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A JOINT SPECTRAL THEOREM FOR UNBOUNDED NORMAL OPERATORS

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Abstract

A joint spectral theorem for an n-tuple of doubly commuting unbounded normal operators in a Hilbert space is proved by using the techniques of GB^*-algebras.

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Introduction

The definition of a joint spectrum for an n-tuple of bounded operators on a Hilbert space H has been given by Harte and others [7, 12]. We generalize this to define the joint spectrum of an n-tuple of unbounded operators in H. By using the techniques of GB^*-algebras (a class if involutive algebras studied by Allan [1] and Dixon [4]), we prove the spectral theorem for an n-tuple of unbounded normal operators (Theorem 2.2). The analogous result for bounded operators is straightforward and has been recently noted by Hastings [8].

1. Preliminaries

Throughout the paper H denotes a complex Hilbert space and $\mathfrak{B}(H)$, the algebra of all bounded linear operators on H.

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(a) *Joint spectrum.*

1.1. DEFINITIONS. Let $T_1, \ldots, T_n$ be closed linear operators in H defined on the same dense domain $\mathfrak{D}$. Suppose that $T_1^*, \ldots, T_n^*$ also have the same dense domain $\mathfrak{D}^*$.

(1) The joint left spectrum $\mathrm{Sp}_l(T)$ of $T = (T_1, \ldots, T_n)$ is the set of all $(\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$ such that for no n-tuple $(B_1, \ldots, B_n)$ of operators in $\mathfrak{B}(H)$, $\sum_{i=1}^n B_i(T_i - \lambda_i) \subset I$ holds.

(2) The joint right spectrum $\mathrm{Sp}_r(T)$ of $T = (T_1, \ldots, T_n)$ is the set $(\mathrm{Sp}_l(T^*))^*$ where $T^* = (T_1^*, \ldots, T_n^*)$ and for $K \subset \mathbf{C}^n$, $K^* = \{(\bar{\lambda}_1, \ldots, \bar{\lambda}_n) : (\lambda_1, \ldots, \lambda_n) \in K\}$.

(3) The joint spectrum $\mathrm{Sp}(T)$ is the set $\mathrm{Sp}_l(T) \cup \mathrm{Sp}_r(T)$.

REMARK. It is easily seen that our definition of the joint spectrum agrees with the usual definition of the spectrum of a closed unbounded operator [10, page 346].

In the remaining part of this section, we assume that $T_1, \ldots, T_n$ are closed linear operators in H defined on the same dense domain $\mathfrak{D}$ such that their adjoints $T_1^*, \ldots, T_n^*$ also have the same dense domain $\mathfrak{D}^*$.

1.2. PROPOSITION. *Let $T = (T_1, \ldots, T_n)$. Then $(\lambda_1, \ldots, \lambda_n) \in \mathrm{Sp}_l(T)$ if, and only if, there is a sequence $\{x_k\}$ of unit vectors in $\mathfrak{D}$ such that $(T_i - \lambda_i)x_k \to 0$ as $k \to \infty$ for each $i = 1, 2, \ldots, n$.*

PROOF. Suppose $(\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$. If there is no sequence $\{x_k\}$ of unit vectors in $\mathfrak{D}$ such that $(T_i - \lambda_i)x_k \to 0$ $i = 1, 2, \ldots, n$, then the operator $\delta : \mathfrak{D} \to \sum_{i=1}^n \oplus H_i$ $(H_i = H)$ defined by $\delta(x) = ((T_1 - \lambda_1)x, \ldots, (T_n - \lambda_n)x)$ is bounded below. Hence there exists a bounded operator $\beta : \sum_{i=1}^n \oplus H_i \to H$ such that $\beta\delta(x) = x$ for all $x \in \mathfrak{D}$. For $i = 1, 2, \ldots, n$, define a operator B_i on H by $B_i(x) = \beta(0, \ldots, 0, x, 0, \ldots, 0)$ where x is in the ith place on the right hand side. Then B_i's are bounded operators and

$$\sum_{i=1}^n B_i(T_i - \lambda_i)x = x \quad \text{for all } x \in \mathfrak{D}.$$

Thus $\sum_{i=1}^n B_i(T_i - \lambda_i) \subset I$ and so $(\lambda_1, \ldots, \lambda_n) \notin \mathrm{Sp}_l(T)$.

The proof of the converse is easy.

