ON GENERALIZATION OF DECOMPOSABILITY

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Let X be a complex Banach space and let T be a bounded linear operator on X. Then T is decomposable if for every finite open cover $\{G_i\}_i^n$ of $\sigma(T)$ there are invariant subspaces $Y_i (i = 1, 2, \ldots, n)$ such that

$$Y_i \text{ is spectral maximal;} \tag{1}$$

$$X = Y_1 + Y_2 + \ldots + Y_n; \tag{2}$$

$$\sigma(T \mid Y_i) \subset G_i \text{ for each } i. \tag{3}$$

(An invariant subspace Y is spectral maximal [for T] if it contains every invariant subspace Z for which $\sigma(T \mid Z) \subset \sigma(T \mid Y)$.)

If condition (1) is suppressed, we say that T has the spectral decomposition property (SDP). It was shown in **[4]** that many properties of decomposable operators generalize to the case of SDP, including the important single-valued extension property. In this note we show that such generalizations are possible because every operator with the SDP is in fact decomposable. Thus decomposability is the most general type of spectral decomposition for operators on Banach space which satisfy condition (2). An example of Albrecht **[1]** shows that a weakening of (2) to

$$X = \overline{Y_1 + Y_2 + \ldots + Y_n}$$

takes one strictly out of the class of decomposable operators.

Recall that for an operator T having the single-valued extension property every vector x in X has a local resolvent, a maximally defined X-valued analytic function f_x satisfying the identity $(\lambda - T)f_x(\lambda) = x$ for all $\lambda \in \operatorname{dom} f_x$. Let $\mathbb{C}$ denote the complex plane. Then $\sigma(x) = \mathbb{C} \setminus \operatorname{dom} f_x$ is called the local spectrum of x. For $F \subset \mathbb{C}$ the set of vectors $X_T(F) = \{x \in X : \sigma(x) \subset F\}$ is a linear manifold in X invariant under the commutant of T. Suppose F is closed. Then $X_T(F)$ is a closed subspace if T is decomposable **[5]**, Corollary 11.4, p. 75. In **[8]**, Corollary 1, we prove that $X_T(F)$ is closed if T has the SDP and F is convex. The main result in the present paper will follow easily once we show that this restriction of convexity can be dropped (Lemma 3).

LEMMA 1. *Let T have the SDP. If $g(\lambda)$ is a polynomial in λ, then $g(T)$ has the SDP.*

Proof. Let $\{G_i\}_i^n$ be an open cover of $\sigma(g(T))$. Since $\{g^{-1}(G_i)\}$ is an open cover of $\sigma(T)$ by the spectral mapping theorem, we can find invariant subspaces Y_i such that $X = Y_1 + Y_2 + \ldots + Y_n$ and $\sigma(T \mid Y_i) \subset g^{-1}(G_i)$ for each i. Now each Y_i is also invariant under $g(T)$; hence

$$\sigma(g(T) \mid Y_i) = g(\sigma(T \mid Y_i)) \subset g(g^{-1}(G_i)) \subset G_i$$

for all i. Therefore, $g(T)$ has the SDP.

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LEMMA 2. *Let T have the SDP and let D be an open convex set in $\mathbb{C}$. Then $X_T(\bar{D})$ and $X_T(\mathbb{C} \setminus D)$ are both closed.*

Proof. Let H be any closed set such that D and interior H cover $\mathbb{C}$. By definition of SDP there are invariant subspaces Y and Z for T such that $X = Y + Z$, $\sigma(T \mid Y) \subset D$ and $\sigma(T \mid Z) \subset H$. Let x be a vector in $X_T(\mathbb{C} \setminus D)$ and let f_x be its local resolvent. Hence $f_x(\lambda)$ is defined for all $\lambda \in D$ and $(\lambda - T)f_x(\lambda) = x$ for $\lambda \in D$. Let $V = Y \cap Z$ and let T^V be the operator on the quotient X/V induced by T. Now X/V may be written as the direct sum $X/V = Y/V \oplus Z/V$ and hence T^V may be written as a direct sum $T^V = R \oplus S$. Moreover, the image of f_x under the canonical surjection $X \to X/V$ may be written as a direct sum $g \oplus h$. We thus have

$$x' = y \oplus z = (\lambda - R)g(\lambda) \oplus (\lambda - S)h(\lambda), \qquad (\lambda \in D),$$

where x' is the coset of x in X/V; y, $g(\lambda) \in Y/V$; and z, $h(\lambda) \in Z/V$. Since R is the operator induced on Y/V by $T \mid Y$, it follows that

