THREE FACIALLY-REGULAR POLYHEDRA

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1. Introduction. H. S. M. Coxeter has shown[1] the existence of three infinite regular polyhedra, and has proved that there are no infinite regular polyhedra other than these. In his paper he gives the definition of regularity of a polyhedron:

> A polyhedron is said to be regular if it possesses two particular symmetries: one which cyclically permutes the vertices of any face c, and one which cyclically permutes the faces that meet at a vertex C, C being a vertex of c.

Among the finite polyhedra there are many of interest other than the regular polyhedra, in particular those known as *uniform* polyhedra:

> A polyhedron is said to be uniform if its faces are regular polygons and it admits symmetries which transform a given vertex into every other vertex in turn.[2]

Such polyhedra may have faces of more than one type; indeed, any non-regular uniform polyhedron which is finite or fills a plane must have two or more different polygons among its faces.

Uniform polyhedra exist, however, with three dimensional content, which are infinite in one, two, or three dimensions, and among these are some (about 30 are known) which have all faces equal. These polyhedra I call *facially-regular*; they are of two types[3]:

(a) having all faces equivalent, in that a symmetry of the polyhedron exists that can transform any face into any other face;

(b) having more than one kind of face, like the two kinds of triangle in the snub cube (which, however, is not facially regular, as it has square faces as well).

To sum up:

A polyhedron is called facially-regular if it is uniform and all its faces are equal regular polygons.

In this note I describe three facially-regular polyhedra, two of type (a) and one of type (b).

2. Notation. For convenience I use the notation a^b to denote a facially-regular polyhedron whose faces are regular a-gons, b of which meet at each

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[1] *Regular skew polyhedra in three and four dimensions*, Proc. London Math. Soc. (2) 43 (1937), 33-62.

[2] This definition is adequate for polyhedra in which faces are allowed to intersect only at edges; otherwise a proviso must be added to exclude compounds such as the five cubes with the vertices of a dodecahedron.

[3] I must thank the referee for pointing this out.

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vertex; but the symbol does not necessarily define the polyhedron uniquely; there are, for instance, an infinity of 3^6.

The three polyhedra described are, in this notation, 3^{12}, 3^9, and 3^8; 3^{12} and 3^9 being of type (a) and 3^8 of type (b). I must thank Mr. S. Melmore for pointing out the existence of the third of these.

3. The polyhedron 3^{12}. The vertices of this polyhedron can best be classified by referring it to rectangular Cartesian coordinates, classifying the integers

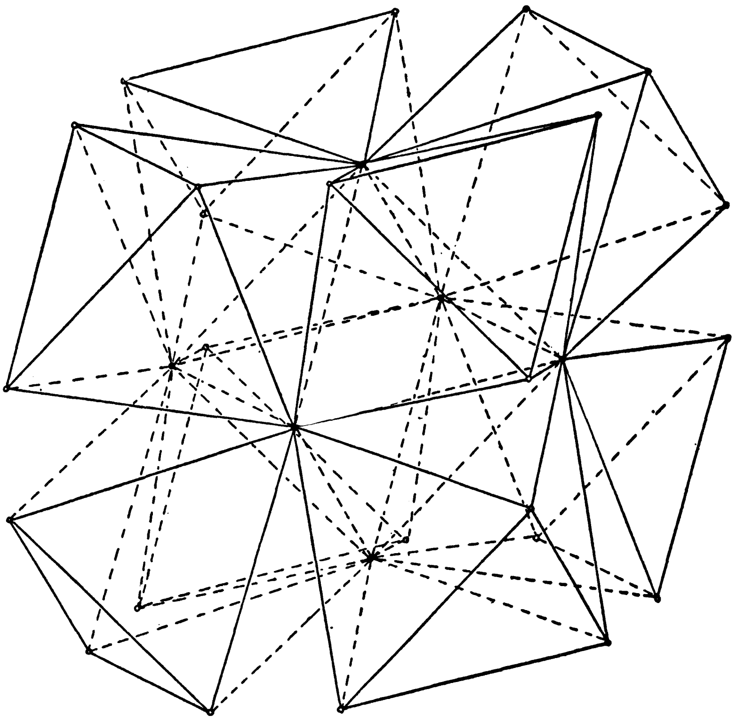

FIGURE 1

by their residues (mod 8), and labelling as white those points having coordinates congruent to 4, 4, 1 in some order, or 4, 4, 7 in some order; and as black those points having coordinates congruent to 0, 0, 3 in some order, or 0, 0, 5 in some order; the aggregate of all points, white and black, gives the vertices of a 3^{12}.

Fig. 1 is a drawing of a cube of side 8 with the part of the relevant 3^{12} inside it; if the axes are taken so that the cube is $|x|, |y|, |z| \leq 4$, then the vertices of the complete 3^{12} are the white and black points.