1.3. COROLLARY. $(\lambda_1, \ldots, \lambda_n) \in \mathrm{Sp}_r(T)$ *if, and only if, there is a sequence $\{x_k\}$ of unit vectors in $\mathfrak{D}^*$ such that $(T_i^* - \bar{\lambda}_i)x_k \to 0$ as $k \to \infty$ for $i = 1, \ldots, n$.*

1.4. COROLLARY. *If* $T_1, \ldots, T_n$ *are normal, then* $\mathrm{Sp}_l(T) = \mathrm{Sp}_r(T)$ *and hence* $\mathrm{Sp}(T) = \mathrm{Sp}_l(T)$.

Since $\|(T_i - \lambda_i)^* x\| = \|(T_i - \lambda_i)x\|$ for $x \in \mathfrak{D}$ and $i = 1, \ldots, n$, the proof of the corollary follows from Proposition 1.2 and Corollary 1.3.

(b) *GB*-algebras.*

A Generalized B^*-algebra (GB^*-algebra) [1, 4] is essentially a symmetric topological *-algebra A which admits a largest bounded *-semigroup (with respect to multiplication) B_0 called its unit ball which is also closed and absolutely convex so that the *-subalgebra $A(B_0)$ of A which B_0 generates algebraically is a Banach algebra with Minkowski functional of B_0 in $A(B_0)$ as the norm. $A(B_0)$ is, in fact, a B^*-algebra. We shall need the following theorem regarding a GB^*-algebra.

1.5. THEOREM [2]. *Let* A *be a* GB^**-algebra with unit ball* B_0. *Then* $A(B_0)$ *is sequentially dense in* A.

In the sequel, we shall need to deal with two important GB^*-algebra—the algebra of measurable functions and the algebra of measurable operators. We discuss below these two algebras.

1.6. EXAMPLE. Let (X, Σ, μ) be a measure space with finite subset property. Let $m(X)$ be the *-algebra consisting of all complex-valued measurable functions (modulo equality a.e.) on X. For each $\varepsilon > 0$, $F \in E$, $\mu(F) < \infty$, consider the set $V(F, \varepsilon) = \{f \in m(X)\colon \mu(\{x \in F\colon |f(x)| > \varepsilon\}) < \varepsilon\}$. Let t_1 be the topology on $m(X)$ for which $\mathcal{V} = \{V(F, \varepsilon)\colon F \in \Sigma,\ \mu(F) < \infty,\ \varepsilon > 0\}$ is a zero neighbourhood base. Then $(m(X), t_1)$ is a complete GB^*-algebra with underlying B^*-algebra $A(B_0) = L^\infty(X, \Sigma, \mu)$ (see [5]).

1.7. EXAMPLE. Let A be a von Neumann algebra acting on a Hilbert space H. Yeadon [13] has discussed the set $m_l(A)$ of locally measurable operators in H defined with respect to A. Dixon [5] has proved that $m_l(A)$ is a complete GB^*-algebra with bounded part $A(B_0) = A$, with respect to a topology t_2, called the topology of convergence locally in measure (see also [13]) which is defined as follows: Let Z be the centre of A. Then Z is * isomorphic to $L^\infty(X, \Sigma, \mu)$ for some measure space (X, Σ, μ). Let d be the Segal's dimension function [11] from the projections in A to nonnegative extended real valued measurable functions on

X. For each $\varepsilon > 0$ and $F \in \Sigma$, $\mu(F) < \infty$, consider the set

$$U(F, \varepsilon) = \{T \in m_l(A)\colon \text{for some projection } P \in A,$$
$$\|TP\| < \varepsilon \text{ and } \mu(\{x \in F\colon d(1-P)(x) > \varepsilon\}) < \varepsilon\}.$$

Then t_2 is a topology on $m_l(A)$ for which $\mathfrak{U} = \{U(F, \varepsilon)\colon F \in \Sigma,\ \mu(F) < \infty,\ \varepsilon > 0\}$ is a zero neighbourhood base.

