21

Perspectives

And it seemed as though in a little while the solution would be found, and then a new and splendid life would begin; and it was clear to both of them that they had still a long, long road before them, and that the most complicated and difficult part was only just beginning.

– A. Chekhov, *The lady with the dog*

Conformal notions provide valuable tools for the analysis of global properties of spacetimes. In Part IV of this book it has been shown how a *conformal point of view* leads to proofs of the global existence and non-linear stability of de Sitter-like spacetimes, of the semiglobal existence and non-linear stability of Minkowski-like spacetimes, and how they provide a systematic procedure for the construction of anti-de Sitter-like spacetimes. Moreover, conformal methods provide a robust framework for the analysis of the gravitational field of isolated systems in a neighbourhood of both null and spatial infinity.

The application of conformal methods in general relativity is a mature area of research with a considerable number of open problems. Several of these have been discussed in various places of this book. Unavoidably, there are other problems and aspects of the subject which, for reasons of space, could not be covered in the main text. This last chapter presents a list of ideas and problems which, in the opinion of the author, may play a role in the future development of the subject.

21.1 Stability of cosmological models

The global non-linear stability of the de Sitter spacetime was discussed in Chapter 15. This exact solution can be regarded as a basic cosmological model. The analysis of Chapter 15 can be extended to include a non-vacuum matter content with good conformal properties: for example, the Maxwell, Yang-Mills and conformally coupled scalar field; see Friedrich (1991) and Lübbe and Valiente

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Kroon (2012). More recently, the ideas behind these proofs have been adapted in Lübbe and Valiente Kroon (2013b) to provide an analysis of the future stability of Friedman-Robertson-Walker cosmological models with a perfect fluid with the equation of state of radiation; see Section 9.4. A natural question is whether conformal methods could be adapted to more general matter models, that is, matter models with a non-vanishing trace. That this may be possible is suggested by the analysis in Friedrich (2015b) where it is shown that massive scalar fields for which the mass parameter is related to the cosmological constant by the condition $3m^2 = -2\lambda$ give rise to a set of regular conformal evolution equations for the Einstein-massive scalar field system. A further indication that conformal methods may be applicable to more general matter models is provided by the observation that for a large class of equations of state, perfect fluid Friedman-Lemaître-Robertson-Walker (FLRW) cosmological models can be smoothly conformally compactified – see, for example, Griffiths and Podolský (2009), section 6.4 – this despite the fact that the "natural" conformal evolution equations for these models are not conformally regular.

An important motivation behind the analysis of the future non-linear stability of cosmological models is the so-called ***cosmic no-hair conjecture*** – the expectation that for a large class of models the late-time evolution approximates, in some sense, a de Sitter state; see, for example, Wald (1983). As the analysis of Lübbe and Valiente Kroon (2013b) exemplifies, conformal methods provide a convenient setting for this discussion – at least for some suitable matter models.

Conformal methods provide a natural tool for the analysis of so-called ***isotropic cosmological singularities***. These are singularities of the physical spacetime that can be removed by means of a conformal rescaling of the metric. The singularity of the rescaled metric is assumed to occur on a spacelike surface. Accordingly, the conformal structure can be extended through the singularity and one can study the Cauchy problem for the cosmological model with data at the location of the singularity; see, for example, Tod (2002) for an introduction into the subject and Anguige and Tod (1999a,b) and Tod (2003) for further details. The Big Bang singularity in FLRW models provides the prototypical example of this type of singularity: as these spacetimes are conformally flat, any curvature singularity must be restricted to the (physical) Ricci tensor. In view of the highly symmetric nature of FLRW spacetimes, the Ricci tensor has only one essential component; combining this observation with the fact that under conformal rescalings the Ricci tensor satisfies a transformation law which is non-homogeneous, one can then see that in FLRW spacetimes the conformal factor can be chosen so as to absorb the singular behaviour of the curvature.

In the analysis of isotropic singularities, the role of the conformal factor is different from the one in the study of asymptotics: the conformal factor diverges at the singularity rather than going to zero – thus, it "blows up" the shrinking physical metric to make it finite. This type of behaviour is not expected to be a general feature of cosmological solutions to the Einstein field equations. This observation is related to Penrose's ***Weyl curvature hypothesis***: the idea that the early geometry of the universe should be such that the Weyl tensor vanishes, singling out a state of *low gravitational entropy*; see Penrose (1979).

