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A NOTE ON A COMPLETE SOLUTION OF A PROBLEM POSED BY K. MAHLER

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Abstract

Let $\rho \in (0, \infty]$ be a real number. In this short note, we extend a recent result of Marques and Ramirez ['On exceptional sets: the solution of a problem posed by K. Mahler', *Bull. Aust. Math. Soc.* **94** (2016), 15–19] by proving that any subset of $\overline{\mathbb{Q}} \cap B(0, \rho)$, which is closed under complex conjugation and contains 0, is the exceptional set of uncountably many analytic transcendental functions with rational coefficients and radius of convergence ρ. This solves the question posed by K. Mahler completely.

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1. Introduction

Denote by $\overline{\mathbb{Q}}$ the field of algebraic numbers. For a function $f : \Omega \to \mathbb{C}$, we define the exceptional set S_f of f as

$$S_f = \{\alpha \in \overline{\mathbb{Q}} \cap \Omega : f(\alpha) \in \overline{\mathbb{Q}}\}.$$

In his book [2], Mahler introduced the basic problem of the theory of transcendental numbers as the study of the set S_f for various classes of functions. After discussing a number of examples, Mahler posed several problems about the admissible exceptional sets for analytic functions, one of which is the following question.

Question 1.1. Let $\rho \in (0, \infty]$ be a real number. Does there exist for any choice of $S \subseteq \overline{\mathbb{Q}} \cap B(0, \rho)$ (closed under complex conjugation and such that $0 \in S$) an analytic transcendental function $f \in \mathbb{Q}[[z]]$, with radius of convergence ρ, for which $S_f = S$?

In fact, Mahler [1] had already proved that if S is a subset of the algebraic numbers in the open unit disc $B(0, \rho)$, with the property that S contains 0 and is invariant under the full absolute Galois group of $\overline{\mathbb{Q}}$, then $S = S_f$ for some transcendental holomorphic function on $B(0, \rho)$. We refer the reader to [2, 4] (and references therein) for more results about the arithmetic behaviour of transcendental functions.

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In 2016, Marques and Ramirez [3] showed that the answer to Question 1.1 is positive when $\rho = \infty$. In this short note, we give an affirmative answer to Mahler's question for any ρ. More precisely, we prove the following theorem.

THEOREM 1.2. *Let $\rho \in (0, \infty]$ be a real number. Then any subset of $\overline{\mathbb{Q}} \cap B(0, \rho)$, closed under complex conjugation and containing 0, is the exceptional set of uncountably many analytic transcendental functions $f \in \mathbb{Q}[[z]]$, with radius of convergence ρ.*

In order to prove Theorem 1.2, we shall prove the following stronger version of [3, Theorem 1.3].

THEOREM 1.3. *Let $\rho \in (0, \infty]$ be a real number and let $\mathbb{K}$ be a dense subset of $\mathbb{C}$. Let A be a countable subset of $B(0, \rho)$. For each $\alpha \in A$, fix a dense subset $E_\alpha \subseteq \mathbb{C}$ (such that if $0 \in A$, then $1 \in E_0 \cap \mathbb{K}$). Then there exist uncountably many analytic transcendental functions $f \in \mathbb{K}[[z]]$, with radius of convergence ρ, such that $f(\alpha) \in E_\alpha$ for all $\alpha \in A$.*

2. The proof that Theorem 1.3 implies Theorem 1.2

The proof that Theorem 1.3 implies Theorem 1.2 is similar to that in [3, page 17]. However, we shall repeat it here for the convenience of the reader.

In the statement of Theorem 1.3, choose $A = \overline{\mathbb{Q}} \cap B(0, \rho)$ and $\mathbb{K} = \mathbb{Q}^* + i\mathbb{Q}$. Write $S = \{\alpha_1, \alpha_2, \ldots\}$ and $\overline{\mathbb{Q}} \cap B(0, \rho) \backslash S = \{\beta_1, \beta_2, \ldots\}$ (one of these sets may be finite). Now define

$$E_\alpha = \begin{cases} \overline{\mathbb{Q}} & \text{if } \alpha \in S, \\ \mathbb{K} \cdot \pi^n & \text{if } \alpha = \beta_n. \end{cases}$$

