GENERATING FUNCTIONS FOR HERMITE FUNCTIONS

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1. Introduction. Hermite's function $H_n(x)$ is defined for all complex values of x and n by

$$H_n(x) = \frac{2^n\Gamma(\frac{1}{2})}{\Gamma\left(\frac{1-n}{2}\right)} F\left(-\frac{n}{2};\frac{1}{2};x^2\right) + \frac{2^n\Gamma(-\frac{1}{2})}{\Gamma\left(-\frac{n}{2}\right)} xF\left(\frac{1-n}{2};\frac{3}{2};x^2\right)$$

$$= 2^n \sum_{k=0}^{\infty} \frac{\binom{n}{k}\Gamma\left(\frac{1}{2}\right) x^k}{\Gamma\left(\frac{1-n+k}{2}\right)},$$

where $F(\alpha;\gamma;x)$ is Kummer's function with the customary indices omitted. It satisfies the differential equation

$$\frac{d^2v}{dx^2} - 2x\frac{dv}{dx} + 2nv = 0, \tag{1.1}$$

of which

$$h_n(x) = e^{x^2}H_{-n-1}(ix)$$

is a second solution. Every solution of (1.1) is an entire function. The only linearly independent polynomial solutions are the Hermite polynomials $H_n(x)$, $n = 0, 1, 2 \ldots$.

The partial differential operator

$$L = \frac{\partial^2}{\partial x^2} - 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$$

annuls $u = y^n v(x)$ if, and only if, $v(x)$ satisfies (1.1). It follows that if $u = u(x,y)$ is annulled by L and is expressible as a series of powers of y, the coefficient of y^n must be a solution of (1.1). It so happens that the equation $Lu = 0$ admits a 5-parameter group of continuous transformations. Following the methods described in a previous paper **(5)** we shall use this group to obtain solutions of $Lu = 0$ and thence generating functions for the Hermite functions.

The results may also be expressed in terms of Weber's function $D_n(x)$ by means of the relation

$$H_n(x) = 2^{\frac{1}{2}n} e^{\frac{1}{2}x^2} D_n(\sqrt{2}\, x).$$

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2. Group of operators. The operators

$$(2.1)\quad A = y\frac{\partial}{\partial y},\ B = y^{-1}\frac{\partial}{\partial x},\ C = y\left(-\frac{\partial}{\partial x} + 2x\right),$$
$$B_2 = \frac{1}{2}y^{-2}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right),\ C_2 = -\frac{1}{2}y^2\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 1 - 2x^2\right)$$

satisfy the commutator relations

$$(2.2)\quad \begin{array}{llll} [A, B] = -B, & [A, C] = C, & [C, B] = -2, & \\ {[A, B_2]} = -2B_2, & [A, C_2] = 2C_2, & [C_2, B_2] = -A - \tfrac{1}{2}, & \\ {[B, B_2]} = 0, & [C, C_2] = 0, & [B, C_2] = C, & [B_2, C] = B \end{array}$$

and therefore generate, with the identity operator, a continuous group Γ.

A generates the trivial group $x' = x$, $y' = ty$ $(t \neq 0)$, which is used for purposes of normalization. The extended forms of the transformation groups generated by the other operators are described by

$$e^{bB}f(x, y) = f(x + by^{-1}, y)$$
$$e^{\beta B_2}f(x, y) = f\left(\frac{xy}{\sqrt{[y^2 - \beta]}}, \sqrt{[y^2 - \beta]}\right)$$
$$e^{cC}f(x, y) = e^{2cxy - c^2y^2}f(x - cy, y)$$
$$e^{\gamma C_2}f(x, y) = (1 + \gamma y^2)^{-\frac{1}{2}}\exp\left\{\frac{\gamma x^2y^2}{1 + \gamma y^2}\right\}f\left(\frac{x}{\sqrt{[1 + \gamma y^2]}}, \frac{y}{\sqrt{[1 + \gamma y^2]}}\right),$$

where b, β, c, γ are arbitrary constants and $f(x, y)$ an arbitrary function. Hence

$$(2.3)\quad e^{cC+\gamma C_2}e^{bB+\beta B_2}f(x, y)$$
$$= (1 + \gamma y^2)^{-\frac{1}{2}}\exp\left\{\frac{2cxy - c^2y^2 + \gamma x^2y^2}{1 + \gamma y^2}\right\}f(\xi, \eta),$$
$$\xi = \frac{b + xy + (b\gamma - c)y^2}{\{(1 + \gamma y^2)[(1 - \beta\gamma)y^2 - \beta]\}^{\frac{1}{2}}},\ \eta = \left\{\frac{(1 - \beta\gamma)y^2 - \beta}{1 + \gamma y^2}\right\}^{\frac{1}{2}}.$$

