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A NILPOTENCY-LIKE CONDITION FOR INFINITE GROUPS

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Abstract

If k is a positive integer, a group G is said to have the FE_k-property if for each element g of G there exists a normal subgroup of finite index $X(g)$ such that the subgroup $\langle g, x\rangle$ is nilpotent of class at most k for all $x \in X(g)$. Thus, FE_1-groups are precisely those groups with finite conjugacy classes (FC-groups) and the aim of this paper is to extend properties of FC-groups to the case of groups with the FE_k-property for $k > 1$. The class of FE_k-groups contains the relevant subclass FE_k^*, consisting of all groups G for which to every element g there corresponds a normal subgroup of finite index $Y(g)$ such that $\langle g, U\rangle$ is nilpotent of class at most k, whenever U is a nilpotent subgroup of class at most k of $Y(g)$.

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1. Introduction

A group G is said to be an *FC-group* if each of its elements has only finitely many conjugates or equivalently if the centralizer $C_G(x)$ has finite index in G for every element x of G. The theory of FC-groups has been introduced and strongly developed in the second half of the last century in order to investigate properties which are common to both finite and abelian groups. We refer to the monograph [11] for results and information on this relevant chapter of the theory of infinite groups.

If k is any positive integer, we shall say that a group G has the *FE_k-property* if for every element g of G there exists a normal subgroup $X(g)$ such that the index $|G : X(g)|$ is finite and the subgroup $\langle g, x\rangle$ is nilpotent of class at most k for all elements x of $X(g)$. Moreover, G will be called an *FE_k^*-group* if to each element g of G there corresponds a normal subgroup of finite index $Y(g)$ such that $\langle g, U\rangle$ is nilpotent of class at most k, whenever U is a subgroup of $Y(g)$ which is nilpotent of class at most k. It is clear that the group properties FE_1, FE_1^* and FC coincide, and the consideration of the group classes FE_k and FE_k^* is an attempt to study properties which are common to both finite and nilpotent groups of class at most k.

The aim of this paper is to show that some of the basic properties of FC-groups can be extended to groups in the classes FE_k and FE_k^*, for $k > 1$, with special attention to the case $k = 2$. Among other results, it will be proved for instance that if G is a finitely generated soluble-by-finite FE_k^*-group, then the factor group $G/\zeta_k(G)$ is finite and that any finitely generated FE_2-group is finite over its second centre.

If k is any positive integer, the definition of the FE_k-property can also be given in terms of the so-called $\mathfrak{N}_k$-*connectivity*, where $\mathfrak{N}_k$ is the class of nilpotent groups of class at most k; here, if $\mathfrak{X}$ is any group class, two subgroups X and Y of a group G are said to be $\mathfrak{X}$-*connected* if the subgroup $\langle x, y\rangle$ belongs to $\mathfrak{X}$ for all elements x of X and y of Y. Thus, a group G has the FE_k-property if and only if every cyclic subgroup of G is $\mathfrak{N}_k$-connected to a (normal) subgroup of finite index. Connectivity with respect to a class of nilpotent groups was introduced by Carocca [4] and later studied by several authors (see for instance [1, 3, 5]).

Most of our notation is standard and can be found in [9].

2. Preliminaries

It is clear that FE_k^* is a subgroup closed group class, while the class FE_k is closed with respect to both subgroups and homomorphic images. Moreover, the inclusion

$$FC = FE_1 = FE_1^* \le FE_k^* \le FE_k \le FE_{k+1}$$

holds for all positive integers k. It should also be remarked that groups in these classes can be seen in relation to groups satisfying Engel-type conditions.

Recall that an element g of a group G is said to be (right) *Engel* if for each element x of G there exists a positive integer $k = k(x)$ such that $[g,_k x] = 1$, where, as usual, the commutator $[g,_k x]$ is defined by setting

$$[g,_1 x] = [g, x] \quad \text{and} \quad [g,_k x] = [[g,_{k-1} x], x] \quad \text{for } k > 1.$$

If the integer k can be chosen independently of x, then g is a (right) *k-Engel element*. Obviously, if the subgroup $\langle g, x\rangle$ is nilpotent of class at most k for each $x \in G$, the element g is k-Engel and hence in particular all elements of an FE_k-group have in some sense a large 'k-Engelizer'. In this context, we also mention that Bastos and Shumyatsky [2] have recently proved that if G is a profinite group and for each element g of G there exists a positive integer n such that g^n is an Engel element, then every finitely generated subgroup of G contains a nilpotent subgroup of finite index.

