−1

A Tour of Solutions

We start off with a tour of some of the solutions to the classical N-body problem, using the number N of bodies as an organizing parameter for our tour. The problem is posed in terms of a system of ODEs whose solutions consist of N synchronized curves in the plane or in space, the curves representing the motion of stars or planets interacting solely through classical gravitational attraction. Solutions can be drawn by indicating the trace of the curves. The reader may wish to consult some animations on the web for a more visceral feel.[1]

−1.1 The One- and Two-Body Problems

Newton posed and solved the gravitational two-body problem. Figure −1.1 shows a solution to this problem when the two masses are equal. The general solutions are as follows. The center of mass of the two bodies moves in a straight line at constant speed. Each body moves around that center along a conic section according to an ODE for a single body known nowadays as Kepler's problem. In this way, the two-body reduces to a one-body problem.

The Kepler problem, our "one-body problem," is the limiting ODE we get from the two-body problem when one of the two masses tends to infinity while the other remains finite. Then the infinite mass, the "Sun," does not budge while the finite mass, the "planet," moves about the Sun according to Kepler's three laws. The first of Kepler's laws says that the planet moves in a conic section having the Sun as one focus. We discuss these laws and derive Kepler's problem from the two-body problem in Chapter 0.

[1] We often get a better sense of solutions through animations instead of stills, but this here is a book. Please see https://peopleweb.prd.web.aws.ucsc.edu/~rmont/Nbdy/NbdyC1.html? for animations.

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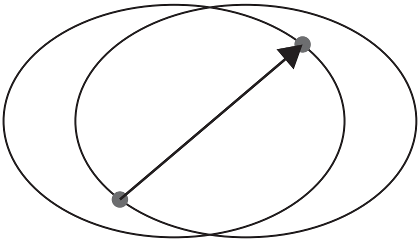

Figure −1.1 A solution to the equal mass two-body problem.

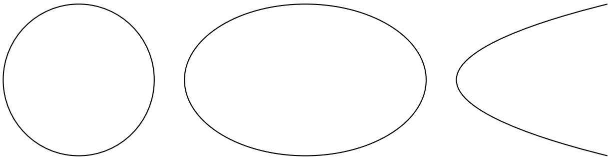

Figure −1.2 Some Kepler conics.

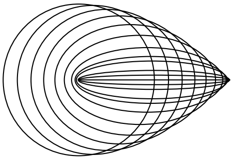

Figure −1.3 A family of constant negative energy Keplerian conics.

Some solutions to Kepler's problem are shown in Figure −1.2. The Kepler problem admits a conserved energy and an angular momentum. Its bounded solutions all have negative energy, the absolute value of which is proportional to the reciprocal of the ellipse's semi-major axis. Figure −1.3 shows a family of Keplerian ellipses having fixed negative energy. As the angular momentum tends to zero in the family, the ellipses degenerate to line segments colliding with the Sun, as indicated in the figure. These line segments are the collision solutions and play a central role in the N-body problem, most notably in Chapter 3. The periods of all these orbits are the same.

Returning to the two-body problem, the difference vector joining the two bodies satisfies Kepler's problem, as does the difference vector joining the center of mass to either body. Our standard Newtonian view of the solar system is as N planets moving around the Sun in bounded Keplerian orbits that are nearly circular. This model ignores the gravitational attraction between planets, which one treats later as a perturbation.

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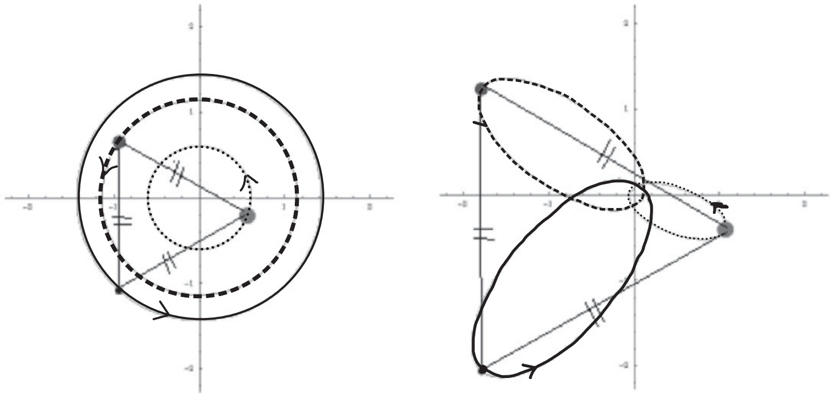

Figure −1.4 Lagrange's solution. Courtesy of Richard Moeckel. See www.scholarpedia.org/article/3-body_problem.

Stars are not infinitely massive. Our own Sun moves in a small Keplerian orbit about the center of mass of our solar system, a point buried inside the Sun. The discovery of the first exoplanets was based on detecting small periodic oscillations in the velocities of distant stars as they moved about the center of mass of their own solar systems. These velocity changes – known as "radial velocity dispersion" – can be seen in the Doppler shifts of the light frequencies detected.

−1.2 The Three-Body Problem

−1.2.1 The Central Solutions of Euler and Lagrange

We have closed-form analytic expressions for five families of solutions to the three-body problem collectively known as the central configuration solutions. Euler [54] found three of these families in 1767. Lagrange found the remaining two families in 1772 [99, 100]. In their solutions, the three bodies travel on three similar Kepler conics, their motion synchronized so that the shape of the triangle they form does not change. The three conics making up such a solution are scaled, rotated versions of each other, as shown in Figure −1.4.