Then, by Theorem 1.3, there exist uncountably many transcendental functions $f(z) = \sum_{k \geq 0} a_k z^k \in \mathbb{K}[[z]]$ with radius of convergence ρ and such that $f(\alpha) \in E_\alpha$ for all $\alpha \in \overline{\mathbb{Q}} \cap B(0, \rho)$. For any such f, we define the function $\psi = \psi_f$ by

$$\psi(z) = \frac{f(z) + \overline{f(\overline{z})}}{2}.$$

Note that $\psi(z) = \sum_{k \geq 0} \Re(a_k) z^k$ has rational coefficients and moreover there are uncountably many of these functions. Since the set of algebraic functions with rational coefficients and positive radius of convergence is countable, then uncountably many of the ψ_f must be transcendental. Clearly, their radius of convergence is ρ. To finish, it suffices to prove that $S_\psi = S$. In fact, if $\alpha \in S$, then $\overline{\alpha} \in S$ and thus $f(\alpha)$ and $f(\overline{\alpha})$ are algebraic numbers and so is $\psi(\alpha)$. In the case of $\alpha = \beta_n$, we must distinguish two cases: when $\beta_n \in \mathbb{R}$, then $\psi(\alpha) = \Re(f(\beta_n))$ is transcendental, since $f(\beta_n) \in \mathbb{K} \cdot \pi^n$. When $\beta_n \notin \mathbb{R}$, then $\overline{\beta_n} = \beta_m$ for some $m \neq n$. Thus, there exist nonzero algebraic numbers γ_1, γ_2 such that

$$\psi(\beta_n) = \frac{\gamma_1 \pi^n + \gamma_2 \pi^m}{2},$$

which is transcendental, since $\overline{\mathbb{Q}}$ is algebraically closed and π is transcendental. This concludes the proof. □

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3. The proof of Theorem 1.3

Consider $g(z) = 1/(1 - z/\rho) = 1 + z/\rho + z^2/\rho^2 + \cdots$ (throughout the paper, when $\rho = \infty$, we consider $1/\rho$ as 0). Write $A = \{\alpha_1, \alpha_2, \ldots\}$. We shall proceed by induction to construct functions $\tilde{f}_n(z)$ such that the first $n+1$ coefficients of $\tilde{f}_n$ belong to $\mathbb{K}$ and $\tilde{f}_n(\alpha_i) \in E_{\alpha_i}$ for $i = 1, \ldots, n+1$.

Suppose, without loss of generality, that $0 \in A$, say $\alpha_1 = 0$. In this case, by hypothesis, $g(0) \in E_{\alpha_1}$. Set $f_1(z) = g(z) + \delta_1 z$, where $\delta_1 \in B(0,1) \setminus \{0\}$ and $1/\rho + \delta_1 \in \mathbb{K}$ (this means that the second coefficient of $f_1(z)$ belongs to $\mathbb{K}$). Now set $\tilde{f}_1(z) = f_1(z) + \epsilon_1 z^2$, where we choose $\epsilon_1 \in B(0,1) \setminus \{0\}$ such that $\tilde{f}_1(\alpha_2) = f_1(\alpha_2) + \epsilon_1 \alpha_2^2 \in E_{\alpha_2}$ (by the density of E_{α_2}).

We proceed by induction. Suppose that

$$\tilde{f}_{n-1}(z) = g(z) + \tilde{P}_{n-1}(z) = \sum_{k \geq 0} c_{n-1,k} z^k$$

has been constructed such that $\tilde{P}_{n-1}(z) \in \mathbb{C}[z]$, $c_{n-1,k} \in \mathbb{K}$ for $k = 0, \ldots, n-1$ and $\tilde{f}_{n-1}(\alpha_j) \in E_{\alpha_j}$ for $j = 1, \ldots, n$. Set

$$f_n(z) = \tilde{f}_{n-1}(z) + \delta_n z^n \prod_{k=2}^{n} (z - \alpha_k),$$

where we choose $\delta_n \in B(0, 1/n^n) \setminus \{0\}$ such that $c_{n-1,n} + (-1)^n \delta_n \alpha_2 \cdots \alpha_n \in \mathbb{K}$. Now we define

$$\tilde{f}_n(z) = f_n(z) + \epsilon_n z^{n+1} \prod_{k=2}^{n} (z - \alpha_k)$$

and we choose $\epsilon_n \in B(0, 1/n^n) \setminus \{0\}$ such that $\tilde{f}_n(\alpha_{n+1}) \in E_{\alpha_{n+1}}$.

In conclusion, we have constructed $f(z) = g(z) + h(z) \in \mathbb{K}[[z]]$, where $h(z) = \lim_{n\to\infty} \tilde{P}_n(z)$. By the choice of the ϵ and the δ as nonzero real numbers in very small balls, h is a nonpolynomial entire function. Hence, h is a transcendental function and so is f. Also, $f(\alpha) \in E_{\alpha_i}$ for all $\alpha \in A$. Clearly, f is an analytic function with radius of convergence ρ. Also, there is an ∞-ary tree of different possibilities for f because in each step we have infinitely many possible choices for ϵ and δ. Thus, we have constructed uncountably many possible functions. The proof is complete. □

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