Let G be an FC-group. As the centre of a group can always be realized as the intersection of all centralizers, it follows from the definition that the factor group $G/Z(G)$ is residually finite and so the finite residual lies in the centre of G. Here the *finite residual* of a group G is the intersection of all (normal) subgroups of finite index of G and a group is *residually finite* if its finite residual is trivial. It is well known that all polycyclic-by-finite (and so in particular all finitely generated nilpotent-by-finite) groups are residually finite.

LEMMA 2.1. *Let G be an FE_k^*-group, where k is a positive integer. Then the finite residual of G is nilpotent of class at most k.*

PROOF. Let g be an arbitrary element of the finite residual J of G. Consider a subgroup M of J which is maximal with respect to the condition of being nilpotent of class at most k. As the index $|G : Y(g)|$ is finite, J is contained in $Y(g)$, so the subgroup $\langle g, M\rangle$ is likewise nilpotent of class at most k and hence $\langle g, M\rangle = M$. Therefore, $J = M$ is nilpotent of class at most k. □

Notice that Lemma 2.1 shows in particular that infinite FE_k^*-groups cannot be simple, as in the case of FC-groups.

COROLLARY 2.2. *Let G be a simple FE_k^*-group, where k is a positive integer. Then G is finite.*

It follows from Lemma 2.1 that any FE_k^*-group with no proper subgroups of finite index is nilpotent of class at most k. On the other hand, Vaughan-Lee and Wiegold [13] exhibited a perfect locally finite group G of exponent 9 in which every n-generator subgroup has nilpotency class at most $8n$ for all n. In particular, G has the FE_{16}-property and so FE_{16}^* is a proper subclass of FE_{16}.

Another direct consequence of Lemma 2.1 is that all FE_k^*-groups are locally graded. Here a group G is said to be *locally graded* if every finitely generated nontrivial subgroup of G contains a proper subgroup of finite index; locally graded groups form a large group class, which is closed with respect to forming extensions and contains all locally (soluble-by-finite) groups.

LEMMA 2.3. *Let G be a locally graded FE_k-group, where k is a positive integer. Then the finite residual of G is locally nilpotent.*

PROOF. Consider the normal subgroup

$$X = \bigcap_{g \in G} X(g)$$

and let x and y be arbitrary elements of X. Since y belongs to $X(x)$, the subgroup $\langle x, y\rangle$ is nilpotent of class at most k and in particular $[x,_k y] = 1$. Thus, X is a locally graded k-Engel group and hence it is locally nilpotent (see [6]). Clearly, X contains the finite residual J of G and so J is locally nilpotent. □

COROLLARY 2.4. *Let G be a simple locally graded FE_k-group, where k is a positive integer. Then G is finite.*

The proof of Lemma 2.3 shows that if G is any FE_k-group, where k is a positive integer, then the finite residual of G is a k-Engel group. On the other hand, it has recently been proved that 4-Engel groups are locally nilpotent (see for instance [12]), so that in particular the finite residual of an arbitrary FE_4-group is locally nilpotent and all simple FE_4-groups are finite.

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Lemma 2.5. *Let G be an FE_k-group, where k is a positive integer, and let x and y be elements of G. Then the normal closure $\langle x\rangle^{\langle y\rangle}$ is finitely generated.*

Proof. It can obviously be assumed that the element y has infinite order. It follows from the property FE_k that there exists a positive integer m such that the subgroup $\langle x, y^m\rangle$ is nilpotent of class at most k. In particular, $[x,_k y^m] = 1$ and hence the normal closure

$$\langle x\rangle^{\langle y\rangle} = \langle x^{y^i} \mid -km \le i \le km\rangle$$

is finitely generated. □

Lemma 2.6. *Let G be a finitely generated FE_k-group, where k is a positive integer. Then the ith term $G^{(i)}$ of the derived series of G is finitely generated for each nonnegative integer i.*