1.8. Definition. For $u_1, \ldots, u_n \in m(X)$, the joint essential range $\mathcal{E}(u)$ of $u = (u_1, \ldots, u_n)$ is defined by

$$\mathcal{E}(u) = \left\{(\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n\colon \mu\left(\left\{x \in X\colon \sum_{i=1}^{n} |u_i(x) - \lambda_i| < \varepsilon\right\}\right) > 0 \text{ for every } \varepsilon > 0\right\},$$

and the joint spectrum $\mathrm{Sp}_{L^\infty(X)}(u)$ of u is defined by

$$\mathrm{Sp}_{L^\infty}(u) = \left\{(\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n\colon \text{for no } n\text{-tuple } (v_1, \ldots, v_n) \text{ of elements in } L^\infty(X),\ \sum_{i=1}^{n} v_i\left(u_i - \lambda_i\right) = 1 \text{ a.e.}\right\}.$$

1.9. Proposition. *Let $n \geq 1$ be a positive integer. Then for $u_1, \ldots, u_n \in m(X)$, $\mathrm{Sp}_{L^\infty}(u) = \mathcal{E}(u)$ where $u = (u_1, \ldots, u_n)$.*

Proof. First we prove the result for $n = 1$. Let $u \in m(X)$. Suppose that $\lambda \notin \mathrm{Sp}_{L^\infty}(u)$. Then there is $v \in L^\infty(X)$ such that $(u - \lambda)v = 1$ a.e. Then $\mu(\{x \in X\colon |u(x) - \lambda| < 1/(2\|v\|_\infty)\}) = 0$. Hence $\lambda \notin \mathcal{E}(u)$. Conversely, if $\lambda \notin \mathcal{E}(u)$ then there is an $\varepsilon > 0$ such that the set $S = \{x \in X\colon |u(x) - \lambda| < \varepsilon\}$ has measure zero. Define v on X by

$$v(x) = \begin{cases} 1/(u(x) - \lambda) & \text{if } x \in X \setminus S, \\ 0 & \text{if } x \in S. \end{cases}$$

Then $v \in L^\infty(X)$ and $v(u - \lambda) = 1$ a.e. Hence $\lambda \notin \mathrm{Sp}_{L^\infty(X)}(u)$.

Next for an arbitrary n, suppose that $(\lambda_1, \ldots, \lambda_n) \notin \mathrm{Sp}(u)$. Then there are $v_1, \ldots, v_n \in L^\infty(X)$ such that $\sum_{i=1}^{n} v_i(u_i - \lambda_i) = 1$ a.e. Let $M = \max\{\|V_i\|\colon i = 1, \ldots, n\}$. Then $\mu(F_{1/2M}) = 0$ where for $\varepsilon > 0$, $F_\varepsilon = \{x \in X\colon \sum_{i=1}^{n} |u_i(x) - \lambda_i| < \varepsilon\}$. For, if $\mu(F_{1/2M}) > 0$, then for some $x \in F_{1/2M}$,

$$1 = \left|\sum_{i=1}^{n} v_i(x)(u_i(x) - \lambda_i)\right| \leq M \sum_{i=1}^{n} |u_i(x) - \lambda_i| < \frac{1}{2},$$

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which is a contradiction. This shows that $(\lambda_1,\dots,\lambda_n) \notin \mathfrak{S}(u)$. Conversely, suppose that $(\lambda_1,\dots,\lambda_n) \notin \mathfrak{S}(u)$. Then for some $\varepsilon > 0$, $\mu(F_\varepsilon) = 0$. If now $(\lambda_1,\dots,\lambda_n) \in \mathrm{Sp}(u)$ then for any $v_1,\dots,v_n$ in $L^\infty(X)$, $\Sigma_{i=1}^n v_i(u_i - \lambda_i)$ is not invertible in $L^\infty(X)$ and hence

$$0 \in \mathrm{Sp}_{L^\infty(X)}\left(\sum_{i=1}^n v_i(u_i - \lambda_i)\right).$$

But then by the result proved above for $n = 1$, $0 \in \mathfrak{S}(\Sigma_{i=1}^n v_i(u_i - \lambda_i))$ for every $v_1,\dots,v_n \in L^\infty(X)$. Hence for each $\eta > 0$ and for each $v = (v_1,\dots,v_n) \in (L^\infty(X))^n$, $\mu(E_\eta(v)) > 0$, where

$$E_\eta(v) = \left\{x \in X: \left|\sum_{i=1}^n (v_i(u_i - \lambda_i))(x)\right| < \eta\right\}.$$

Take

$$v_i = \frac{\overline{(u_i - \lambda_i)}}{1 + |u_i - \lambda_i|^2} \quad \text{and} \quad \eta = \frac{\varepsilon^2}{n^2 + \varepsilon^2}.$$

Then $E_\eta(v) \subset F_\varepsilon$, so that $\mu(E_\eta(v)) = 0$, which is a contradiction. It follows that $(\lambda_1,\dots,\lambda_n) \notin \mathrm{Sp}(u)$ and the proof is complete.