In the discussion of isotropic cosmological singularities one pursues conformal rescalings of the form $\tilde{\boldsymbol{g}} = \mho^2 \boldsymbol{g}$ where $\tilde{\boldsymbol{g}}$ denotes the physical metric, while the unphysical metric $\boldsymbol{g}$ extends the conformal structure through the singularity characterised by the condition $\mho \to 0$. Under these conventions the Einstein field equations, written in a suitable gauge, lead to *conformal evolution equations* having a well-understood singular behaviour at the Big Bang. These evolution equations are an example of so-called ***Fuchsian differential equations*** – a class of equations with a well-defined theory. Using this theory, a number of statements concerning isotropic singularities can be obtained; see again Tod (2002) and references within for further details. More recently, it has been shown that a duality property of the conformally coupled scalar field equation allows one to analyse isotropic singularities in a framework involving the conformal Einstein field equations; see Lübbe (2014). It is of interest to see whether these ideas can be pursued further and extended to more general contexts.

21.2 Stability of black hole spacetimes

One of the outstanding open problems in mathematical general relativity is the question of the ***non-linear stability of the Kerr spacetime***; see, for example, Dafermos and Rodnianski (2010) for an entry point into the literature of the subject. The expectation associated with this question is that perturbations of a Kerr metric should dynamically approach a member of the Kerr family of solutions in the exterior of the black hole region. This problem involves both an orbital and an asymptotic stability analysis; see the discussion in Section 14.4. The non-linear stability of the Kerr spacetime poses both technical and conceptual challenges. On the technical side, it requires the development of robust partial differential equation (PDE) techniques to control the behaviour of the Einstein field equations in the strong gravitational field regime of a black hole. Current efforts in this direction have involved a detailed analysis of linear wave equations whose solutions propagate on the domain of outer communication of a Kerr background. This analysis makes systematic use of so-called *vector field methods*. The expectation is that these wave equations provide a suitable model for the Einstein equations written, say, in harmonic coordinates; see again Dafermos and Rodnianski (2010) for an account of this approach. On the conceptual side, the problem needs a detailed specification, compatible with the needs of PDE theory, of what is meant by the statement that a given spacetime is *close* to the Kerr spacetime; some ideas on how to address this issue are discussed in Bäckdahl and Valiente Kroon (2010a,b).

Given that conformal methods, as discussed in this book, provide a tool for the analysis of the non-linear stability of asymptotically simple spacetimes – compare

Chapters 15 and 16 – it is natural to wonder whether they could also provide an avenue for the analysis of the non-linear stability of black hole spacetimes. The stability proofs discussed in this book start from the premise that a detailed understanding of the conformal geometry of a background solution is key to the analysis. Once this has been achieved, the existence and stability results follow by means of general results of the theory of PDEs – namely, the Cauchy stability guaranteed by Theorem 12.2. From the perspective of conformal geometry, the essential difference between the basic asymptotically simple spacetimes and the exact solutions describing black hole spacetimes is that while the former are conformally regular, the latter have a conformal structure with singular regions. This observation rules out the possibility of directly using arguments based solely on the notion of Cauchy stability to prove global existence and stability of black hole spacetimes. In order to go any further, it seems necessary to analyse the structure of the singularities in the conformal structure of the background solutions so as to obtain, if possible, conformal representations of the black hole spacetimes for which the conformal Einstein field equations acquire a form which is amenable to a PDE analysis. An example of the *regularisation* of singularities in the conformal structure is provided by the analysis of the ***problem of spatial infinity*** in Section 20.3 where a detailed knowledge of the singular behaviour of the various conformal fields led to the construction of a regular Cauchy problem for the conformal field equations. It is possible that some singular regions in the conformal structure of black hole spacetimes – such as neighbourhoods of $i^{\pm}$ in the extreme Reissner-Nordström and extreme Kerr spacetimes – are amenable to an analogous discussion; see, for example, Lübbe and Valiente Kroon (2014).