The relation of the group Γ to the operator L of § 1 is indicated by the operator identities

$$(2.4)\quad \begin{array}{ll} -L = CB - 2A, & -x^2L = 4C_2B_2 - A^2 + A, \\ 4B_2 = B^2 - y^{-2}L, & 4C_2 = C^2 - y^2L, \end{array}$$

from which it follows that L is commutative with A, B, C and x^2L is commutative with A, B_2, C_2. Therefore every operator of the group Γ converts each solution of $Lu = 0$ into a solution. In particular we note that

$$(2.5)\quad \begin{array}{ll} A\{H_n(x)y^n\} = nH_n(x)y^n, & A\{h_n(x)y^n\} = nh_n(x)y^n; \\ B\{H_n(x)y^n\} = 2nH_{n-1}(x)y^{n-1}, & B\{h_n(x)y^n\} = -ih_{n-1}(x)y^{n-1}; \\ C\{H_n(x)y^n\} = H_{n+1}(x)y^{n+1}, & C\{h_n(x)y^n\} = 2i(n+1)h_{n+1}(x)y^{n+1}. \end{array}$$

3. Conjugate operators of the first order. We shall examine the functions annulled by L and

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$$R = r_1A + r_2B + r_3C + r_4B_2 + r_5C_2 + r_6,$$

where the r's are arbitrary constants, of which the first five do not vanish simultaneously. To this end we separate the operators R into conjugate classes with respect to the group Γ. We find as in (**5**, p. 1035) that

$$e^{aA}Be^{-aA} = e^{-a}B,\ e^{aA}Ce^{-aA} = e^{a}C,\ e^{aA}B_2e^{-aA} = e^{-2a}B_2,\ e^{aA}C_2e^{-aA} = e^{2a}C_2;$$
$$e^{bB}A\,e^{-bB} = A + bB,\ e^{bB}Ce^{-bB} = C + 2b,\ e^{bB}C_2e^{-bB} = bC + C_2 + b^2;$$
$$e^{cC}Ae^{-cC} = A - cC,\ e^{cC}Be^{-cC} = B - 2c,\ e^{cC}B_2e^{-cC} = -cB + B_2 + c^2;$$
$$e^{\beta B_2}Ae^{-\beta B_2} = A + 2\beta B_2,\ e^{\beta B_2}Ce^{-\beta B_2} = C + \beta B,\ e^{\beta B_2}C_2e^{-\beta B_2} = \beta A + \beta^2 B_2 + C_2 + \tfrac{1}{2}\beta;$$
$$e^{\gamma C_2}Ae^{-\gamma C_2} = A - 2\gamma C_2,\ e^{\gamma C_2}Be^{-\gamma C_2} = B - \gamma C,\ e^{\gamma C_2}B_2e^{-\gamma C_2} = -\gamma A + B_2 + \gamma^2 C_2 - \tfrac{1}{2}\gamma.$$

It follows that $I = r_1^2 - r_4r_5$ is an invariant of R with respect to Γ.

Setting $S = e^{cC+\gamma C_2}e^{bB+\beta B_2}$, we have

$$\begin{aligned}
SAS^{-1} &= (1-2\beta\gamma)A + (b-2c\beta)B + (2c\beta\gamma - c - b\gamma)C + 2\beta B_2 \\
&\quad + 2\gamma(\beta\gamma - 1)C_2 + 2c^2\beta - 2bc - \beta\gamma,\\
SBS^{-1} &= B - \gamma C - 2c,\\
SCS^{-1} &= \beta B + (1-\beta\gamma)C + 2(b - c\beta),\\
SB_2S^{-1} &= -\gamma A - cB + c\gamma C + B_2 + \gamma^2C_2 + c^2 - \tfrac{1}{2}\gamma,\\
SC_2S^{-1} &= \beta(1-\beta\gamma)A + \beta(b-c\beta)B + (1-\beta\gamma)(b-c\beta)C + \beta^2B_2\\
&\quad + (1-\beta\gamma)^2C_2 + (b-c\beta)^2 + \tfrac{1}{2}\beta(1-\beta\gamma).
\end{aligned}$$

From these formulae it follows that for suitable choices of the constants $a, b, c, \beta, \gamma, \lambda, \nu, p$, and q, R is a conjugate of

(a) $\lambda A - \nu$ if $I \neq 0$;
(b) $pC + qB_2$ if $I = 0,\ r_1r_2 \neq r_3r_4$;
(c) $\lambda B_2 - \nu$ if $I = 0,\ r_1r_2 = r_3r_4,\ r_4 \neq 0$ or $r_5 \neq 0$;
(d) $\lambda B - \nu$ if $I = 0,\ r_1 = r_4 = r_5 = 0,\ r_2 \neq 0$ or $r_3 \neq 0$.