$$\sigma(R) \subset \sigma(T \mid Y) \cup \sigma(T \mid V)$$

(see [**5**], Proposition 1.14, p. 12); and since $T \mid V$ is a restriction of $T \mid Y$, the spectrum $\sigma(T \mid V)$ is contained in the convex hull of $\sigma(T \mid Y)$. Hence $\sigma(R) \subset D$. Because x is fixed y is also fixed; thus the analyticity of g on the neighborhood D of $\sigma(R)$ implies that g must vanish. Hence $y = 0$ and $x \in Z$. Since H was arbitrary, it follows that

$$\sigma(T \mid \overline{X_T(\mathbb{C} \setminus D)}) \subset \mathbb{C} \setminus D, \quad \text{or} \quad \overline{X_T(\mathbb{C} \setminus D)} \subset X_T(\mathbb{C} \setminus D).$$

This proves that $X_T(\mathbb{C} \setminus D)$ is closed.

To prove that $X_T(\bar{D})$ is closed, we note that the above argument may be applied to the manifold $X_T(K)$, where K is a closed halfplane, to prove that $X_T(K)$ is closed. Let $\{K_\alpha\}$ be a family of closed halfplanes whose intersection is $\bar{D}$. Hence $X_T(\bar{D}) = X_T(\cap_\alpha K_\alpha) = \cap_\alpha X_T(K_\alpha)$ is also closed.

The key to our main theorem is Lemma 3 below. Its proof follows closely one of Apostol [**3**], Lemma 3.2, p. 436.

LEMMA 3. *Let T have the SDP. If F is closed in $\mathbb{C}$, then $X_T(F)$ is closed.*

Proof. Suppose that $X_T(F)$ is not closed. Then we find $x \in X$ and a sequence $\{x_n\} \subset X_T(F)$ such that $\sigma(x) \not\subset F$ with

$$x = \sum_{n=0}^{\infty} x_n \quad \text{and} \quad \sum_{n=0}^{\infty} \|x_n\| < \infty. \tag{4}$$

We shall derive a contradiction. Let $\lambda_0 \in \sigma(x) \setminus F$ and let g be a nonconstant polynomial such that $g^{-1}(g(\lambda_0))$ has more than one point. Put $\mu = g(\lambda_0)$. Then there is some $r > 0$ sufficiently small such that the closed disc D centered at μ and of radius r has the property that $g^{-1}(D) = K_1 \cup K_2$, where the K_j $(j = 1, 2)$ are disjoint and closed, and $K_1 \cap F = \emptyset$ (see, for example, [**3**], Lemma 1.5, p. 434). Clearly $\lambda_0 \in K_1$.

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Let H be the complement in $\mathbb{C}$ of the open disc centered at μ and having radius $r/2$. Since the interiors of D and H together cover $\mathbb{C}$, it follows by Lemma 1 that there are two subspaces Y_1, Y_2 invariant under $S = g(T)$ such that $X = Y_1 + Y_2$, $\sigma(S \mid Y_1) \subset D$ and $\sigma(S \mid Y_2) \subset H$. Thus $X = X_S(D) + X_S(H)$, and by Lemma 2 each of the latter linear manifolds is closed. By [**5**], Corollary 1.7, p. 7, $X_S(D) = X_T(g^{-1}(D)) = X_T(K_1 \cup K_2)$ and $X_S(H) = X_T(K_3)$, where $K_3 = g^{-1}(H)$. Hence $X_T(K_1 \cup K_2)$ and $X_T(K_3)$ are closed; moreover $X_T(K_1 \cup K_2) = (X_T(K_1) + X_T(K_2)$ by [**3**], Lemma 2.3, p. 435. Now, since $X = X_T(K_1) + X_T(K_2) + X_T(K_3)$, we may apply the closed graph theorem to the mapping $y^1 \oplus y^2 \oplus y^3 \to y^1 + y^2 + y^3$ $(y^j \in X_T(K_j), j = 1, 2, 3)$ to obtain a constant $R > 0$ such that for $y \in X$ there are $y^j \in X_T(K_j)$ such that

$$y = y^1 + y^2 + y^3 \quad \text{and} \quad R\,\|y\| \le \|y^1\| + \|y^2\| + \|y^3\|. \tag{5}$$

For each of the above $x_n \in X_T(F)$ let $x_n = x_n^1 + x_n^2 + x_n^3$ be its decomposition (5). Then for each $n = 1, 2, \ldots,$

$$\sigma(x_n^1) \subset (F \cup K_2 \cup K_3) \cap K_1 \subset K_3,$$

since $K_1 \cap K_2 = \emptyset$. By (4) and (5) the series $\sum_{n=0}^{\infty} x_n^j$ converge in X to respective sums x^j $(j = 1, 2, 3)$. Obviously $x = x^1 + x^2 + x^3$ and, since the $X_T(K_j)$ are all closed,

$$\lambda_0 \in \sigma(x) \subset \sigma(x^1) \cup \sigma(x^2) \cup \sigma(x^3) \subset K_2 \cup K_3.$$

Now $\lambda_0 \notin K_2$; hence $\lambda_0 \in K_3$; but then this implies the contradiction $\mu = g(\lambda_0) \in H$. Thus $X_T(F)$ must be closed.