A systematic approach to the analysis of the conformal structure of black holes is through the study of suitable congruences of conformal geodesics. In Friedrich (2003a) it is shown that it is possible to construct a non-intersecting congruence of conformal geodesics that covers the whole of the Kruskal extension of the Schwarzschild spacetime. This congruence is prescribed by initial data on the time symmetric hypersurface of the spacetime, and it provides a preferred conformal representation of the spacetime as well as a global conformal Gaussian gauge system from which, say, a global numerical evaluation of the spacetime can be undertaken; see the discussion in the next section. In addition, this type of construction sheds some light on the singular behaviour of the conformal structure at the timelike infinity; see Friedrich (2002), section 1.4.4. A similar analysis has been carried out in the Reissner-Nordström spacetime (including the extremal case) using so-called ***conformal curves*** in Lübbe and Valiente Kroon (2013a) and the Schwarzschild-de Sitter and Schwarzschild-anti de Sitter spacetime with conformal geodesics in García-Parrado et al. (2014). It would be of great interest extend this type of analysis to stationary black holes, that is, the Kerr spacetime.

The expectation driving the constructions described in the previous paragraph is that they will lead to a suitable conformal representation of the background black hole spacetimes which, in turn, lends itself to the formulation of an initial value problem allowing the analysis of the non-linear stability of black hole spacetimes. Nevertheless, the presence of singular points of the conformal structure of the background solution will require considerations of *asymptotic stability* – rather than just *orbital stability* as in the case of the proofs of stability of the de Sitter and Minkowski spacetimes given in Chapters 15 and 16. The development of methods that allow this type of analysis for the conformal field equations is an interesting and challenging problem.

Finally, it should be mentioned that the notion of conformal compactification of spacetimes, as introduced in Chapter 7, has been used as the starting point of a programme to construct a *theory of peeling and scattering of fields* (including gravity) on black hole spacetimes; see Nicolas (2015) and references within. It would be of great interest to combine this approach to the asymptotic analysis of spacetimes with the methods for the conformal Einstein field equations developed in this book.

21.3 Conformal methods and numerics

Numerical relativity, the study of the Einstein field equations by means of numerical methods, has undergone a great development in recent years. Extended numerical simulations of coalescing black holes have become almost routine; see, for example, Alcubierre (2008), Pretorious (2009) and Baumgarte and Shapiro (2010). To a great extent, these numerical simulations have been concerned with astrophysical aspects of black holes – most notably the extraction of gravitational wave forms; see, for example, Lehner and Pretorious (2014). In addition to this important application aimed at the detection of gravitational waves, numerical relativity offers a powerful tool in mathematical investigations of the equations of general relativity. Some promising areas for this type of interaction have been described in, for example, Jaramillo et al. (2008); for an alternative perspective, see Andersson (2006).

The conformal field equations suggest the possibility of performing *global numerical evaluations of spacetimes*, that is, evaluations which are not limited in their spatial and temporal dimensions by the finiteness of the computational grids. In addition, one would expect such evaluations to be free, to some extent, of the problems posed by the presence of *unphysical* boundary conditions required to obtain a discretisation in a finite grid without periodic boundary conditions.

There have been a number of efforts geared towards the construction of global numerical evaluations of spacetimes using the conformal Einstein field equations. An early implementation of these ideas for the spherically symmetric Einstein-conformally invariant scalar field system can be found in Hübner (1995). A programme to construct a computer code for numerical simulations using the metric vacuum conformal field equations has been reported in Hübner (1999a,b, 2001b) and culminated in Hübner (2001a) where a numerical demonstration of the semi-global existence result for hyperboloidal data discussed in Chapter 16

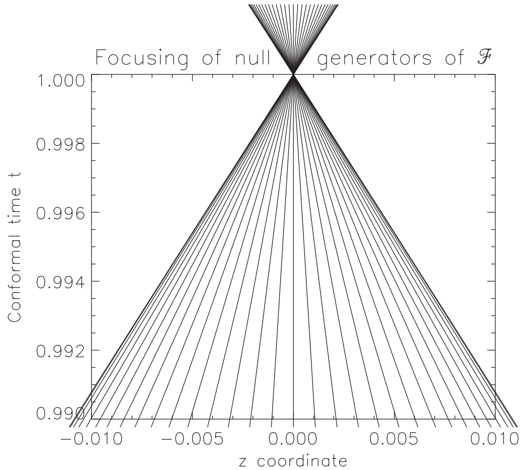

Figure 21.1 Focusing of the generators of null infinity in a numerical evaluation of hyperboloidal initial data close to Minkowski initial data. From Figure 2 in P. Hübner (2001), *From now to timelike infinity in a finite grid*, Class. Quantum Grav. **18**, 1871–1884. © IOP Publishing. Reproduced by permission of IOP Publishing. All rights reserved.