Lagrange's Solution

In Lagrange's solution the bodies form an equilateral triangle at each instant. If we take the Kepler conic to be a circle, then this equilateral triangle rotates rigidly about the center of mass of the triangle. If we take a general Keplerian

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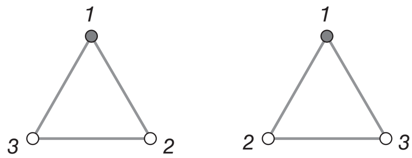

Figure −1.5 The two orientations of a labeled planar triangle.

conic, then each body moves in a Keplerian conic about the center of mass, with the motion of the conics synchronized so that at each instant they form an equilateral triangle. Included in the family is the degenerate zero-angular momentum solution where the bodies are dropped from rest in an equilateral configuration, in which case each body moves on a line segment ending in simultaneous triple collision at the center of mass. The Lagrange families exists for all mass ratios. We count the Lagrange solutions as two families of solutions since we distinguish between "right-handed" and "left-handed" planar labeled triangles. See Figure −1.5. The circular Lagrange solutions are linearly stable only when one of the three masses is much greater than the other two.

Euler's Solutions

In Euler's solutions the bodies are collinear at every instant, and the line they lie on is typically spinning. They comprise three families labeled according to which of the three bodies sits between the other two during the motion. Throughout the motion the ratio between the distances of the bodies remains constant. (That ratio is determined as the positive real solution to a fifth-order polynomial whose coefficients depend on the masses.) When the masses are all equal this middle mass must lie at at the midpoint of the segment formed by the other two. If we choose the circular solution to Kepler's equations, then the ends of the segment rotate as the ends of a diameter of a circle about this center. For an Euler solution when the masses are all distinct see Figure −1.6. The Euler solutions are linearly unstable for all mass distributions. (See Chapter 2 regarding linear stability and instability.)

−1.2.2 The Circular Restricted Limit

The Euler and Lagrange solutions persist if we let one of the three masses tend to zero while the other two are fixed. If we take the circular Kepler element of the central configuration family, then these limiting solutions become special solutions to the *circular restricted planar three-body problem*,

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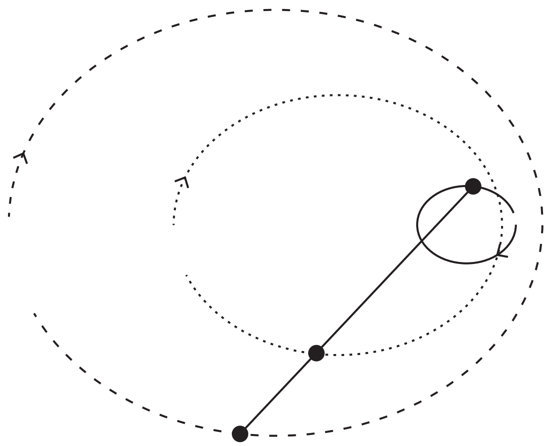

Figure −1.6 Euler's solution.

which is probably the most studied of all three-body problems. In this limiting version, the two big masses, or "primaries," rotate in a circular Keplerian orbit. Newton's equations, written in coordinates rotating at the angular frequency of the primaries, reduces to a second-order autonomous ODE in the rotating plane for the remaining infinitesimal mass. The corresponding ODE in the rotating frame has five fixed points corresponding to these five limiting central configurations. These fixed points are named L1–L5 and are indicated in Figure −1.7. (Somehow Euler's name disappeared in the labeling.)

−1.2.3 The Eight

At the other extreme of having one of the three masses close to zero, take all three masses to be equal. In what is called the "figure eight," or simply "eight," solution, three *equal* masses chase each other around a fixed figure of eight curve drawn (here laid on its side) on the plane. See Figure −1.8. You can generate the figure eight yourself by using a numerical integrator to integrate the three-body equations (0.1)–(0.2). Take $m_1 = m_2 = m_3 = G = 1$ in these equations. For inital positions take $q_1 = -q_2, q_3 = 0$, and for initial velocities take $\dot{q}_1 = \dot{q}_2, \dot{q}_3 = -2\dot{q}_1$ where

$$q_1 = (-0.97000436, 0.24308753), \dot{q}_3 = (0.93240737, 0.86473146).$$

I'm indebted to Carlés Simó for zeroing in on these initial conditions. See [192].

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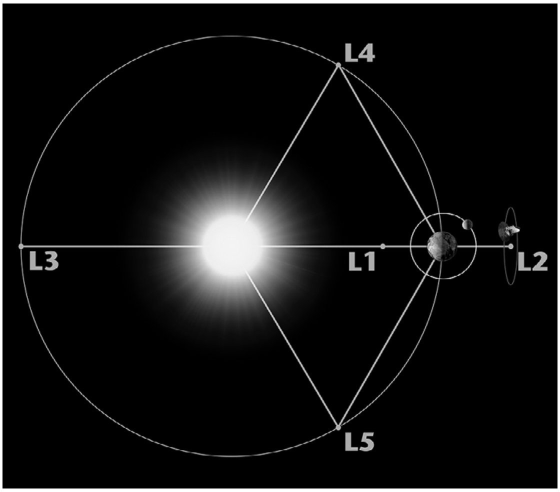

Figure −1.7 The central configurations marked within the circular restricted three-body problem. In this context, the central configurations are usually labeled L1, L2, L3, L4, and L5. Image credit: NASA/WMAP Science Team.

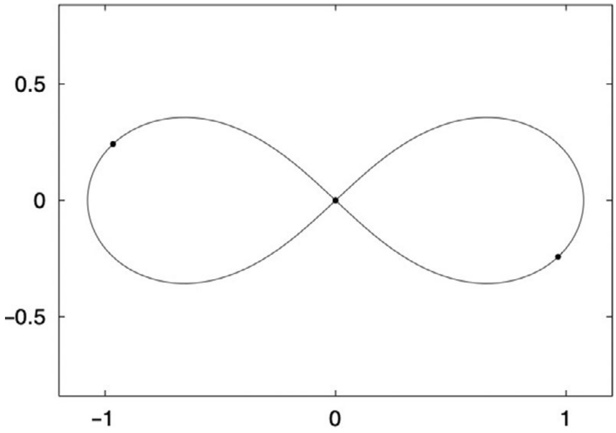

Figure −1.8 The figure eight, a periodic zero-angular momentum KAM stable solution to the three-body problem in which three equal masses chase each other around a figure eight–shaped curve in the plane. From Chenciner et al. [35].