Proof. By Lemma 2.5, the subgroup $\langle x\rangle^{\langle y\rangle}$ is finitely generated for all elements x and y of G and so it follows that also the commutator subgroup G' is finitely generated (see [6, Corollary 4]). As the class of FE_k-groups is subgroup closed, the statement is proved. □

Corollary 2.7. *Let G be a finitely generated soluble FE_k-group, where k is a positive integer. Then G is polycyclic.*

Proof. Each term of the derived series of G is finitely generated by Lemma 2.6 and hence the soluble group G is polycyclic. □

3. Main results

Recall that the *Baer radical* of a group G is the subgroup generated by all abelian subnormal subgroups of G, and G is called a *Baer group* if it coincides with its Baer radical. Thus, G is a Baer group if and only if all its cyclic subgroups are subnormal. Of course, any Baer group is locally nilpotent and so the Baer radical of a group G is contained in the largest locally nilpotent normal subgroup, the so-called *Hirsch–Plotkin radical*. As every finitely generated subgroup of a group with finite conjugacy classes is contained in a finitely generated normal subgroup, it turns out that in any FC-group G the Hirsch–Plotkin radical, the Baer radical and the Fitting subgroup coincide.

Our next result shows that a corresponding statement holds for any soluble-by-finite FE_k-group G and also that the factor group $G/H(G)$ has a restricted structure; a sharper description is given when $k = 2$.

Theorem 3.1. *Let G be a group and let $H(G)$ and $F(G)$ be the Hirsch–Plotkin radical and the Fitting subgroup of G, respectively.*

(a) *If G is a soluble-by-finite FE_k-group, for some positive integer k, then $H(G)$ is a Baer group and $G/H(G)$ is a periodic FC-group.*
(b) *If G is an FE_2-group, then $G/F(G)$ is an FC-group and $G/H(G)$ is periodic.*

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Proof. (a) Let h be any element of the Hirsch–Plotkin radical $H = H(G)$. Then h is a left k-Engel element of $\langle h, X(h)\rangle$ and so also of the soluble group

$$K = \langle h, X(h)\rangle \cap H.$$

It follows that $\langle h\rangle$ is subnormal in K (see for instance [9, Part 2, Theorem 7.35]) and so also in H, since H is locally nilpotent and the index $|H : K|$ is finite. Therefore, all cyclic subgroups of H are subnormal and H is the Baer radical of G. Consider now an arbitrary element g of G and let $S(g)$ be the largest soluble normal subgroup of $X(g)$. Since $S(g)$ has finite index in G, there exists a positive integer n such that g^n belongs to $S(g)$. Clearly, g^n is a left k-Engel element of $S(g)$, so that the cyclic subgroup $\langle g^n\rangle$ is subnormal in $S(g)$ and hence also in G. It follows that g^n belongs to H and so G/H is periodic. Moreover, g is a left k-Engel element of the soluble group $\langle g, S(g)\rangle$, so that $\langle g\rangle$ is a subnormal subgroup of $\langle g, S(g)\rangle$ and hence the normal closure $\langle g\rangle^{S(g)}$ is locally nilpotent. Then also the subnormal subgroup $[g, S(g)]$ of G is locally nilpotent and so it lies in H. On the other hand, the subgroup $S(g)$ has finite index in G, so that the coset gH has only finitely many conjugates and G/H is an FC-group.

(b) Let g be any element of G. For each element x of $X(g)$, the subgroup $\langle g, x\rangle$ is nilpotent of class at most 2, so that $[g, g^x] = 1$, and hence the normal closure $\langle g\rangle^{X(g)}$ is abelian. It follows that also the subgroup $A_g = [X(g), g]$ is abelian. Moreover, A_g is normal in $X(g)$, so that it has only finitely many conjugates in G and hence its normal closure A_g^G is nilpotent. It follows that A_g is contained in $F = F(G)$ and so the coset gF has only finitely many conjugates in G/F. Therefore, the group G/F has the FC-property. Put $Z/F = Z(G/F)$. As G/F is an FC-group, the factor group G/Z is periodic. Let g be any element of G and let r be a positive integer such that g^r belongs to Z. Consider a transversal $\{y_1, \ldots, y_t\}$ to $X(g^r) \cap Z$ in Z. As the index $|G : X(y_i)|$ is finite for each $i = 1, \ldots, t$, there is a positive integer s such that g^{rs} belongs to the subgroup