has been provided. Remarkably, the numerical simulations obtained by means of this code show how the generators of null infinity intersect, up to numerical precision at a single point, future timelike infinity i^+; see Figure 21.1. An alternative approach based on the frame version of the standard vacuum conformal field equations has been described in Frauendiener (1998a,b, 2002) and implemented in Frauendiener and Hein (2002); see also the review by Frauendiener (2004). A critical discussion of the numerical implementation of the standard conformal Einstein field equation can be found in Husa (2002).

Conformal Gaussian gauge systems provide an alternative approach to the numerical implementation of the conformal Einstein field equations. As shown in Chapter 13, the evolution equations implied by the extended conformal Einstein equations in this type of gauge splits into a subsystem of transport equations for the components of the frame, connection and Schouten tensor and a symmetric hyperbolic system for the components of the rescaled Weyl tensor. This remarkable structure, highlighting the special role of the Weyl tensor as describing the *free gravitational field*, may facilitate the numerical implementation of the system. An added advantage of this formulation of the conformal field equations is, in the vacuum case, the a priori knowledge of the conformal factor linking the unphysical spacetime with the physical spacetime for which the Einstein equations hold.

A programme to analyse the global dynamics of cosmological spacetimes by numerical methods using the extended conformal field equations has been pursued in Beyer (2007, 2008, 2009a,b). This work has provided valuable insights

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into the *cosmic no-hair* conjecture – see Section 21.1 – and the role of the so-called *Nariai solution*. Cosmological spacetimes provide a convenient testbed for the numerical implementation of the conformal field equations as they allow the use of compact spatial domains – say, the 3-sphere $\mathbb{S}^3$, the 3-torus $\mathbb{S} \times \mathbb{S} \times \mathbb{S}$ or the 3-handle $\mathbb{S}^2 \times \mathbb{S}$ – so that no boundary conditions in the spatial domain are required. In addition, compact spatial sections are naturally amenable to the use of spectral methods; see, for example, Beyer (2009c).

A further application of the extended conformal Einstein field equations is the global numerical evaluation of spherically symmetric static black hole spacetimes. This idea was first investigated in Zenginoglu (2006, 2007) for the Schwarzschild spacetime and later extended to the electrovacuum case (i.e. the Reissner-Nordström spacetime) in Valiente Kroon (2012). The assumption of spherical symmetry implies a great simplification in the equations so as to render a reduced evolution system consisting of transport equations solely. Notice, however, that the conformal gauge in terms of which the evolution equations are expressed is not adapted to the orbits of the static Killing vectors, and, thus, one has non-trivial *gauge dynamics*. An important property of these spherically symmetric reduced equations is that their essential dynamics is governed by a *core system* consisting of three equations in the vacuum case (for a component of the connection, a component of the Schouten tensor and the non-vanishing component of the rescaled Weyl tensor) and four equations in the electrovacuum case (connection, Schouten tensor, rescaled Weyl tensor and the single non-vanishing component of the Faraday tensor). These equations can be easily implemented and numerically solved with present-day desktop computers and allow the global computation of a privileged conformal representation of the black hole spacetime from an initial hypersurface to either the singularity or null infinity and beyond; see Figure 21.2. These small-scale numerical simulations could be used, in the future, as the first step in the global numerical evaluation of dynamic, non-spherically symmetric spacetimes.

More recently, there have been efforts aimed at the numerical implementation of the construction of the cylinder at spatial infinity described in Section 20.3.2. The ultimate goal of this programme is the numerical computation of hyperboloidal data from Cauchy data and to obtain insight into the numerical consequences of the obstructions to the smoothness of null infinity discussed in the later sections of Chapter 20. At the time of writing, the analysis has been restricted to the analysis of the spin-2 field equation on a Minkowski background – in the spirit of Section 20.3.4 – with the expectation that this situation contains the essential difficulties in the implementation; see Beyer et al. (2012) and Frauendiener and Hennig (2014).