The figure eight makes guest appearances in Chapter 2 on stability and Chapter 3 on braids. Unexpectedly, the eight is about as stable as one can hope for a periodic solution in celestial mechanics. Figure −1.9 depicts a perturbed eight, a solution resulting from near-eight initial conditions. We can imagine this orbit as being "nestled between" two KAM 2-tori near the eight.

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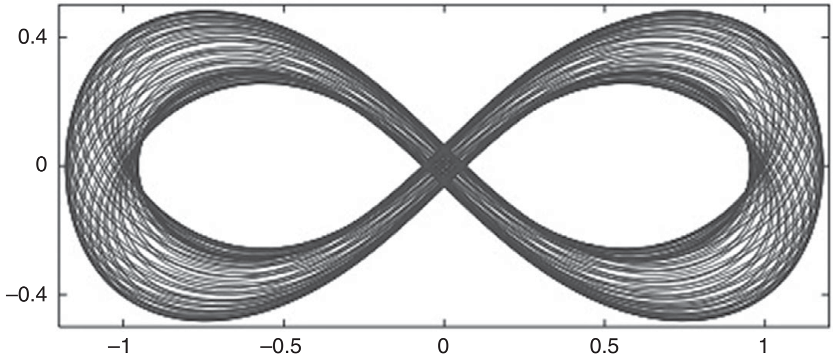

Figure −1.9 A near-eight. From Simó [192].

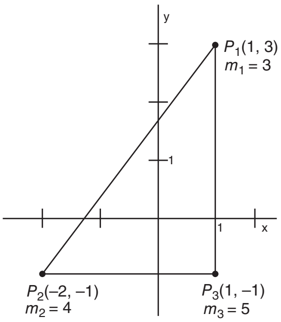

Figure −1.10 Initial conditions for the Pythagorean three-body problem. Reprinted from Szebehely and Peters [204] ©AAS. Reproduced with permission.

−1.2.4 Escape and Scattering

Drop three bodies. In other words, let them go from rest. What can happen? A special case of this problem was investigated for nearly a century in a long and convoluted series of papers for a specific case known as the Pythagorean three-body problem, and which became a kind of benchmark problem for N-body integrators. For initial conditions place three masses at the vertices of a triangle whose edge lengths are in the ratio 3:4:5. Choose the masses to have the same ratio, and place mass 5 opposite the edge of length 5, and so on. See Figure −1.10. Now drop the bodies: Take all velocities to be zero, and numerically integrate. What happens? See Figure −1.11.

Now throw a binary system at a distant isolated third body. Hut and collaborators [85] did this numerically in order to understand questions

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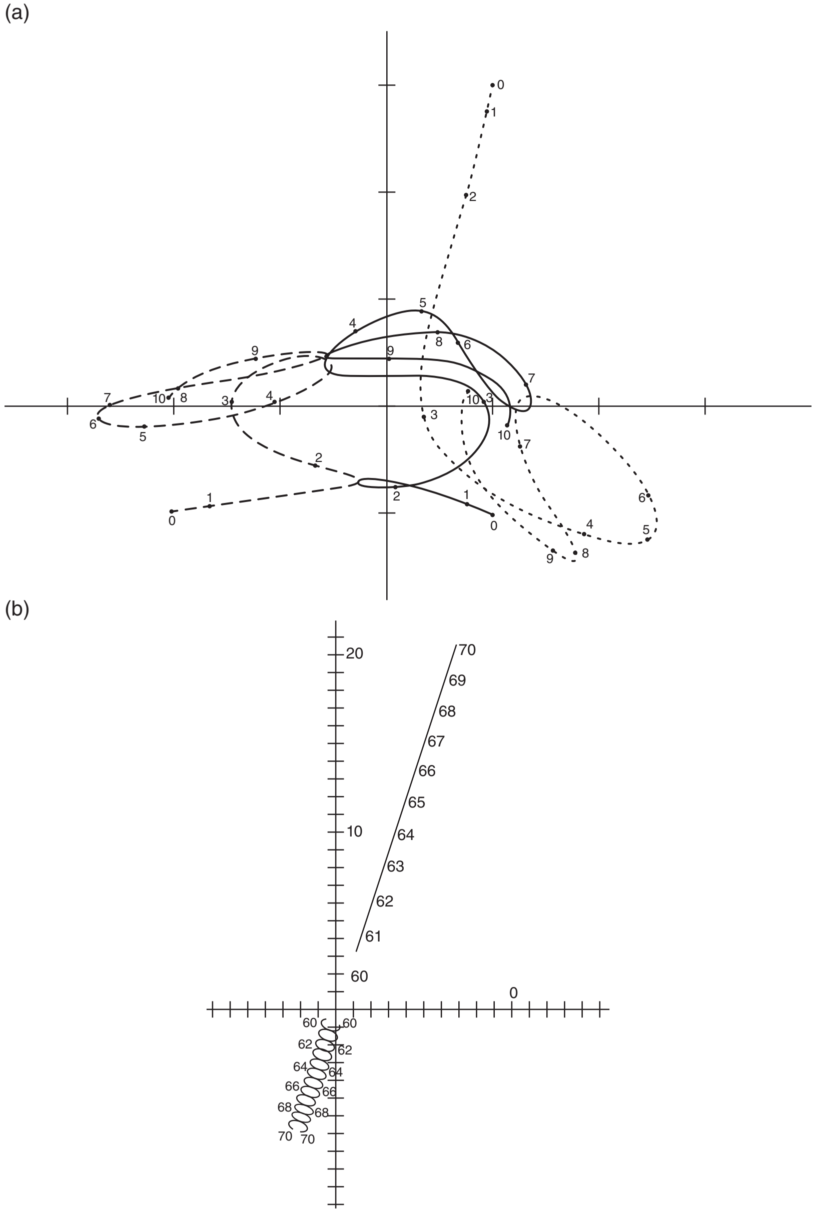

Figure −1.11 The Pythagorean three-body problem. (a) The solution from being dropped at $t = 0$ to time $t = 10$ with one body's curve depicted as a solid line, one as a dashed curve, and the other as a dotted curve. (b) The motion from $t = 50$ to 70 is depicted. We see that two of the masses form a tight binary and escape to infinity. Reprinted from Szebehely and Peters [204] ©AAS. Reproduced with permission.