The identities (2.4) show that the last two cases do not require special consideration.

4. Generating functions for functions annulled by conjugates of $A - \nu$. Since $u_1 = y^\nu H_\nu(x)$, $u_2 = y^\nu e^{x^2}H_{-\nu-1}(ix)$ are linearly independent solutions of $Lu = 0$, $(A - \nu)\,u = 0$, where ν is an arbitrary constant, it follows from (2.3) that

$$G_1(x,y) = (1+\gamma y^2)^{-(\nu+1)/2}\{(1-\beta\gamma)y^2 - \beta\}^{\frac{1}{2}\nu} \cdot \exp\left\{\frac{2cxy - c^2y^2 + \gamma x^2y^2}{1+\gamma y^2}\right\} H_\nu(\xi),$$

$$G_2(x,y) = (1+\gamma y^2)^{-(\nu+1)/2}\{(1-\beta\gamma)y^2 - \beta\}^{\frac{1}{2}\nu} \cdot \exp\left\{\frac{(1-\beta\gamma)x^2y^2 + (\beta c^2 - 2bc + b^2\gamma)y^2 + 2(b-\beta c)xy + b^2}{(1-\beta\gamma y)^2 - \beta}\right\} H_{-\nu-1}(i\xi)$$

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are linearly independent solutions of $Lu = 0$, $\{S(A - \nu)S^{-1}\}u = 0$. It suffices to examine G_1.

Case 1. $\beta = \gamma = c = 0$. Setting $b = 1$, we obtain, after simplification

$$(4.1) \qquad H_\nu(x+y) = \sum_{n=0}^{\infty} \binom{\nu}{n} H_{\nu-n}(x)(2y)^n,$$

a Taylor expansion which may be derived directly from $H_\nu'(x) = 2\nu H_{\nu-1}(x)$.

Case 2. $\beta = \gamma = b = 0$. Setting $c = 1$, we have

$$y^\nu e^{2xy-y^2} H_\nu(x-y) = \sum_{n=0}^{\infty} \{a_n H_{\nu+n}(x) + b_n h_{\nu+n}(x)\} y^{\nu+n}.$$

Since the left member is annulled by $S(A - \nu)S^{-1} = A - C - \nu$, we obtain the recurrence relations

$$na_n = a_{n-1}, \quad nb_n = 2i(n+1)b_{n-1} \qquad (n = 1, 2, \ldots)$$

with the aid of (2.5). Cancelling y^ν and setting $y = 0$, we have $a_0 = 1$, $b_0 = 0$, whence $a_n = 1/n!$, $b_n = 0$. Hence (**4**, p. 85)

$$(4.2) \qquad e^{2xy-y^2} H_\nu(x-y) = \sum_{n=0}^{\infty} \frac{1}{n!} H_{\nu+n}(x) y^n.$$

Case 3. $\beta = \gamma = 0$, $c \neq 0$. Setting $c = 1$, $b = -w/2$, we obtain with the aid of (4.1) and (4.2)

$$(4.3) \qquad e^{2xy-y^2} H_\nu\left(x - y - \frac{w}{2y}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} F(-\nu; n+1; w) H_{\nu+n}(x) y^n$$
$$+ \sum_{n=1}^{\infty} (-1)^n \binom{\nu}{n} F(n-\nu; n+1; w) H_{\nu-n}(x) w^n y^{-n}, \qquad (y \neq 0).$$

If ν is a non-negative integer, this result may be written

$$(4.4) \qquad \frac{y^\nu}{\nu!} e^{2xy-y^2} H_\nu\left(x - y - \frac{w}{2y}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} L_\nu^{(n-\nu)}(w) H_n(x) y^n,$$

where $L_\nu^{(\alpha)}(w)$ is the generalized Laguerre polynomial of degree ν.