2. Main theorems

In this section, we prove two main results of the paper.

2.1. THEOREM. *Let $T_1,\dots,T_n$ be doubly commuting (that is $T_iT_j^* = T_j^*T_i$ and $T_iT_j = T_jT_i$ for $i, j = 1,\dots,n$) normal operators in H with the same domain $\mathfrak{D}$. Then the bounded bicommutant A of $\{T_1,\dots,T_n\}$ is a commutative von Neumann algebra such that $m_f(A)$ contains $T_1,\dots,T_n$.*

PROOF. Since each T_i is closed, $(1 + T_i^*T_i)^{-1}$ exists and is a bounded operator with $\|(1 + T_i^*T_i)^{-1}\| \leq 1$. Let E^i be the resolution of identity on Borel subsets of $[0,1]$ for the operator $(1 + T_i^*T_i)^{-1}$. Let $w_0 = \{0\}$, $w_k = (1/(k+1), 1/k]$ for $k = 1,2,\dots$. Then $\{w_k\}$ is a sequence of disjoint Borel subsets of $[0,1]$ with union $[0,1]$. Then $T_{ik} = T_iE^i(w_k)$ is bounded and normal for each $i = 1,\dots,n$ and $k = 0,1,2,\dots$. Hence using ([3], Theorem 15.12.8 page 389), $x \in \mathfrak{D}$ $(= \mathfrak{D}(T_i)$, domain of $T_i)$ if, and only if, $\Sigma_{k=0}^\infty \|T_{ik}x\|^2 < \infty$, and for such x, $\Sigma_{k=0}^\infty T_{ik}x = T_ix$ and $\Sigma_{k=0}^\infty T_{ik}^*x = T_i^*x$ for $i = 1,\dots,n$.

Now let $S \in \{T_1,\dots,T_n\}'$, the bounded commutant of $T_1,\dots,T_n$ (that is $S \in \mathfrak{B}(H)$ and $ST_i \subset T_iS$ for each i). By the Fuglede theorem [6], it is easy to see that

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$(1 + T_i^*T_i)^{-1}S = S(1 + T_i^*T_i)^{-1}$. Therefore, $E^i(w_k)S = SE^i(w_k)$; and so $ST_{ik}x = T_{ik}Sx$ $(x \in H,\ i = 1, 2, \ldots, n;\ k = 0, 1, \ldots)$. Hence $S \in \{T_{ik}\colon i = 1, \ldots, n;\ k = 0, 1, 2, \ldots\}'$. Conversely, let $S \in \{T_{ik}\colon i = 1, \ldots, n;\ k = 0, 1, 2, \ldots\}'$. Then $ST_{ik} = T_{ik}S$ for $i = 1, \ldots, n$ and $k = 0, 1, 2\ldots$. Let $x \in \mathfrak{D}$. Then $Sx \in \mathfrak{D}$, for

$$\sum_{k=0}^{\infty} \|T_{ik}Sx\|^2 = \sum_{k=0}^{\infty} \|ST_{ik}x\|^2 \leqslant \|S\|^2 \sum_{k=0}^{\infty} \|T_{ik}x\|^2 < \infty$$

for $i = 1, \ldots, n$. Also,

$$\begin{aligned} ST_ix = S\sum_{k=0}^{\infty} T_{ik}x &= \sum_{k=0}^{\infty} ST_{ik}x \\ &= \sum_{k=0}^{\infty} T_{ik}Sx = T_iSx \end{aligned}$$

for $i = 1, \ldots, n$. Hence $S \in \{T_1, \ldots, T_n\}'$. Thus $\{T_1, \ldots, T_n\}' = \{T_{ik}\colon i = 1, \ldots, n;\ k = 0, 1, 2, \ldots\}'$.