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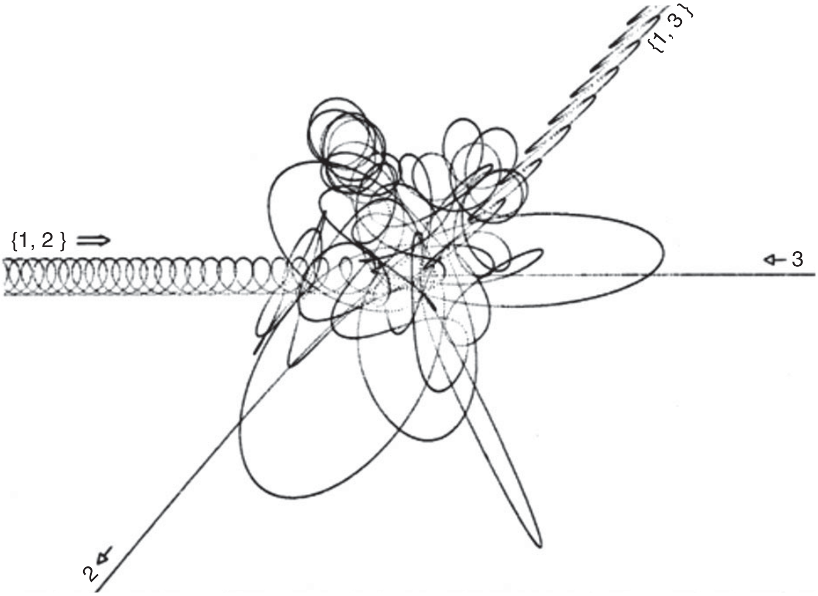

Figure −1.12 Chaotic interchange. Reprinted from Hut & Bahcall [85] ©AAS. Reproduced with permission.

regarding the prevalence of binary stars in galaxies and their effects on galactic evolution. A sample solution is depicted in Figure −1.12.

−1.2.5 Schubart and Broucke–Henon

In 1956, Schubart [188] discovered the periodic solution to the equal mass collinear three-body problem sketched in Figure −1.13. The middle body shuttles back and forth between the two on the extremes, colliding with each once per period. These collisions are regularized according to Levi-Civita, a transformation of space–time that analytically extends the planar N-body flow through isolated binary collisions. See Appendix G. Schubart's orbit is stable within the context of the collinear problem and plays a central role in the phase portrait of the negative energy equal mass collinear three-body problem. See [205, 206].

In 1974, Broucke [22] found many new orbits for the three-body problem. See the 3rd and 4th orbits in Figure −1.14. Henon [80] connected the Schubart and Broucke orbits through an analytic family of relative periodic orbits by using the angular momentum as a bifurcation parameter. In the process, Henon discovered orbits now called the Broucke–Henon orbits.

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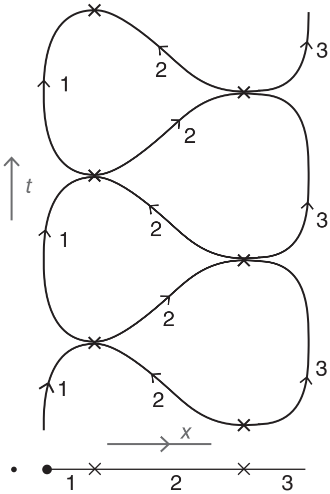

Figure −1.13 Space–time diagram of Schubart's collinear equal mass three-body solution. The x's mark collisions.

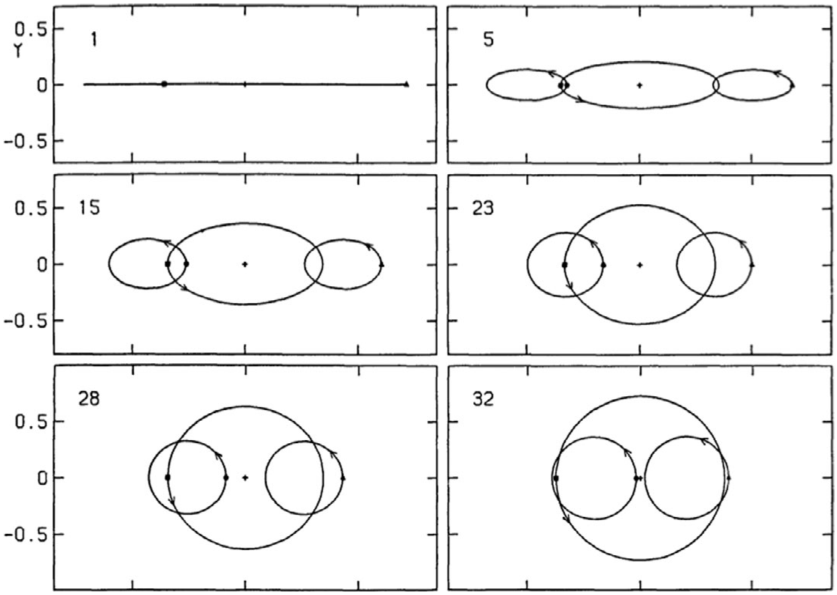

Figure −1.14 Broucke and Henon's continuation of the Schubart orbit (top left). The orbits are shown in a rotating frame with respect to which the orbit is periodic. For the Schubart collinear orbit the frame does not rotate. From [80].

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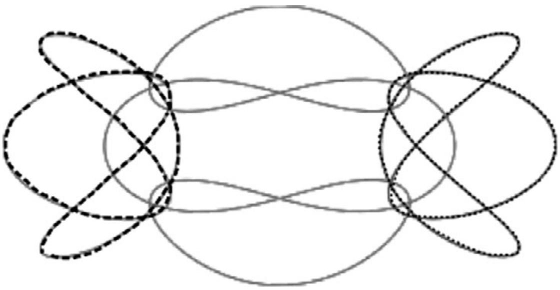

Figure −1.15 Another equal mass zero angular momentum solution, one of hundreds found by Danya Rose.