Case 4. $\beta \neq 0$. Setting $\beta = -1$, $b = w$, $c = z$, we obtain

$$(4.5) \qquad (1+\gamma y^2)^{-(\nu+1)/2} \{1 + (1+\gamma)y^2\}^{\frac{1}{2}\nu} \exp\left\{\frac{2xyz - y^2z^2 + \gamma x^2 y^2}{1+\gamma y^2}\right\} H_\nu(\xi)$$
$$= \sum_{n=0}^{\infty} g_n H_n(x) y^n, \qquad |y| < \mathrm{Min}(|\gamma|^{-\frac{1}{2}}, |1+\gamma|^{-\frac{1}{2}}),$$

where

$$\xi = \frac{w + xy + (\gamma w - z)y^2}{\{(1+\gamma y^2)[1 + (1+\gamma)y^2]\}^{\frac{1}{2}}}.$$

By inspection of the left member it is evident that the coefficient of y^n is a polynomial in x; hence the second solution does not occur. Replacing x by $1/x$, y by xy, and then setting $x = 0$, we obtain

$$(4.6)\qquad e^{2yz+\gamma y^2}H_\nu(w+y) = \sum_{n=0}^{\infty} g_n(2y)^n,$$

a simple generating function for g_n. The explicit form of g_n may be found with the aid of (4.1) and (4.2):

$$(4.7)\qquad g_n = \sum_{k=0}^{n} \binom{\nu}{n-k} \frac{(-\gamma)^{k/2}}{2^k k!} H_k\left(\frac{z}{(-\gamma)^{\frac{1}{2}}}\right) H_{\nu-n+k}(w) \qquad (\gamma \neq 0)$$

$$g_n = \sum_{k=0}^{n} \binom{\nu}{n-k} \frac{1}{k!} H_{\nu-n+k}(w) z^k \qquad (\gamma = 0).$$

In particular, when $\gamma = z = 0$ (**1**, p. 890)

$$(4.8)\qquad (1+y^2)^{\frac{1}{2}\nu} H_\nu\left(\frac{w+xy}{\sqrt{[1+y^2]}}\right) = \sum_{n=0}^{\infty} \binom{\nu}{n} H_{\nu-n}(w) H_n(x) y^n \quad (|y| < 1).$$

When $\gamma = -1$ and $z = -w$, the value of g_n may be obtained by comparing (4.6) with (4.2). Thus

$$(4.9)\qquad (1-y^2)^{-(\nu+1)/2} \exp\left\{\frac{2wxy - (x^2+w^2)y^2}{1-y^2}\right\} H_\nu\left(\frac{w-xy}{\sqrt{[1-y^2]}}\right)$$
$$= \sum_{n=0}^{\infty} \frac{H_{\nu+n}(w) H_n(x) y^n}{2^n n!}, \qquad (|y| < 1),$$

which reduces to Mehler's formula (**3**, p. 173) when $\nu = 0$ and to Feldheim's formula (**2**, p. 233) when $x = w$ and ν is an even number.

Case 5. $\beta = 0$, $\gamma \neq 0$. Setting $\gamma = -1$, $b = z$, $c = w$, we obtain with the aid of (4.1) and (4.9)

$$(4.10)\qquad (1-y^2)^{-\frac{1}{2}(\nu+1)} \exp\left\{\frac{2wxy - (x^2+w^2)y^2}{1-y^2}\right\}$$
$$\cdot H_\nu\left(\frac{x-wy}{\sqrt{[1-y^2]}} + \frac{z\sqrt{[1-y^2]}}{y}\right) = \sum_{n=-\infty}^{\infty} g_n H_{\nu+n}(x)(y/2)^n,$$

where

$$g_n = \sum_{k=0}^{\infty} \binom{\nu}{k} \frac{1}{\Gamma(k+n+1)} H_{k+n}(w) z^k \qquad (n = 0, \pm 1, \pm 2 \ldots).$$

Moreover g_n has the generating function

$$(1+z/y)^\nu e^{2wy-y^2} = \sum_{n=-\infty}^{\infty} g_n y^n, \qquad (|y| > |z|).$$

5. Generating functions annulled by conjugates of 3C − B₂. In accordance with the analysis of § 3 we examine next the functions annulled by L and $pC + qB_2$, $pq \neq 0$. Only the ratio p/q is essential, and it proves convenient to choose $p = 3$, $q = -1$.