Next we show that the set $\{T_{ik}\colon i = 1, \ldots, n;\ k = 0, 1, 2, \ldots\}$ is a commuting set. For $1 \leqslant i, j \leqslant n$, we have $T_jT_i^*T_i = T_jT_iT_i^* = T_iT_jT_i^* = T_iT_i^*T_j = T_i^*T_iT_j$. Hence $T_j(1 + T_i^*T_i) = (1 + T_i^*T_i)T_j$. Therefore $(1 + T_i^*T_i)^{-1}T_j(1 + T_i^*T_i) = (1 + T_i^*T_i)^{-1}(1 + T_i^*T_i)T_j \subset T_j$; and so $(1 + T_i^*T_i)^{-1}T_j \subset T_j(1 + T_i^*T_i)^{-1}$. By Fuglede's theorem, $(1 + T_i^*T_i)^{-1}T_j^* \subset T_j^*(1 + T_i^*T_i)^{-1}$ also. After a little computation, we get $(1 + T_i^*T_i)^{-1}(1 + T_j^*T_j)^{-1} \subset (1 + T_j^*T_j)^{-1}(1 + T_i^*T_i)^{-1}$. Since $(1 + T_i^*T_i)^{-1}$ and $(1 + T_j^*T_j)^{-1}$ are bounded, $(1 + T_i^*T_i)^{-1}(1 + T_j^*T_j)^{-1} = (1 + T_j^*T_j)^{-1}(1 + T_i^*T_i)^{-1}$. It follows that $E^i(w_k)E^j(w_l) = E^j(w_l)E^i(w_k)$ for $1 \leqslant i, j \leqslant n$ and $k, l = 0, 1, 2, \ldots$. Since for $1 \leqslant i,\ j \leqslant n$ $(1 + T_i^*T_i)^{-1}T_j \subset T_j(1 + T_i^*T_i)^{-1}$, we have for $x \in H$, and $k, l = 0, 1, 2, \ldots$

$$\begin{aligned} (1 + T_i^*T_i)^{-1}T_{jl}x &= (1 + T_i^*T_i)^{-1}T_jE^j(w_l)x \\ &= T_j(1 + T_i^*T_i)^{-1}E^j(w_l)x \\ &= T_jE^j(w_l)(1 + T_i^*T_i)^{-1}x \\ &= T_{jl}(1 + T_i^*T_i)^{-1}x \end{aligned}$$

which implies that $E^i(w_k)$ commutes with T_{jl}. Hence for $x \in H$,

$$\begin{aligned} T_{ik}T_{jl}x = T_iE^i(w_k)T_{jl}x &= T_iT_{jl}E^k(w_k)x \\ &= T_iT_jE^j(w_l)E^i(w_k)x \\ &= T_jT_iE^i(w_k)E^j(w_l)x \\ &= T_jT_{ik}E^j(w_l)x \\ &= T_jE^j(w_l)T_{ik}x = T_{jl}T_{ik}x. \end{aligned}$$

Therefore, $T_{ik}T_{jl} = T_{jl}T_{ik}$ and the set $\{T_{ik}: i = 1,\dots,n;\ k = 0,1,2,\dots\}$ is a commuting set of bounded normal operators. Hence

$$A = \{T_1,\dots,T_n\}'' = \{T_{ik}: i = 1,\dots,n;\ k = 0,1,2,\dots\}''$$

is a commutative von Neumann algebra. Now let $Q_{ik} = \sum_{j=0}^{k} E^i(w_j)$. Then Q_{ik} is a projection in A and $A_{ik} \nearrow I$ as $k \to \infty$. Also, $T_i Q_{ik} = T_i \sum_{j=0}^{k} E^i(w_j) = \sum_{j=0}^{k} T_{ij}$. Hence $T_i Q_{ik} \in A$ and so $T_i Q_{ik}$ is measurable for $i = 1,2,\dots,N$; $k = 0, 1, 2,\dots$. Therefore by [13, Theorem 2.1], $T_1,\dots,T_n \in m_l(A)$.

REMARK. If A is as above and B is a von Neumann algebra containing A, then it is clear that each $T_i \in m_l(B)$.

2.2. THEOREM (*Spectral Theorem*). *Let $T = (T_1,\dots,T_n)$ be an n-tuple of doubly commuting normal operators with the same domain $\mathfrak{D}$. Then there is a resolution of identity on Borel subsets of joint spectrum* $\mathrm{Sp}(T)$ *of T such that for each Borel function f on* $\mathrm{Sp}(T)$, *there is an operator $f(T)$ with*

$$f(T) = \int_{\mathrm{Sp}(T)} f(\lambda)\,dE(\lambda).$$

PROOF. Let A be the maximal abelian self-adjoint algebra containing the bounded bicommutant of $\{T_1,\dots,T_n\}$. Then by the remark following Theorem 2.1, A is a von Neumann algebra such that $m_l(A)$ contains $T_1,\dots,T_n$. Since A is maximal abelian, there exists a measure space (X, Σ, μ) such that H can be identified with $L^2(X)$ and $L^\infty(X, \Sigma, \mu)$ is W^*-isomorphic to A.