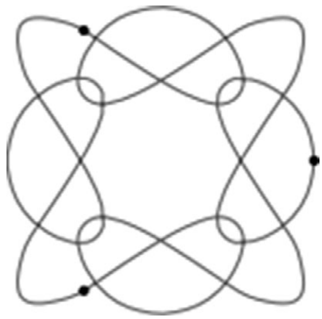

Figure −1.16 The three-body "Celtic Knot" choreography denoted $D(3,4)$ by Montaldi and Steckles [143].

−1.2.6 A Bestiary

In 2015, Danya Rose [177] combined simultaneous regularization of all binary collisions, shape space thinking, and careful numerical integrations to catalogue hundreds of equal-mass zero angular momentum solutions to the three-body problem. Figure −1.15 is an unstable orbit that Rose christened F1.2.7.

In 2013, James Montaldi and Katrina Steckles [143] compiled an artistic bestiary of a different nature, based on equivariant homotopy ideas combined with the symmetry ideas that led to the figure eight. Figure −1.16 shows a representative orbit of theirs whose existence has not yet been rigorously established.

−1.3 The Restricted Three-Body Problem

When the value of one of the masses tends to zero in the three-body equations (see Equations (0.1) at the beginning of Chapter 0) the limiting equations

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decouple. The two nonzero masses – called "the primaries" – move according to the two-body problem. So the primaries move in Keplerian orbits about their common center of mass, unaffected by the third infinitesimal mass. The infinitesimal mass feels the changing gravitational pull of the primaries. How does it move? This is the restricted three-body problem.

The restricted three-body problem is a singular limit of the full three-body problem. It is also, historically speaking, the most studied three-body problem. Satellite and space mission planning problems are treated as restricted three-body problems, with perturbations added as needed. The Earth–Moon–Sun system is also typically treated as a restricted three-body problem,[2] the ratios of masses Sun : Earth : Moon being of the order of $1 : 10^{-6} : 10^{-8}$. Much of Poincaré's work on the three-body problem and, in particular, his three-volume "Methodes Nouvelles de Mecanique Celeste" [169] is aimed at the restricted three-body problem.

The restricted three-body problem is really a family of problems parameterized by the choice of conic section for the Keplerian orbit of the primaries, the mass ratios of the primaries, the residual energy of the third body (called the Jacobi constant), and whether or not the tiny body moves in the plane of the primaries or is allowed to wander in space. The most studied problems within this family of problems are those where the orbits of the primaries are circles. This sub-problem is called the circular restricted planar three-body problem By going to a rotating frame, rotating at the rotation rate of the primaries, this problem becomes a nonautonomous two-degree of freedom system, while the full planar three-body problem, after reduction by symmetries, is three-degree of freedom. We saw solutions to this problem arising from central solutions earlier. See Figure −1.7. In the rotating frame these central configuration solutions of Euler and Lagrange became equilibria. Another solution to the circular restricted planar three-body problem is represented in Figure −1.17.

In the early 1930s, Strömgren organized a group of human computers to survey some periodic solutions for the planar circular restricted three-body problem in the case where two masses are equal [200]. The remarkable variety of orbits he found are summarized in Figure −1.18. The variety, complicated character, and implications of the orbits he depicted have not been explored.

The Sitnikov problem (Figure −1.19) concerns a perturbed circular restricted *spatial* three-body problem, again with the two primaries having equal mass. The infinitesimal mass moves on the line orthogonal to the plane of motion of the primaries, passing through their center of mass. By symmetry,

[2] The Earth–Moon–Sun system turns out to be especially challenging since, due to the distances involved, the magnitudes of the force exerted on the Moon by the Sun and by the Earth are of roughly the same order.

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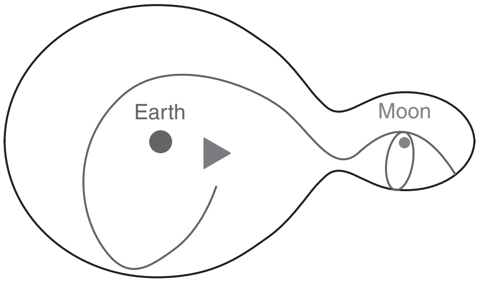

Figure −1.17 Transit orbit. The primaries, Earth and Moon, move in circular orbits and are viewed in a frame rotating with them so they become stationary. The infinitesimal mass, which we may think of as a moon mission, transits back and forth. Courtesy of Richard Moeckel www.scholarpedia.org/article/3-body_problem [137].

the tiny mass stays on this line, and this remains true if the conics on which the primaries move are not circular.

A good chunk of the book by Moser [159] is dedicated to the Sitnikov problem. If the primaries move in circular orbits, then the force on the infinitesimal is not time-dependent, and the problem becomes a one-degree-of-freedom autonomous Hamiltonian system, and hence is integrable. Let them move in eccentric orbits and the problem becomes a nonautonomous time-periodic one-degree-of-freedom system. Moser uses the eccentricity of the ellipse as a perturbation parameter away from the integrable case. He uses analysis at infinity and a study of Smale horseshoes to prove the following. Let time be counted by full revolutions of the primary once around their center of mass. For negative energies (Jacobi constant) and for most solutions, the infinitesimal mass crosses the plane of motion of the primaries over and over. Every time it crosses, mark down the integer number of years passed since the last crossing. Then there is a (large) integer N_0 such that every infinite sequence $n_{-1}n_0n_1\cdots$ with $n_i \geq N_0$ is realized: There is some solution that ticks off this sequence in its travels.