$$X(y_1) \cap \cdots \cap X(y_t).$$

Let z be any element of Z, so that $z = xy_i$ for some $i \le t$ and a suitable element x of $X(g^r) \cap Z$. Consider now the subgroup $V = \langle x, y_i, g^{rs}\rangle$ of Z. As $V/V \cap F \simeq VF/F$ is a finitely generated abelian group, the subgroup $V \cap F$ is contained in the normal closure of a finitely generated subgroup of F and so it is nilpotent. Moreover,

$$[V, g^{rs}] \le V \cap Z' \le V \cap F,$$

so that the subgroup $[V, g^{rs}]$ is nilpotent and hence the set

$$L_V(g^{rs}) = \{v \in V \mid [g^{rs},_n v] = 1 \text{ for some } n = n(g^{rs}, v)\}$$

is a subgroup (see [10, Corollary 3*]). On the other hand, x and y_i obviously belong to $L_V(g^{rs})$ and so also z lies in $L_V(g^{rs})$. Thus, g^{rs} is a left Engel element of Z and hence it belongs to the Hirsch–Plotkin radical of Z (see [9, Part 2, Theorem 7.34]), which is contained in $H(G)$. Therefore, the group $G/H(G)$ is periodic. □

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It is well known that any finitely generated FC-group is finite over the centre. The following result describes the behaviour of finitely generated groups in the classes FE_k and FE_k^*. Recall here that for each positive integer k the symbols $\zeta_k(G)$ and $\gamma_k(G)$ denote the kth term of the upper central series and the kth term of the lower central series of the group G, respectively. In particular, $\zeta_1(G) = Z(G)$, $\gamma_1(G) = G$ and $\gamma_2(G) = G'$.

THEOREM 3.2. *Let G be a finitely generated soluble-by-finite group and let k be a positive integer.*

(a) *If G is an FE_k-group, then there exists a positive integer m, depending only on k, such that the group $G/\zeta_m(G)$ is finite.*
(b) *If G is an FE_k^*-group, then the factor group $G/\zeta_k(G)$ is finite.*

PROOF. (a) Let S be the largest soluble normal subgroup of G. Then the index $|G : S|$ is finite, so that S is finitely generated, and it follows from Corollary 2.7 that G is polycyclic-by-finite. Assume for a contradiction that G is not finite-by-nilpotent and let M be a normal subgroup of G which is maximal with respect to the condition that the factor group $\overline{G} = G/M$ is not finite-by-nilpotent. Then all proper homomorphic images of $\overline{G}$ are finite-by-nilpotent. Since $\overline{G}$ is an infinite polycyclic-by-finite group, it contains a torsion-free abelian nontrivial normal subgroup $\overline{A}$. Let $\overline{a}$ be an arbitrary element of $\overline{A}$. It follows from the FE_k-property that for each element $\overline{g}$ of $\overline{G}$ there exists a positive integer m such that the subgroup $\langle \overline{a}^m, \overline{g}\rangle$ is nilpotent of class at most k. Then

$$[\overline{a},_k \overline{g}]^m = [\overline{a}^m,_k \overline{g}] = 1,$$

so that $[\overline{a},_k \overline{g}] = 1$ and $\overline{a}$ is a right k-Engel element of $\overline{G}$. Therefore, $\overline{A}$ is contained in $\zeta_n(\overline{G})$ for some positive integer n (see [9, Part 2, Theorem 7.21]). Moreover, the factor group $\overline{G}/\overline{A}$ is finite-by-nilpotent and hence a combined application of results by Baer and Hall yields that $\overline{G}$ itself is finite-by-nilpotent (see [9, Part 1, Theorems 4.21 and 4.25]). This contradiction shows that G is finite-by-nilpotent, so that it contains a finite normal subgroup N such that $\widetilde{G} = G/N$ is a torsion-free nilpotent group. If $\widetilde{x}$ and $\widetilde{y}$ are arbitrary elements of $\widetilde{G}$, there is a positive integer s such that $\langle \widetilde{x}^s, \widetilde{y}\rangle$ has class at most k and so also the subgroup $\langle \widetilde{x}, \widetilde{y}\rangle$ has class at most k, since it is contained in the isolator of $\langle \widetilde{x}^s, \widetilde{y}\rangle$ (see [7, Section 2.3.9]). It follows now from a result of Zel'manov [14] that the nilpotency class m of $\widetilde{G}$ is bounded by a function of k. Finally, as $\gamma_{m+1}(G)$ is finite and G is residually finite, we have that also the group $G/\zeta_m(G)$ is finite.