THEOREM 1. *Every operator with the SDP is decomposable.*

Proof. Let T have the SDP and let $\{G_i\}$ be a finite open cover of $\sigma(T)$. Now let $\{H_i\}$ be a second open cover of $\sigma(T)$ such that $\bar{H}_i \subset G_i$. Then there are invariant subspaces Y_i such that $X = Y_1 + Y_2 + \ldots + Y_n$ and $\sigma(T \mid Y_i) \subset H_i$. The subspaces $X_i = X_T(\bar{H}_i)$ are closed by Lemma 3; hence each X_i is spectral maximal such that $\sigma(T \mid X_i) \subset G_i$ by [**5**], Theorem 3.11, p. 31. Since $Y_i \subset X_i$ for each i, T satisfies conditions (1)–(3), and, by definition, T is decomposable. This completes the proof.

Theorem 1 gives a negative answer to Problem 4 posed in [**5**], p. 115. Aside from its theoretical interest, this theorem also provides a sufficient condition for decomposability which avoids the necessity of producing spectral maximal spaces explicitly. We now give three such applications of Theorem 1. The first two corollaries provide simple proofs of some standard, basic results in the theory of decomposable operators; the last result (Theorem 2) is new.

COROLLARY 1. *If T is decomposable and f is a scalar-valued analytic function on some neighborhood of $\sigma(T)$, then $f(T)$ is decomposable* ([**5**], p. 2).

Proof. Let $\{G_i\}$ be an open cover of $f(\sigma(T))$ so that $\{f^{-1}(G_i)\}$ is an open cover of $\sigma(T)$. Since $\{f^{-1}(\bar{G}_i)\}$ also covers $\sigma(T)$, if we put $Y_i = X_T(f^{-1}(\bar{G}_i))$ then each subspace Y_i is

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invariant under $f(T)$ and $\sigma(f(T) \mid Y_i) = f(\sigma(T \mid Y_i)) \subset f(f^{-1}(\bar{G}_i)) \subset \bar{G}_i$. Now $X = Y_1 + Y_2 + \ldots + Y_n$, since T is decomposable; hence $f(T)$ has the SDP. By Theorem 1, $f(T)$ is decomposable.

COROLLARY 2. *Let T be decomposable and suppose that A is a bounded operator in X such that $A(X_T(F)) \subset X_T(F)$ and $\sigma(A \mid X_T(F)) \subset F$ for each closed F. Then A is decomposable.*

Proof. We first prove that $\sigma(A) = \sigma(T)$. By hypothesis $\sigma(A) = \sigma(A \mid X_T(\sigma(T))) \subset \sigma(T)$. It $\lambda \in \sigma(T) \setminus \sigma(A)$, then there is a closed disc D such that (interior D) $\cap\, \sigma(T) \neq \emptyset$ and $D \cap \sigma(A) = \emptyset$. Since $X_T(D) \neq (0)$ we have $\emptyset \neq \sigma(A \mid X_T(D)) \subset D$; hence $\sigma(A) \cap \sigma(A \mid X_T(D)) = \emptyset$. But the last equality is impossible for any nonzero invariant subspace of A. It follows that $\sigma(A) = \sigma(T)$; hence every open cover $\{G_i\}$ of $\sigma(A)$ is also one for $\sigma(T)$. In this case $X = X_T(\bar{G}_1) + \ldots + X_T(\bar{G}_n)$. By hypothesis each $X_T(\bar{G}_i)$ is A-invariant and such that $\sigma(A \mid X_T(\bar{G}_i)) \subset \bar{G}_i$. This proves that A has the SDP, and hence is decomposable.

LEMMA 4. *Let T be decomposable and let $G \subset \mathbb{C}$ be open such that $\overline{G \cap \sigma(T)} \neq \sigma(T)$. Let $Y = X_T(G)$, and let T^Y be the operator induced by T on X/Y. Then (1) $\sigma(T \mid Y) = \overline{G \cap \sigma(T)}$ and (2) $\sigma(T^Y) = \overline{\sigma(T) \setminus \sigma(T \mid Y)}$.*

Proof. Since $X_T(G) = X_T(G \cap \sigma(T))$, it is clear that $\sigma(T \mid Y) \subset \overline{G \cap \sigma(T)}$. To see the opposite inclusion, let $\lambda \in G \cap \sigma(T)$. Then for $\varepsilon > 0$ sufficiently small there is a disc D_ε of radius ε and center λ such that $D_\varepsilon \cap \sigma(T) \subset \sigma(T \mid Y)$. Hence $\lambda \in \sigma(T \mid Y)$ and statement (1) is proved.