Let Φ be the W^*-isomorphism of $L^\infty(X, \Sigma, \mu)$ onto A. We show that Φ extends uniquely to a toplogical *-isomorphism of $m(X)$ onto $m_l(A)$. For this, since $L^\infty(X)$ and A are sequentially dense in $m(X)$ and $m_l(A)$ respectively, it is sufficient to prove that the induced topologies on $L^\infty(X)$ from $m(X)$ and on A from $m_l(A)$ are identical via Φ. We prove this below.

Let $\{f_k\}$ be a sequence in $L^\infty(X)$ converging to zero in the induced topology. Let $F \in \Sigma$, $\mu(F) < \infty$ and let $\varepsilon > 0$. Take ε' such that $0 < \varepsilon' < \varepsilon$. Since $f_k \to 0$ there exists an integer k_0 such that $f_k \in V(F, \varepsilon')$ for all $k \geqslant k_0$. Hence if $R_k = \{x \in F: |f_k(x)| > \varepsilon'\}$, then $\mu(R_k) < \varepsilon'$ for all $k \geqslant k_0$. Let $E_k = \Phi(\chi_{R_k^c})$ be a projection in A. Then $\|\Phi(f_k)E_k\| = \|\Phi(f_k\chi_{R_k^c})\| = \|f_k\chi_{R_k^c}\| \leqslant \varepsilon' < \varepsilon$ for all $k \geqslant k_0$. Let $d = \Phi^{-1}$ restricted to projections in A. Then d satisfies all the properties of Segal's dimension function and so

$$\begin{aligned}\mu(\{x \in F: d(1 - E_k)(x) > \varepsilon\}) &\leqslant \mu(\{x \in F: d(1 - E_k)(x) > \varepsilon'\}) \\ &= \mu\left(\left\{x \in F: \Phi^{-1}\left(1 - \Phi(\chi_{R_k^c})\right)(x) > \varepsilon'\right\}\right) \\ &= \mu\left(\left\{x \in F: \chi_{R_k}(x) > \varepsilon'\right\}\right) \\ &\leqslant \mu(R_k) < \varepsilon' < \varepsilon \quad \text{for all } k \geqslant k_0.\end{aligned}$$

Therefore $\Phi(f_k) \in U(F, \varepsilon)$ for all $k \geqslant k_0$, and hence $\Phi(f_k) \to 0$ in the topology induced from $m_l(A)$. Thus $\Phi: L^\infty(X) \to A$ is continuous in the induced topologies of $m(X)$ and $m_l(A)$.

Conversely, let $\{S_k\}$ be a sequence in A such that $S_k \to 0$ in $m_l(A)$. Let $F \in \Sigma$, $\mu(F) < \infty$ and let $0 < \varepsilon < 1$. Then there exists k_0 such that $S_k \in U(F, \varepsilon)$ for all $k \geqslant k_0$, that is, there exists a projection P in A such that for all $k \geqslant k_0$, $\| S_k P \| < \varepsilon$ and $\mu(\{x \in F: d(1-P)(x) > \varepsilon\}) < \varepsilon$ where d is the Segal's dimension function Φ^{-1} restricted to projections in A. Since Φ is onto, there exists $W \in \Sigma$ such that $\Phi(\chi_W) = P$. Also Φ being an isometry, $\|\Phi^{-1}(S_k)\chi_W\| = \|S_k P\| < \varepsilon$ for all $k \geqslant k_0$. Hence $\{x \in X: |\Phi^{-1}(S_k)(x)| \geqslant \varepsilon\} \subset W^c$ for all $k \geqslant k_0$. Therefore, for $k \geqslant k_0$,

$$\begin{aligned}\{x \in F: |\Phi^{-1}(S_k)(x)| > \varepsilon\} \subset W^c \cap F &= \{x \in F: \Phi^{-1}(1-P)(x) > \varepsilon\} \\ &= \{x \in F: d(1-P)(x) > \varepsilon\};\end{aligned}$$

and so

$$\begin{aligned}\mu(\{x \in F: |\Phi^{-1}(S_k)(x)| > \varepsilon\}) &\leqslant \mu(\{x \in F: d(1-P)(x) > \varepsilon\}) \\ &< \varepsilon \quad \text{for all } k \geqslant k_0.\end{aligned}$$

Hence $\Phi^{-1}(S_k) \in V(F, \varepsilon)$ for all $k \geqslant k_0$ and so $\Phi^{-1}(S_k) \to 0$ in $m(X)$. Thus Φ^{-1} is continuous.