It was within the context of the circular restricted three-body problem that Poincaré established his famous non-integrability results: that the three-body problem is not an integrable system, as that term is meant nowadays in Hamiltonian dynamics. He did so by establishing what would later be called “homoclinic tangles,” which imply the presence of “Hamiltonian chaos.” Often, Poincaré took as the perturbation parameter the ratio of masses between the primaries, with the case where one is infinitely massive relative to the other limiting the problem to an integrable problem: two uncoupled Kepler problems.

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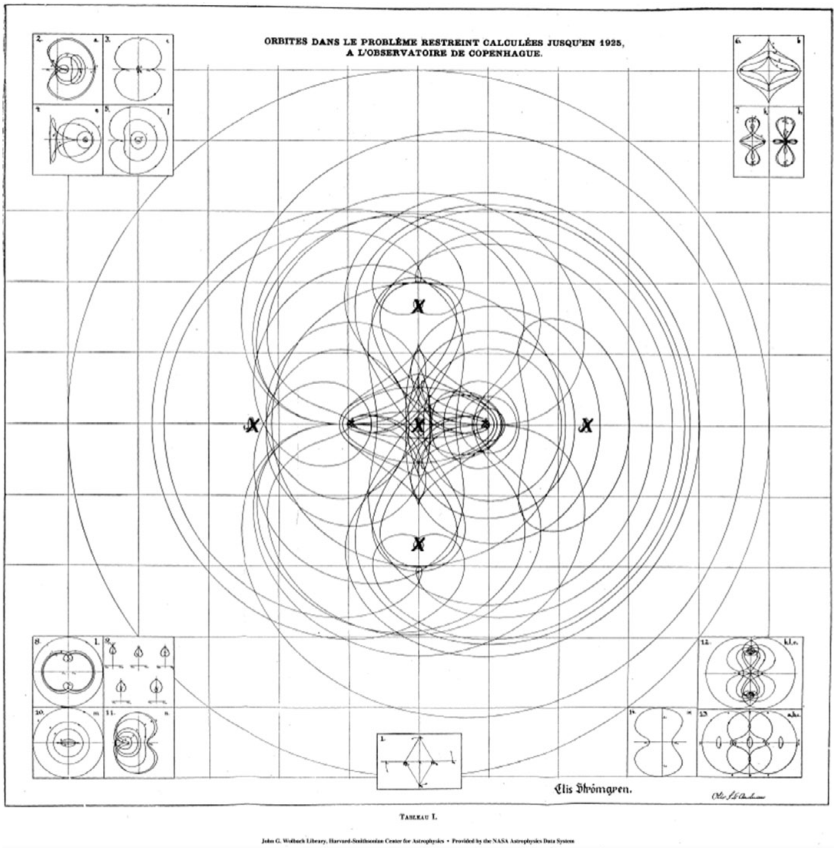

Figure −1.18 A few dozen periodic orbits for the planar restricted circular three-body problem in the case where the two primaries have equal mass, viewed in the rotating frame. The filled-in gray circles are centered on the primaries, the X's indicate the central configurations of Euler and Lagrange. Strömgren [200] found these orbits in 1933 through numerical integration before digital computers were available.

−1.4 The Four-Body Problem

−1.4.1 Central Configurations

Solutions such as the Euler and Lagrange three-body solutions in which the N bodies maintain their shape throughout their motion are called central configuration solutions. The shapes themselves are referred to as "central configurations." For the equal-mass planar four-body problem there are precisely four possible shapes. See Figure −1.20(a). When the vertices of the

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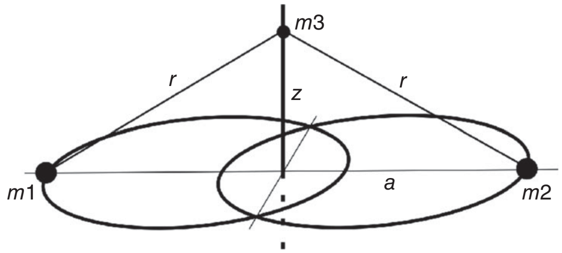

Figure −1.19 The Sitnikov problem.

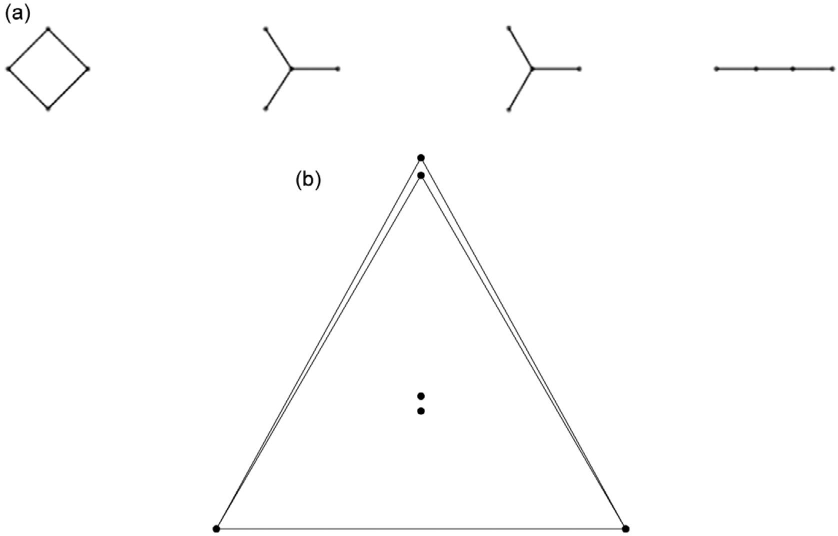

Figure −1.20 Four-body central configurations for equal masses. (a) Two of the four shapes look the same. One of these two is an equilateral triangle with the fourth body placed at the center of mass. The other is isosceles but not equilateral. (b) We have enlarged the two identical-looking configurations and super-imposed them after a rotation to show that they are actually different. Reprinted from Hampton [74].

4-gons shapes are labeled they yield 50 distinct labeled shapes modulo scaling and rotation, and hence 50 families of solutions, the four-body analogues of the Euler and Lagrange three-body solutions. Counting the number of central configurations for $N \geq 4$ is the subject of Question 1 of the book (see Chapter 1). A simple four-body central configuration solution to visualize is a rotating square for the equal-mass four-body problem (Figure −1.20(a)).

