap \mathfrak{Z}(F))$ is certainly contained in the kernel of the canonical epimorphism $\Gamma \rightarrow \Gamma^*$.

Conversely, if $(\mathbf{x}F)^{\gamma(A)} = \mathbf{x}F$ for all points $\mathbf{x}F$, then this is true in particular for interior points, whence $A = tI$ with $t \in \mathfrak{Z}(K)$, by 3.6. We have to show that t is even in $\mathfrak{Z}(F)$, i.e. $tf = ft$ for any $f \in F$. For this, let $\mathbf{a} = (1, f, 0)$; then $(\mathbf{a}F)^{\gamma(tI)} = \mathbf{a}F$ implies that $\mathbf{a}^{\gamma(tI)} = \mathbf{a}g$ for some $g \in F$. But then

$$(t, tf, 0) = (tI)\mathbf{a} = \mathbf{a}^{\gamma(tI)} = \mathbf{a}g = (g, fg, 0),$$

whence $t = g$ and $tf = fg = ft$. This proves 5.1.

The following is an obvious corollary of 3.6 and 5.1.

5.2. *The following statements about* Γ^* *are equivalent:*

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(i) $\Gamma^* \cong \mathrm{PGL}_3(K)$,
(ii) Γ^* *is faithful on the interior points,*
(iii) $K \cap \mathfrak{Z}(F) = \mathfrak{Z}(K)$.

For the remainder, we again assume condition (H) and therefore, by 4.5, that $\mathbf{H}(F, K)$ is a projective plane. We know how Γ operates on the substructure of the interior points and lines, by 3.6; here we investigate how Γ acts on the exterior elements. We shall improve 4.3 by showing that Γ is transitive on exterior flags, i.e. on incident pairs of exterior points and exterior lines.

As in (25), we denote by E the exterior point $(1, 0, e)F$, with $e \in F - K$. Also, we use again a pair of nearfield functions u, v as in § 4, so that in particular we again have (26). We first show the following.

5.3. *The stabilizer* Γ_E *in* Γ *of the point* E *consists of all* $\gamma(A) \in \Gamma$ *with* $A \in \mathrm{GL}_3(K)$ *of the form*

$$\begin{pmatrix} x & a_{12} & y \\ 0 & a_{22} & 0 \\ u(x, y) & a_{32} & v(x, y) \end{pmatrix}. \tag{29}$$

Proof. $E^{\gamma(A)} = E$ means that $A\mathbf{e} = \mathbf{e}f$ for some $f \in F$. (Here $\mathbf{e} = (1, 0, e)$ as in § 4.) Written more explicitly, as in (12), this is

$$a_{11} + a_{13}e = f, \qquad a_{21} + a_{23}e = 0, \qquad a_{31} + a_{33}e = ef,$$

whence immediately $a_{21} = a_{23} = 0$. On the other hand, if we write $f = x + ye$ with $x, y \in K$ as in (23), it is clear that $a_{11} = x, a_{13} = y$ and, by (26), that $a_{31} = u(x, y)$ and $a_{33} = v(x, y)$. Thus A is of the form (29), and one verifies easily that, conversely, every matrix of that form satisfies the condition $E^{\gamma(A)} = E$.

5.4. Γ_E *is transitive on the interior points not on* $L(\infty)$.

Proof. Any such point can be represented by an element $\mathbf{x} = (x_1, x_2, x_3) \in V$, i.e. with $x_i \in K$ $(i = 1, 2, 3)$, and $x_2 \neq 0$. In particular, the point

$$P = \mathbf{p}F, \quad \text{with } \mathbf{p} = (0, 1, 0) \in V, \tag{30}$$

is of this sort. Our claim is now proved by the fact that the matrix A of the form (29) with $a_{i2} = x_i$ $(i = 1, 2, 3)$ satisfies $A\mathbf{p} = \mathbf{x}$.

5.5. THEOREM. *If* F *has rank* 2 *over the sub-division ring* K *of* $\mathfrak{K}(F)$, *then* Γ *acts transitively on the flags of exterior points and lines of* $\mathbf{H}(F, K)$.

Proof. In view of 4.3, it suffices to show that Γ_E is transitive on the exterior lines through E. This, however, follows immediately from 5.4 because, by 4.5, each exterior line through E carries a unique interior point not on $L(\infty)$.

This theorem gives the possibility of representing the incidence structure $\mathbf{H}(F, K) - \mathbf{H}_0(F, K)$ within the group $\mathrm{GL}_3(K)$; cf. [**1**, p. 15, result **17**].

Such a representation is of particular interest because the structure of $\mathbf{H}(F, K)$ is uniquely determined by that of the exterior elements. (For the finite case, this follows from [**1**, p. 317]; the general proof follows from similar ideas and will not be given here.)

In order to give this representation, we have to determine the stabilizer in Γ of some exterior line through an exterior point, say E. Now the line $L(e)^{\gamma(S)}$ defined by (17) is exterior because $e \notin K$, and it consists of all elements $\mathbf{x}$ with $x_3 = ex_1$ of W. Hence it contains in particular the point E (incidentally, also P) and is, therefore, an exterior line of the required kind. We denote this line by L.

5.6. *The stabilizer* Γ_L *in* Γ *of the line* L *consists of all* $\gamma(B) \in \Gamma$ *with* $B \in \mathrm{GL}_3(K)$ *of the form*

$$\begin{pmatrix} x & 0 & y \\ b_{21} & b_{22} & b_{23} \\ u(x, y) & 0 & v(x, y) \end{pmatrix}. \tag{31}$$

Proof. The line L is characterized by the relation $x_3 = ex_1$ for all its elements; cf. (17). Hence we have to find all matrices B preserving this relation. But $B = (b_{ij})$ has this property if and only if

$$b_{31}x_1 + b_{32}x_2 + b_{33}ex_1 = e(b_{11}x_1 + b_{12}x_2 + b_{13}ex_1)$$

for all $x_1, x_2 \in F$. For $x_1 = 0$ and $x_2 = 1$, this yields $b_{32} = eb_{12}$ and therefore $b_{32} = b_{12} = 0$, since $e \notin K$. On the other hand, for $x_1 = 1$ and $x_2 = 0$, we obtain

$$b_{31} + b_{33}e = e(b_{11} + b_{13}e),$$

and this shows, because of (26), that B is of the form (31). Conversely, one easily verifies that all matrices (31) preserve the line L.

As in [**1**, p. 15], let P and B be two subgroups of the group G and define the incidence structure $\mathbf{K}(G, P, B)$ as follows: points are the cosets Px, blocks the cosets By $(x, y \in G)$, and incidence of Px and By is defined by $Px \cap By \neq \emptyset$. In view of 5.3, 5.5, 5.6 as well as of [**1**, p. 15, result 17], we can now state our final result.

5.7. THEOREM. *The incidence structure of the exterior points and lines of the generalized Hughes plane* $\mathbf{H}(F, K)$, *where* F *is of rank* 2 *over* K, *is isomorphic to* $\mathbf{K}(G, P, B)$, *where* $G = \mathrm{GL}_3(K)$, *and* P *and* B *consist of the matrices* (29) *and* (31), *respectively.*

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