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ON ASPECTS OF NUMERICAL ERGODIC THEORY: STABILITY OF ULAM'S METHOD, COMPUTING OSELEDETS SUBSPACES AND OPTIMAL MIXING

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A dynamical system is a pairing between a set of states $X \subset \mathbb{R}^d$ and a map $T : X \circlearrowleft$ which describes how the system evolves from state to state over time. The Perron–Frobenius, or transfer, operator is a natural extension of the point-by-point dynamics defined by T to an ensemble theory which describes the evolution of *distributions* of points. It features heavily in dynamical systems theory and in a wide array of numerical methods including the approximation of invariant densities, physical measures, almost-invariant partitionings, coherent structures, Lyapunov exponents and topological entropy.

This thesis focuses on numerical methods that take advantage of the Perron–Frobenius operator and its statistical representation of dynamical systems. Central to these is Ulam's method—the most commonly used scheme to produce a finite-rank approximation of the Perron–Frobenius operator. Of interest are the circumstances in which Ulam's method fails to produce an approximation whose spectral values converge to the spectral values of the Perron–Frobenius operator. We discuss a simple advection–diffusion dynamical system whose eigenvalues exhibit unusual behaviour. The eigenvalues of the Perron–Frobenius operator acting on L^2 functions depend on the diffusion parameters, while the eigenfunctions do not. However, if the eigenfunctions are ordered by the magnitudes of their corresponding eigenvalues, the same eigenfunctions appear in a different order depending on the diffusion parameters. Finally, as Ulam's method has its own diffusive effect, the same effect is observed by keeping the diffusion parameters constant, but altering the shape of the partition elements used in Ulam's method.

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For nonautonomous systems—those whose dynamics also depend on time—the map T is replaced by a series of maps $\{T_n\}$ for $n \in \mathbb{Z}$, or even a collection of maps $\{T_\omega\}$, with ω belonging to a measure space Ω. The Perron–Frobenius operator is replaced by a collection of Perron–Frobenius operators $\{\mathcal{P}_n\}$, or $\{\mathcal{P}_\omega\}$, and their Ulam approximations. If the ω are generated by an invertible, ergodic process the d-dimensional Ulam approximation matrices satisfy the multiplicative ergodic theorem, which guarantees the existence of expansion rates called Lyapunov exponents and a splitting of $\mathbb{R}^d$ into so-called Oseledets subspaces. The Lyapunov exponents capture the limiting, exponential growth of vectors in $\mathbb{R}^d$ over time and the Oseledets subspaces the directions of expansion. Four numerical methods were recently developed which approximate the Oseledets subspaces given a finite sequence of data. We provide an improvement on one and, using three example dynamical systems, compare each method on precision, accuracy and speed of convergence.

Finally, we approach the problem of increasing the rate at which a dynamical system mixes by applying local perturbations. We introduce a quadratic, convex optimisation problem based around the transfer operator which can be efficiently solved using standard convex optimisation techniques. We compare the solution of the proposed optimisation problem with other fixed diffusion protocols such as uniform or Gaussian diffusion, and demonstrate significant speed-ups. We also indicate how one might alternatively use local perturbations to push a mass density toward a particular region of the domain.

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