The general solution of $(3C - B_2)\, u = 0$, or

$$(x+6y^3)\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 12xy^3 u$$

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is

$$u = e^{-6y^3(x+y^3)}f(\zeta), \qquad \zeta = 2xy + 3y^4.$$

This function is annulled by L if

$$\frac{d^2f}{d\zeta^2} - 3\zeta f = 0.$$

Two linearly independent solutions are given by

$$f = {}_0F_1\left(-; \frac{2}{3}; \frac{1}{3}\zeta^3\right), \qquad f = \zeta\,{}_0F_1\left(-; \frac{4}{3}; \frac{1}{3}\zeta^3\right).$$

Therefore, omitting the indices,

$$u_1 = e^{-6y^3(x+y^3)}F\left(-; \frac{2}{3}; \frac{1}{3}\zeta^3\right).$$

$$u_2 = e^{-6y^3(x+y^3)}\zeta F\left(-; \frac{4}{3}; \frac{1}{3}\zeta^3\right)$$

are linearly independent solutions of $Lu = 0$, $(3C - B_2)\,u = 0$. Their expansions in powers of y are readily obtained. On replacing y by $y^{\frac{1}{3}}$, we obtain

$$(5.1) \quad e^{-6y(x+y)}F\left(-; \frac{2}{3}; \frac{y}{3}(2x + 3y)^3\right) = \sum_{n=0}^{\infty} \frac{\Gamma(2/3)H_{3n}(x)}{n!\Gamma(n + 2/3)}\left(\frac{y}{3}\right)^n,$$

$$(5.2) \quad e^{-6y(x+y)}(2x + 3y)F\left(-; \frac{4}{3}; \frac{y}{3}(2x + 3y)^3\right) = \sum_{n=0}^{\infty} \frac{\Gamma(4/3)H_{3n+1}(x)}{n!\Gamma(n + 4/3)}\left(\frac{y}{3}\right)^n.$$

Applying S to u_1 and u_2, and setting $w = c + 3\beta$, $z = 2b + 3\beta^2$, we obtain the following functions annulled by L and $S(3C - B_2)S^{-1} = \gamma A + wB + (3 - \gamma w)C - B_2 - \gamma^2 C_2 + 3z - w^2 + \frac{1}{2}\gamma$:

$$(5.3) \quad (1 + \gamma y^2)^{-\frac{1}{2}}\, e^Y F\left(-; \frac{2}{3}; \frac{1}{3}X^3\right) = \sum_{n=0}^{\infty} a_n H_n(x) y^n \qquad (|y| < |\gamma|^{-\frac{1}{2}})$$

$$(5.4) \quad (1 + \gamma y^2)^{-\frac{1}{2}}\, e^Y X F\left(-; \frac{4}{3}; \frac{1}{3}X^3\right) = \sum_{n=0}^{\infty} b_n H_n(x) y^n \qquad (|y| < |\gamma|^{-\frac{1}{2}}),$$

where

$$X = z + \frac{2y(x - wy)}{1 + \gamma y^2} + \frac{3y^4}{(1 + \gamma y^2)^2},$$

$$Y = x^2 - \frac{3y^2 z + (x - wy)^2}{1 + \gamma y^2} - \frac{6y^3(x - wy)}{(1 + \gamma y^2)^2} - \frac{6y^6}{(1 + \gamma y^2)^3}.$$

Replacing x by $1/x$ and y by xy, and then setting $x = 0$, we obtain the following generating functions for a_n and b_n:

$$e^{2wy+\gamma y^2}F\left(-; \frac{2}{3}; \frac{1}{3}(2y + z)^3\right) = \sum_{n=0}^{\infty} a_n (2y)^n,$$

$$e^{2wy+\gamma y^2}(2y + z)F\left(-; \frac{4}{3}; \frac{1}{3}(2y + z)^3\right) = \sum_{n=0}^{\infty} b_n (2y)^n.$$

References

1. H. Bateman, *A partial differential equation connected with the functions of the parabolic cylinder*, Bull. Amer. Math. Soc., *41* (1935), 884–93.
2. E. Feldheim, *Développements en série de polynomes d'Hermite et de Laguerre à l'aide des transformations de Gauss et de Hankel*, Neder. Akad. Wetensch. *43* (1940), 224–48, 378–89.
3. F. G. Mehler, *Ueber die Entwicklung einer Funktion von beliebig vielen Variablen nach Laplaceschen Functionen höherer Ordnung*, J. Reine Angew. Math., *66* (1866), 161–76.
4. C. Truesdell, *An essay toward a unified theory of special functions*, Ann. Math. Studies, no. 18 (Princeton, 1948).
5. L. Weisner, *Group-theoretic origin of certain generating functions*, Pacific J. Math. *5* (1955), 1033–9.

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