(b) The group G is finite-by-nilpotent by part (a) and so it contains a finite normal subgroup N such that G/N is a torsion-free nilpotent group. Let $g_1, \dots, g_t$ be elements of G such that $G = \langle g_1, \dots, g_t\rangle$ and put

$$Y = Y(g_1) \cap \cdots \cap Y(g_t).$$

Then Y is a subgroup of finite index of G and hence there exist positive integers $m_1, \dots, m_t$ such that $g_i^{m_i}$ belongs to Y for each $i = 1, \dots, t$. Suppose that the subgroup

$$\langle g_1^{m_1}, \dots, g_i^{m_i}\rangle$$

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is nilpotent of class at most k for some $i < t$. Since $\langle g_1^{m_1}, \ldots, g_i^{m_i} \rangle$ is contained in $Y(g_{i+1})$, it follows that also

$$\langle g_1^{m_1}, \ldots, g_i^{m_i}, g_{i+1} \rangle$$

is nilpotent of class at most k and in particular its subgroup

$$\langle g_1^{m_1}, \ldots, g_i^{m_i}, g_{i+1}^{m_{i+1}} \rangle$$

has nilpotency class at most k. Therefore, the subgroup

$$U = \langle g_1^{m_1}, \ldots, g_t^{m_t} \rangle$$

is nilpotent of class at most k. But G/N is a torsion-free nilpotent group and it is the isolator of its subgroup UN/N, so that G/N itself has class at most k (see [7, Section 2.3.9]) and hence the subgroup $\gamma_{k+1}(G)$ is finite. As G is residually finite, it follows that the factor group $G/\zeta_k(G)$ is finite. □

Corollary 3.3. *Let G be a locally (soluble-by-finite) group and let k be a positive integer.*

(a) *If G has the FE_k-property, there exists a positive integer m, depending only on k, such that the subgroup $\gamma_{m+1}(G)$ is locally finite.*
(b) *If G has the FE_k^*-property, the subgroup $\gamma_{k+1}(G)$ is locally finite.*

It is well known that torsion-free FC-groups are abelian and it follows from the latter result that a corresponding statement holds for torsion-free FE_k^*-groups, at least in the universe of locally (soluble-by-finite) groups.

Corollary 3.4. *Let G be a torsion-free locally (soluble-by-finite) FE_k^*-group, where k is a positive integer. Then G is nilpotent of class at most k.*

Notice that part (a) of Theorem 3.2 cannot be improved by showing that the group G is finite over its kth centre. In fact, Newman [8] constructed, for any integer $k > 2$, a finitely generated torsion-free nilpotent group G_k of class $k + 1$ in which all two-generator subgroups have class at most k. In particular, G_k has the FE_k-property, but the factor group $G_k/\zeta_k(G_k)$ is infinite. However, our next result proves that any finitely generated FE_2-group is finite over its second centre.

Theorem 3.5. *Let G be a finitely generated FE_2-group. Then the factor group $G/\zeta_2(G)$ is finite.*

Proof. Let $H = H(G)$ be the Hirsch–Plotkin radical of G. The group G/H is finite by Theorem 3.1, so that H is a finitely generated nilpotent group and in particular G is polycyclic-by-finite, of Hirsch length h, say. We shall prove that G is finite-by-nilpotent, this claim being obvious if $h = 0$, that is, if G is finite. Suppose that G is infinite, so that it contains a torsion-free abelian nontrivial normal subgroup A. As in the proof of part (a) of Theorem 3.2, it can be shown that A is contained in $\zeta_n(G)$ for some positive integer n. Moreover, the factor group G/A has the FE_2-property

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and Hirsch length smaller than h, so that G/A is finite-by-nilpotent and then G itself is finite-by-nilpotent. Thus, G contains a finite normal subgroup N such that G/N is torsion-free and nilpotent. By a result of Hall, it is enough to show that the subgroup $\gamma_3(G)$ is finite, so that without loss of generality it can be assumed that G is a torsion-free nilpotent group.