We denote the extension of Φ to $m(X)$ again by Φ. Since Φ is from $m(X)$ onto $m_l(A)$, there exist $u_1, \ldots, u_n$ in $m(X)$ such that $\Phi(u_i) = T_i$ $(i = 1, \ldots, n)$. We shall show that $\mathrm{Sp}(T) = \mathrm{Sp}_A(T)$ where $\mathrm{Sp}_A(\mathrm{T}) = \{(\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$: there are no $B_1, \ldots, B_n$ in A satisfying $\sum_{i=1}^n (T_i - \lambda_i)B_i = I\}$. By Corollary 1.4, $\mathrm{Sp}(T) = \mathrm{Sp}_l(T)$ and as is easily seen $\mathrm{Sp}_l(T) \subset \mathrm{Sp}_A(\mathrm{T})$. Also, since Φ is an isomorphism which maps $L^\infty(X)$ onto A, we have $\mathrm{Sp}_A(T) = \mathrm{Sp}_{L^\infty(X)}(u) = \mathcal{E}(u)$ where $u = (u_1, \ldots, u_n)$. Thus to show that $\mathrm{Sp}(T) = \mathrm{Sp}_A(T)$, it is enough to show that $\mathcal{E}(u) \subset \mathrm{Sp}_l(T)$. Let, therefore, $(0, \ldots, 0) \in \mathcal{E}(u)$. Let $E_k = \{x \in X: \sum_{i=1}^n |u_i(x)| \leqslant 1/k\}$ where k is a positive integer. Then $\mu(E_k) > 0$. Let $\{f_k\}$ be the sequence of unit vectors in $L^2(X)$ defined by $f_k = \chi_{E_k} / \sqrt{u(E_k)}$. Since

$$\int_X |u_i f_k|^2 \, d\mu = \int_X |u_i|^2 |f_k|^2 \, d\mu < \frac{1}{k^2}, \qquad f_k \in \mathfrak{D}.$$

Also,

$$\begin{aligned}\sum_{i=1}^n \|T_i f_k\|^2 = \sum_{i=1}^n \|u_i f_k\|^2 &= \sum_{i=1}^n \int_X |u_i|^2 |f_k|^2 \, d\mu \\ &= \sum_{i=1}^n \frac{1}{\mu(E_k)} \int_{E_k} |u_i|^2 \, d\mu \\ &\leqslant \frac{1}{\mu(E_k)} \frac{n}{k^2} \mu(E_k) \to 0 \quad \text{as } k \to \infty,\end{aligned}$$

proving that $(0, \ldots, 0) \in \mathrm{Sp}_l(T)$. Thus $\Phi(u) \subset \mathrm{Sp}_l(T)$, hence $\mathrm{Sp}(T) = \mathrm{Sp}_A(T)$.

If f is a Borel function on $\mathrm{Sp}(T)$, then $f \circ u \in m(X)$. Now a resolution of identity E on $\mathrm{Sp}(T)$ is defined as follows: For each Borel subset w of $\mathrm{Sp}(T)$, define $E(w) = \Phi(\chi_w \circ u)$. It is easy to verify that

(i) $E(\varnothing) = 0$, $E(\mathrm{Sp}(T)) = I$.

(ii) $E(w_1 \cap w_2) = E(w_1)E(w_2)$ for all Borel subsets w_1, w_2 of $\mathrm{Sp}(T)$.

(iii) $E(w_1 \cup w_2) = E(w_1) + E(w_2)$ whenever w_1, w_2 are disjoint Borel subsets of $\mathrm{Sp}(T)$.

It only remains to show:

(iv) For each $\zeta, \eta \in H$, $w \to (E(w)\zeta, \eta)$ is a complex Borel measure on $\mathrm{Sp}(T)$, or equivalently, for each $\zeta \in H$, $w \to (E(w)\zeta, \zeta)$ is a complex Borel measure.