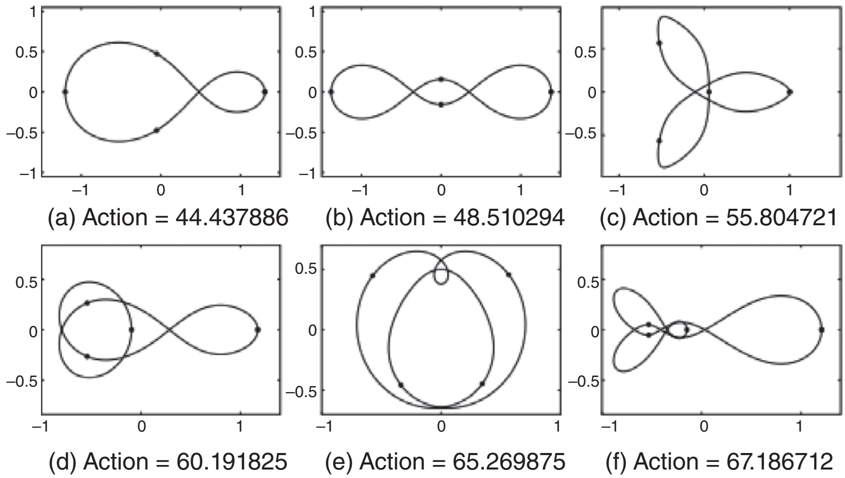

Figure −1.21 Four-body choreographies. Panel b) is known as the Gerver super eight and was discovered by Joseph Gerver at a conference during the last week of the twentieth century. From Chenciner et al. [35].

−1.4.2 Dancing Quadrilaterals

A vast array of equal-mass periodic orbits with high symmetry have been discovered in the last 25 years by combining symmetry and variational methods. Figure −1.21 illustrates some four-body choreographies. A *choreography* is an N-body solution for which all N masses travel the same curve in d-space. Figure −1.22 depicts the *hip-hop* solution, which enjoys the symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_4$.

−1.4.3 A Non-collision Singularity

A solution has a singularity if there is a finite time t_* beyond which it cannot be continued as a solution. Along a singular solution the lim inf of at least one pair r_{ij} of relative distances must tend to zero. The singularity is called a collision singularity provided all the bodies have limiting positions as t_* is approached, in which case at least two of the bodies have collided. Otherwise, the solution is called a non-collision singularity. In 2022, Joseph Gerver, Guan Huang, and Jinxin Xue [67] described solutions with non-collision singularities for the four-body problem. We give a cartoon of their solutions in Figure −1.23.

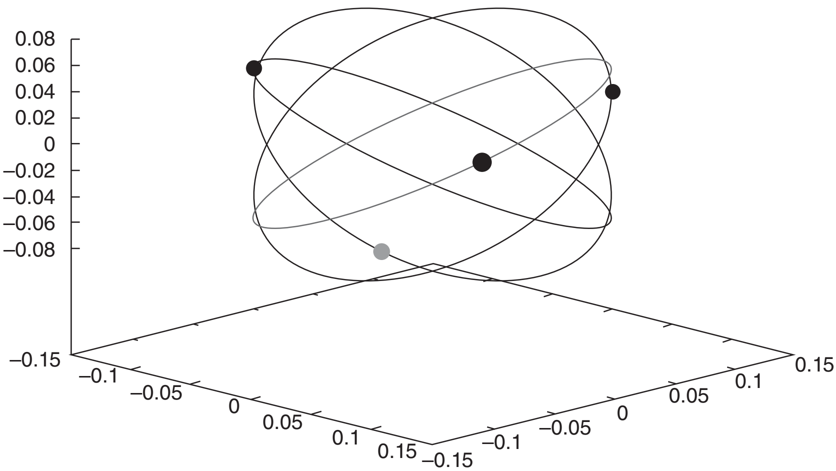

Figure −1.22 The hip-hop solution discovered by Chenciner and Venturelli [34] has four equal masses oscillating between the square and tetrahedral configurations and enjoys the symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_4$. Courtesy of Davide Ferrario.

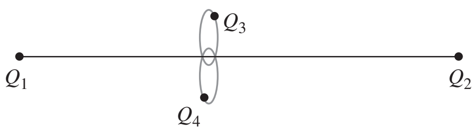

Figure −1.23 Cartoon of non-collision singularity solutions with four bodies found by [67]. None of the bodies have finite limiting positions as $t \to t_*$, the singularity time. $Q_3(t)$ and $Q_4(t)$ accumulate onto the entire x axis with the distance between them going to zero, while $Q_1(t) \to (-\infty, 0)$ and $Q_2(t) \to (+\infty, 0)$ as $t \to t_*$. If $r_{ij}(t) = |Q_i(t) - Q_j(t)|$ is the distance between bodies i and j at time t then $\limsup_{t \to t_*} r_{ij}(t) = \infty$ and $\liminf_{t \to t_*} r_{ij}(t) = 0$ provided $i = 1$ or 2 and $j = 3$ or 4. Think $\frac{1}{|t|}|\sin(1/t)|$ with $t_* = 0$. Reprinted from [67].

−1.5 The Many-Body Problem

−1.5.1 Central Configurations and Dark Matter

The ansatz leading the central configurations (CCs) of Euler and Lagrange works for any N. See Chapter 1. The number of distinct similarity classes of CCs grows at least as fast as $(3/4)N!$ for planar CCs. Is this number always finite? That's our first open question.