Foliations of spacetimes by means of hyperboloidal hypersurfaces have been used in numerical simulations aimed at analysing radiative processes in gravitational collapse and perturbations of black hole spacetimes; see, for example, Zenginoglu (2008, 2011a,b), Rinne (2010, 2014), Zenginoglu and Kidder (2010)

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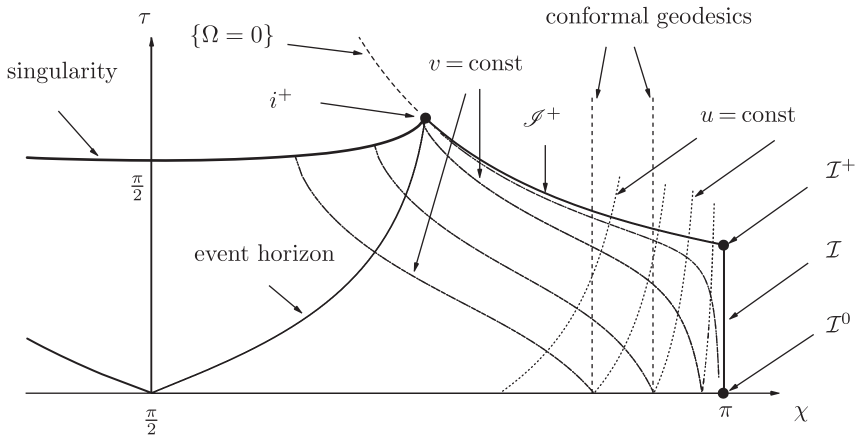

Figure 21.2 The Schwarzschild spacetime in a conformal Gaussian gauge system; see the discussion in the main text and compare also Section 20.3.3. From Figure 3.3, page 57, of A. Zenginoglu, *A conformal approach to numerical computations of asymptotically simple spacetimes*, PhD thesis, University of Postdam (2006). Reproduced courtesy of the author.

and Rinne and Moncrief (2013). These numerical investigations make use of formulations of the Einstein field equations alternative to the ones discussed in this book. Finally, hyperboloidal foliations have also been used in the implementation of fully spectral (i.e. in time and space) evolution schemes for various fields; see Hennig and Ansorg (2009) and Macedo and Ansorg (2014).

21.4 Computer algebra

The analysis of hyperbolic reductions in Chapter 13 shows that despite their elegance and appealing geometric nature, the study of the consequences of the conformal field equations requires a considerable amount of algebraic manipulations. Modern computer algebra systems provide a natural way of performing these manipulations in an effective and efficient way. At the time of writing, the suite of packages `xAct` for `Mathematica` provides a robust and versatile framework for the type of tensorial and spinorial manipulations discussed in this book; see Martín-García (2014). The packages in the suite `xAct` allow one to perform tensorial and spinorial abstract index computations on generic tensors as well as explicit component computations for a given metric. At the core of `xAct` is a *canonicalisation routine* which allows one to simplify large tensorial and spinorial expressions by identifying “dummy” indices and exploiting the symmetries of the various objects involved. In addition, `xAct` allows one to carry out cumbersome operations such as the decomposition of spinors into irreducible parts. An additional appeal of this system is that it provides its output in standard index notation. The system also provides facilities to produce Latex output of the calculations

The capabilities of modern computers have reached the point that, for example, using `xAct`, it is possible to perform certain types of analyses which would have been impractically long otherwise. As an example, the study of asymptotic expansions using the framework of the cylinder at spatial infinity described in Chapter 20 and reported in Valiente Kroon (2004a,b,c, 2005) depended, in a crucial manner, on computer algebra calculations. These calculations were carried out with purpose-built routines in the computer algebra system `Maple V`.

21.5 Concluding remarks

This book has discussed a particular approach to the use of conformal methods in mathematical general relativity. Clearly, the approach presented is not the only one possible nor are the potential applications restricted to the ones discussed in these pages. It constitutes a body of work extending over a period of more than 30 years starting with the work of H. Friedrich in the early 1980s – or 50 years if one considers the seminal work by R. Penrose in the 1960s. This extended period of time is proof of the vitality of the subject. Nevertheless, a more exacting assessment of its vitality and relevance should come from its influence in the whole of mathematical general relativity and its ability to foster new ideas and research problems. Time will be the ultimate judge on this matter.

This book is an attempt to bring to the fore the relevance of conformal methods in modern research in general relativity and to make the subject as accessible as possible to those interested in using these ideas in their own research. The reader is left to decide whether this goal has been achieved.