Let x and y be arbitrary elements of G. Then there exists a positive integer m such that the subgroup $\langle x, y^m\rangle$ has class at most 2. Since G is torsion-free and $\langle x, y\rangle$ is contained in the isolator of $\langle x, y^m\rangle$, it follows that also $\langle x, y\rangle$ has class at most 2. Therefore, G is a 2-Engel group and hence it has class at most 2 (see [9, Part 2, Theorem 7.14]). □

Corollary 3.6. *If G is an FE_2-group, then the subgroup $\gamma_3(G)$ is locally finite. In particular, the set of elements of finite order of any FE_2-group is a subgroup, and torsion-free FE_2-groups are nilpotent of class at most 2.*

It is also known that any FC-group containing an abelian subgroup of finite index is finite over the centre. This fact can be seen as a special case of the following result.

Theorem 3.7. *Let G be an FE_k^*-group, where k is a positive integer. If G contains a subgroup of finite index which is nilpotent of class at most k, then the factor group $G/Z(G)$ has a subgroup of finite index which is nilpotent of class at most $k-1$.*

Proof. Let N be a nilpotent normal subgroup of G of class at most k such that the index $|G : N|$ is finite and let $\{g_1, \dots, g_t\}$ be a transversal to N in G. Consider the subgroup

$$Y = N \cap Y(g_1) \cap \cdots \cap Y(g_t).$$

If g is any element of G, there exists $a \in N$ such that $g = ag_i$ for some $i = 1, \dots, t$, and the subgroup $\langle a, Y\rangle$ is nilpotent of class at most k, since it is contained in N. Moreover, as Y is contained in $Y(g_i)$, it follows from the FE_k^*-property that also $\langle g_i, Y\rangle$ is nilpotent of class at most k. In particular, the subgroup $\gamma_k(Y)$ centralizes both a and g_i, so that $[g, \gamma_k(Y)] = \{1\}$ and hence $\gamma_k(Y)$ lies in the centre of G. Therefore, $YZ(G)/Z(G)$ is a normal subgroup of finite index of $G/Z(G)$ which is nilpotent and has class at most $k-1$. □

Corollary 3.8. *Let G be an FE_2^*-group and let C be the FC-centre of G. If G contains a subgroup of finite index which is nilpotent of class at most 2, then the factor group G/C is central-by-finite.*

Proof. Let N be a nilpotent normal subgroup of G of class at most 2 such that the index $|G : N|$ is finite and let g be any element of G. The subgroup $\langle g, N \cap Y(g)\rangle$ is nilpotent of class at most 2 by the FE_2^*-property, so that in particular $[g, N \cap Y(g)]$ is centralized by $N \cap Y(g)$. It follows that $[g, N \cap Y(g)]$ lies in C and hence the coset gC has only finitely many conjugates in G/C. Therefore, G/C is an FC-group. On the other hand, the group G/C is abelian-by-finite by Theorem 3.7 and so it is even central-by-finite. □

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Any locally soluble FC-group is hyperabelian (see [11, Corollary 1.15]) and our results allow us to prove that also locally soluble FE_2-groups admit an ascending normal series with abelian factors.

Corollary 3.9. *Let G be a locally soluble FE_2-group. Then G is hyperabelian.*

Proof. Clearly, the Fitting subgroup F of G has an ascending G-invariant series with abelian factors. On the other hand, G/F is an FC-group by Theorem 3.1 and hence it is hyperabelian. It follows that G itself is hyperabelian. □

We notice finally that the example of Vaughan-Lee and Wiegold [13] quoted in Section 2 shows in particular that locally nilpotent groups with the FE_k-property need not be hypercentral, in contrast to the behaviour of FC-groups.

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