The vertex figure is given, for example, by the vertices adjacent to the black point (0, 0, 3), which are, in order,

$$\begin{array}{cccccc} (3,0,0) & (4,1,4) & (1,4,4) & (0,3,0) & (-1,4,4) & (-4,1,4) \\ (-3,0,0) & (-4,-1,4) & (-1,-4,4) & (0,-3,0) & (1,-4,4) & (4,-1,4) \end{array}$$

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It is clear that the vertex figures of all the black points are congruent; and since the two colours are interchanged by the transformation which adds 4 to each coordinate, the vertex figures of all the white points are congruent to each other and to the vertex figures of the black points.

From a consideration of the vertex figure of (0, 0, 3), it is seen that all the twelve vertices adjacent to this point are distant $3\sqrt{2}$ from it; similarly with

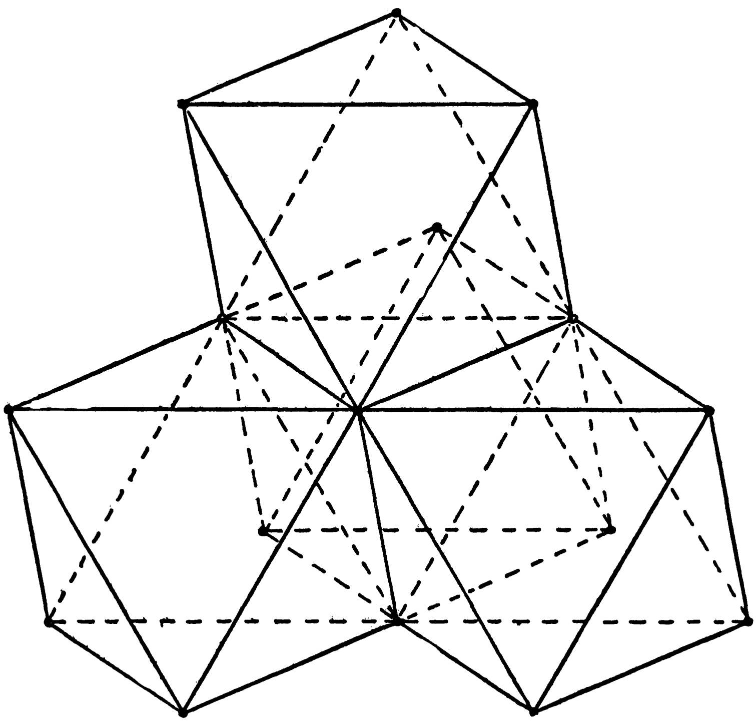

FIGURE 2

the other vertices of the polyhedron. Hence the faces of the polyhedron are equilateral triangles.

Consequently the polyhedron is facially regular. (See Fig. 4.)

As a method of practical construction of this polyhedron, it may be most convenient to consider a building together of octahedra of two types, say A and B, different only in that two opposite faces of each B octahedron are distinguished from the others and called, say, *internal* faces. (Let the remaining faces of the B octahedra be called *external*.) Then, if to every face of an A octahedron is attached a B by an internal face, and to each internal face of each B octahedron is attached an A octahedron; the aggregate of the external faces of the B octahedra gives the faces of a 3^{12}.

4. The polyhedron 3^9. If we view a polyhedron as a solid body, this polyhedron can be considered either as part of the honeycomb $[3^4]$ (in Rouse Ball's

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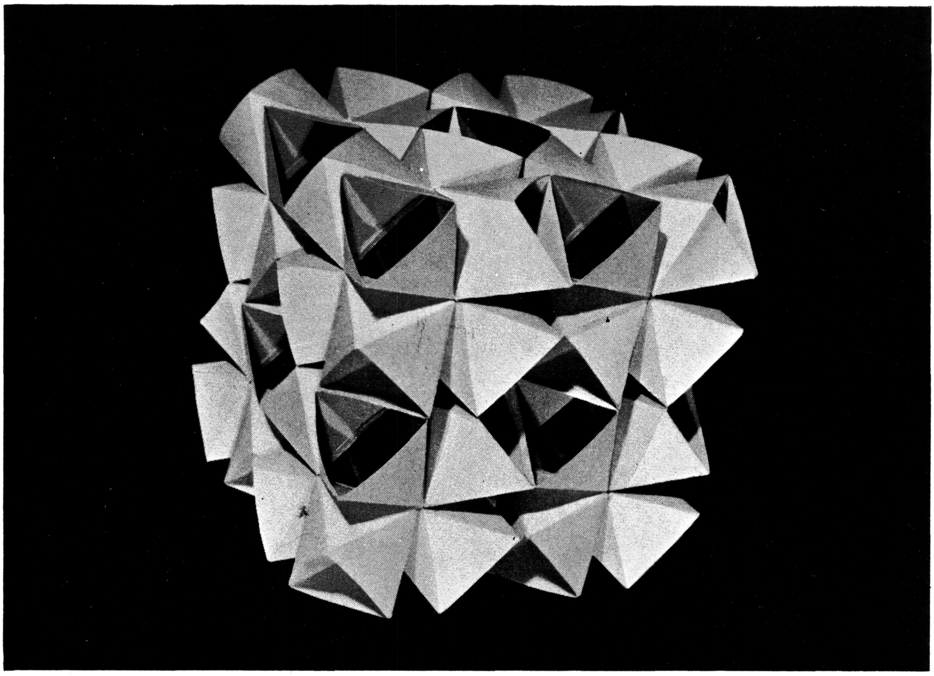