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Our galaxy contains about 10^{11} stars. Saari [186] argues that CCs involving this number of bodies might organize the structure of many galaxies and thereby dispense with the need for dark matter. Imagine a galaxy roughly as a spinning disc full of stars. For many galaxies, as we go out radially away from the galactic center, after a certain distance, the velocities v of the stars have been observed to increase with radius r in a nearly linear manner with distance, as in $v = \omega r$ – as if the stars were glued to a rotary saw rotating at angular velocity ω. This graph of v versus r is called the rotation curve. In contrast, if we take a continuum approach, supposing the stars move under the gravitational potential induced by a spherically or cylindrically symmetric potential whose mass density is estimated by counting luminous stars in the galaxy, we get a very different rotation curve. The observed linear behavior of the rotation curve in a certain range of r and its discrepancy from the one derived by continuum models is one of the arguments for the existence of dark matter. Plunk enough dark matter into the density distribution and you can adjust the rotation curve to fit the data and also estimate how much dark matter there is. Saari doesn't buy this. He says, in essence, "drop this Keplerian – spherically symmetric, star soup – thinking and use central configurations to explain the rotational curve data." I don't buy Saari's counterargument among other reasons because equal mass CCs all seem to be dynamically unstable (see [89] for a detailed critique). But it is fun to think about.

−1.5.2 Large Dances

For any positive integer N, there are choreographic solutions for the equal mass N-body problem. The existence of infinite families of choreographies has been established. Figure −1.24 is a sample. One such family consists of an odd number N of bodies traveling a chain made of $N - 1$ loops, the eight being the case $N = 3$. For any fixed finite postive integer N there may be an infinite number of choreographies with N bodies. We don't know.

The variational and symmetry methods that established the existence of the figure eight solution have allowed researchers to discover and establish the existence of solutions having the symmetries of each of the Platonic solids. Figure −1.25 depicts a 60-body solution enjoying dodecahedral symmetry, 60

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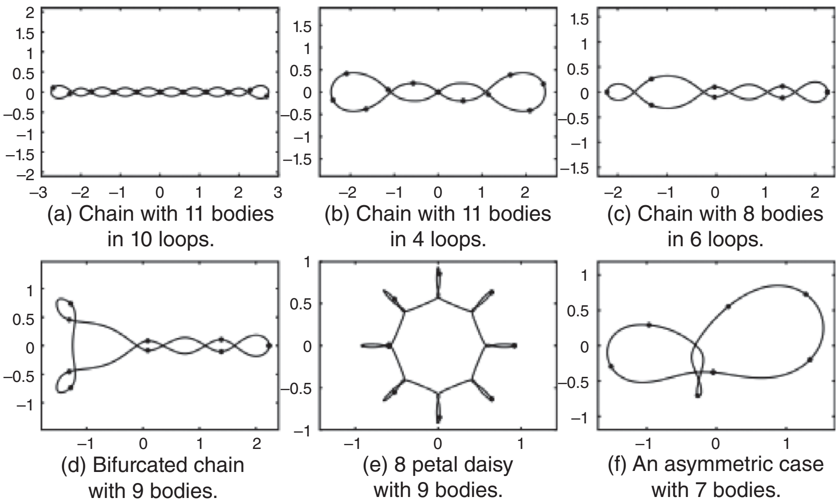

Figure −1.24 A sampling of N-body choreographies. From Chenciner et al. [35].

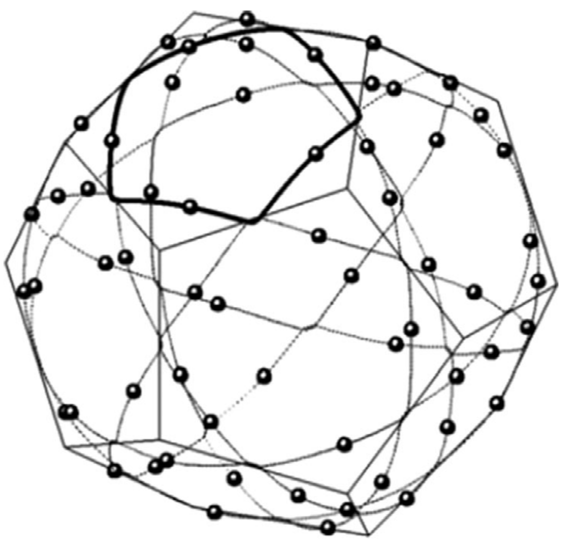

Figure −1.25 A 60-body solution enjoying dodecahedral symmetry. Associated to each of 12 faces are 5 bodies moving in a choreography. Reprinted from [65].

being the order of the subgroup of symmetries of the dodecahedron that are orientation preserving.

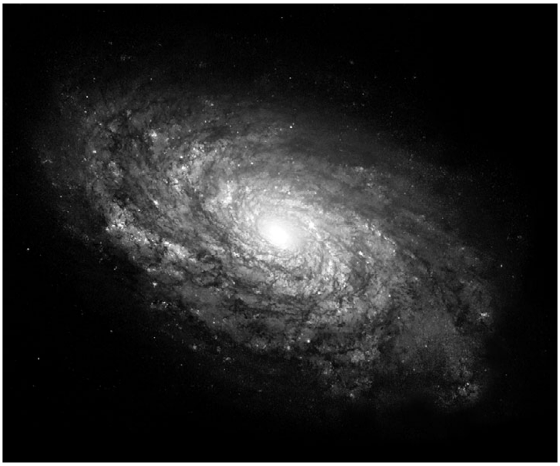

Figure −1.26 Galaxy NGC 1414 as viewed from the Hubble space telescope. Image credit: The Hubble Heritage Team (AURA/STScI/NASA).

−1.5.3 Globular Clusters and Galaxies

For astronomers, N is often quite large. See Figure −1.26. Our own Milky Way contains around 10^{11} stars. Galaxies contain up to 10^{12} stars. Globular clusters are spherical collections of around 10^6 stars that are found orbiting galaxies.

There is an enormous body of work on galactic dynamics. Researchers tend to use continuum models and statistical mechanical thinking. See the classic book by Binney and Tremaine [18] for a sense of this field. Closer to the spirit of the book in your hands, large N-body simulators have been an important tool. See the books "The Gravitational Million-Body Problem" by Heggie and Hut [79] and "Gravitational N-Body Simulations" by Aarseth [1] for an overview and surveys of this kind of work.

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