Let $\zeta \in H$. Define ψ_ζ on $L^\infty(X)$ by $\psi_\zeta(f) = (\Phi(f)\zeta, \zeta)$. Then ψ_ζ is a positive linear functional and hence norm continuous on $L^\infty(X)$. Therefore there is a finitely additive measure $\mu_{\zeta,\zeta}$ on X such that

$$\psi_\zeta(f) = (\Phi(f)\zeta, \zeta) = \int_X f\, d\mu_{\zeta,\zeta} \qquad (f \in L^\infty(X)), \tag{1}$$

[9, page 357]. In fact, ψ_ζ is weak* continuous on $L^\infty(X)$. To see this, let $\{g_\alpha\}$ be a net in $L^\infty(X)$ converging to g in weak* topology. Then by continuity of Φ, $\Phi(g_\alpha)$ converges to $\Phi(g)$ in σ-weak topology and so $\Phi(f_\alpha)$ converges to $\Phi(g)$ weakly. In particular, $\psi_\zeta(g_\alpha) \to \psi_\zeta(g)$, which proves the continuity of ψ_ζ in weak* topology. It follows that $\mu_{\zeta,\zeta}$ is a countably additive measure on X. Thus

$$\begin{aligned} w \to (E(w)\zeta, \zeta) &= (\Phi(\chi_w \circ u)\zeta, \zeta) \\ &= \int_X \chi_w \circ u\, d\mu_{\zeta,\zeta} = \int_{\mathrm{Sp}(T)} \chi_w\, d\mu^u_{\zeta,\zeta}, \end{aligned}$$

where $\mu^u_{\zeta,\zeta}(w) = \mu_{\zeta,\zeta}(u^{-1}(w))$, defines a Borel measure on $\mathrm{Sp}(T)$. Hence E is a resolution of identity on Borel subsets of $\mathrm{Sp}(T)$, and by (1)

$$(\Phi(f \circ u)\zeta, \zeta) = \int_X f \circ u\, d\mu_{\zeta,\zeta} = \int_{\mathrm{Sp}(T)} f\, d\mu^u_{\zeta,\zeta} = \int_{\mathrm{Sp}(T)} f\, dE_{\zeta,\zeta},$$

where $E_{\zeta,\zeta}(w) = \mu^u_{\zeta,\zeta}(w)$ for all $\zeta \in H$ and for all bounded Borel functions f on $\mathrm{Sp}(T)$. Let f be a nonnegative Borel function on $\mathrm{Sp}(T)$. Then, as is well known, for some sequence $\{f_k\}$ of nonnegative simple functions on $\mathrm{Sp}(T)$, $f_k \nearrow f$. Then, since $\mathfrak{S}(u) = \mathrm{Sp}(T)$, $f_k \circ u \nearrow f \circ u$ on X and hence $\sup_k f_k \circ u = f \circ u$.

Let $\zeta \in H$ for which $\int_{\mathrm{Sp}(T)} |f|^2\, dE_{\zeta,\zeta} < \infty$. With this ζ,

$$\begin{aligned} \int_{\mathrm{Sp}(T)} f\, dE_{\zeta,\zeta} &= \int_X f \circ u\, d\mu_{\zeta,\zeta} = \int_X \sup_k f_k \circ u\, d\mu_{\zeta,\zeta} \\ &= \sup_k \int_X f_k \circ u\, d\mu_{\zeta,\zeta} = \sup_k (\Phi(f_k \circ u)\zeta, \zeta). \end{aligned}$$

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By [13, Theorem 3.5], $\sup_k \Phi(f_k \circ u)$ exists and it is in $m_l(A)^+$. Let $B = \sup_k \Phi(f_k \circ u)$. Then $B \leqslant \Phi(f \circ u)$. If $B \neq \Phi(f \circ u)$, there exists $g \in m(X)$ such that $B = \Phi(g)$; $g \leqslant f \circ u$ and $g \neq f \circ u$. Since $\Phi(g) \geqslant \Phi(f_k \circ u)$, $g \geqslant f_k \circ u$ which contradicts $f \circ u = \sup_k f_k \circ u$. Thus $B = \Phi(f \circ u)$. Therefore

$$\int_{\mathrm{Sp}(T)} f \, dE_{\zeta,\zeta} = (\Phi(f \circ u)\zeta, \zeta).$$

Hence $\int_{\mathrm{Sp}(T)} f \, dE \subset \Phi(f \circ u)$. Both being normal, their maximal normality gives

$$\int_{\mathrm{Sp}(T)} f \, dE = \Phi(f \circ u).$$

3. Concluding remarks

(a) From [10, Theorem 13.24] it is easily seen that our joint spectral theorem holds for $T = (T_1, \ldots, T_n)$ if, and only if, all the operators in the n-tuple T are doubly commuting.

(b) Also our technique yields a new proof with much algebraic flavour of the classical spectral theorem for unbounded normal operator.

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