To prove (2), let $F = \overline{\sigma(T) \setminus \sigma(T \mid Y)}$. The inclusion $F \subset \sigma(T^Y)$ follows from the identity $\sigma(T) = \sigma(T^Y) \cup \sigma(T \mid Y)$ by [**5**], Proposition 2.2, p. 15. In order to see that $\sigma(T^Y) \subset F$, suppose $\lambda \in \sigma(T^Y) \setminus F$. Let $\{H_1, H_2\}$ be an open cover of $\sigma(T)$ such that $H_1 \supset F$, $\lambda \notin H_1$ and $\bar{H}_2 \cap F = \emptyset$. Note first that $X = X_T(H_1) + X_T(H_2)$ because T is decomposable. Further, $H_2 \cap (\sigma(T) \setminus \sigma(T \mid Y)) = \emptyset$; hence $H_2 \cap \sigma(T) \subset \sigma(T \mid Y) = \overline{G \cap \sigma(T)}$ by (1). Hence we may suppose that $H_2 \cap \sigma(T) \subset G \cap \sigma(T)$. It follows that $X_T(H_2) \subset Y$. Let $x \in X$ be arbitrary so that $x = x_1 + x_2$, $x_i \in X_T(H_i)$. By choice of H_1 there is $y_1 \in X_T(H_1)$ such that $x_1 = (\lambda - T)y_1$. Letting u' denote the coset $u + Y$ of u, we see that $x' = (\lambda - T^Y)y'_1$; that is $\lambda - T^Y$ is surjective. We obtain a contradiction if we show that $\lambda - T^Y$ is also injective. But $\lambda - T^Y$ has the single-valued extension property [**7**], Theorem 3 and Lemma 1; thus $\lambda - T^Y$ is injective by [**6**], Corollary 7. Therefore (2) is proved.

Conclusion (2) of Lemma 4 fails without the restriction $\overline{G \cap \sigma(T)} \neq \sigma(T)$. For let X be the Banach space of complex-valued continuous functions on the closed unit disc D and let G be the open unit disc. The operator T defined on X by $(Tx)(\lambda) = \lambda x(\lambda)$, $x \in X$, $\lambda \in D$, is decomposable, and $Y = X_T(G)$ generates the spectrum $\sigma(T \mid Y) = \bar{G} = D = \sigma(T)$. Now $Y \neq X$ and $\sigma(T^Y) \neq \emptyset$, but $\sigma(T) \setminus \sigma(T \mid Y) = \emptyset$.

THEOREM 2. *Let T be decomposable such that $T \mid Z$ is also decomposable for every spectral maximal Z. Let G be open such that $\overline{G \cap \sigma(T)} \neq \sigma(T)$ and let $Y = X_T(G)$. Then the induced operator T^Y is decomposable.*

Proof. Let $\{H_i\}_1^n$ be an open cover of $\sigma(T^Y)$. Without loss of generality we may suppose that $\{H_i\}$ actually is a cover for $\mathbb{C}$. For each $i = 1, 2, \ldots, n$ let $Z_i = X_T(\bar{H}_i \cup \sigma(T \mid Y))$. Then each Z_i is spectral maximal for T, $Y \subset Z_i$ for each i and $\underline{X = Z_1 + Z_2 +} \ldots + Z_n$. Since each H_i is open, it is evident from Lemma 4 that $\overline{\sigma(T \mid Z_i) \cap G} \neq \sigma(T \mid Z_i)$ for each i. Put $Z = Z_i$ for some fixed i. An easy argument shows that $Y = \overline{X_T(G)} = \overline{Z_S(G)}$, where $S = T \mid Z$. If we now apply Lemma 4 to $T \mid Z_i$ (each i), we obtain the inclusions

$$\sigma(T^Y \mid (Z_i/Y)) = \sigma((T \mid Z_i)^Y) = \overline{\sigma(T \mid Z_i \setminus \sigma(T \mid Y)} \subset \overline{[\bar{H}_i \cup \sigma(T \mid Y)] \setminus \sigma(T \mid Y)} \subset \bar{H}_i.$$

since $X/Y = (Z_1/Y) + \ldots + (Z_n/Y)$, T^Y has the SDP. Thus T^Y is decomposable by Theorem 1.

NOTE. After this paper had been submitted for publication, the author discovered that the main result had been obtained independently and a different method by Ernst Albrecht [**2**].

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