FIGURE 4

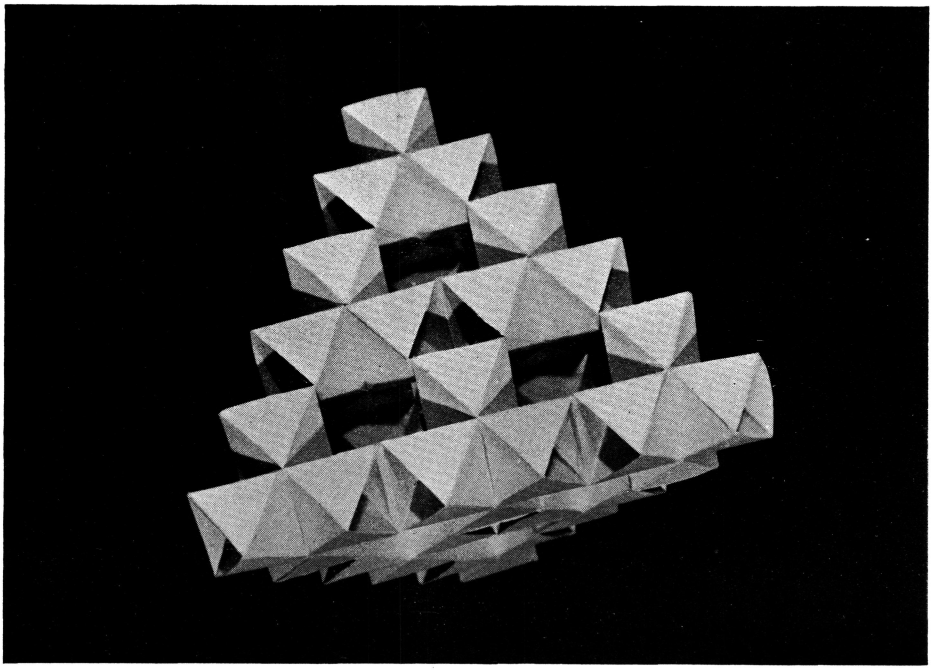

FIGURE 5

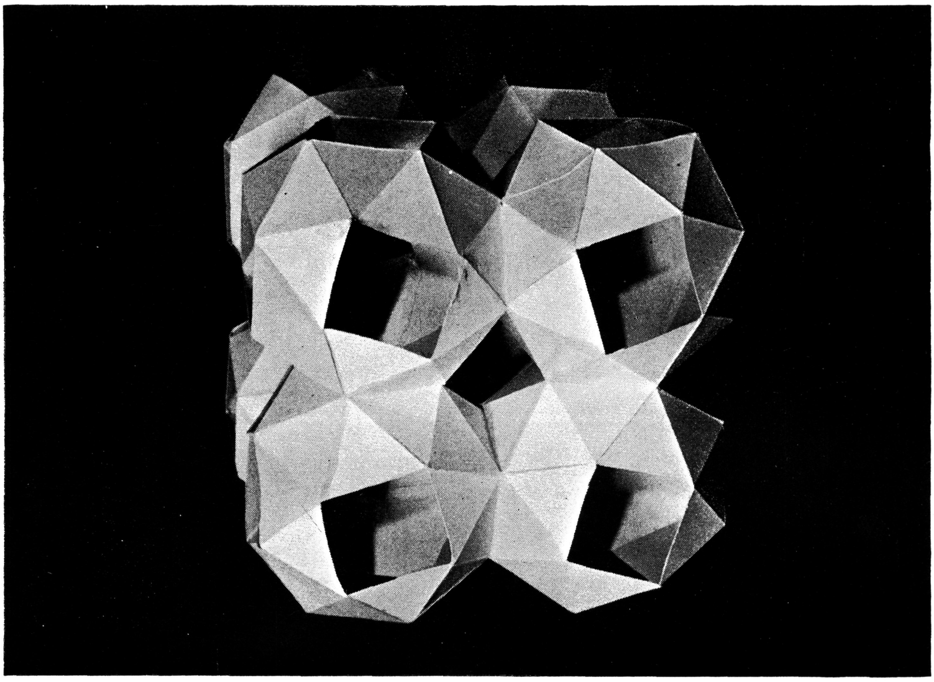

FIGURE 6

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notation; $\left\{3, \begin{smallmatrix}3\\4\end{smallmatrix}\right\}$ in Coxeter's) which consists of tetrahedra and octahedra, two of each (arranged alternately) surrounding each edge; or as part of the regular polyhedron 6^6 (see Coxeter, *loc. cit.*; in his notation $\{6, 6|3\}$). Since a truncated tetrahedron can be built up of four octahedra and seven tetrahedra, we can consider the honeycomb as being composed of tetrahedra and truncated tetrahedra, and therefore, further, as composed of two 6^6. If we take one of these 6^6 and remove from each of the truncated tetrahedra the external six of the seven tetrahedra referred to (see Fig. 2), we are left with a building of tetrahedra and octahedra, such that on each face of each tetrahedron is an octahedron,

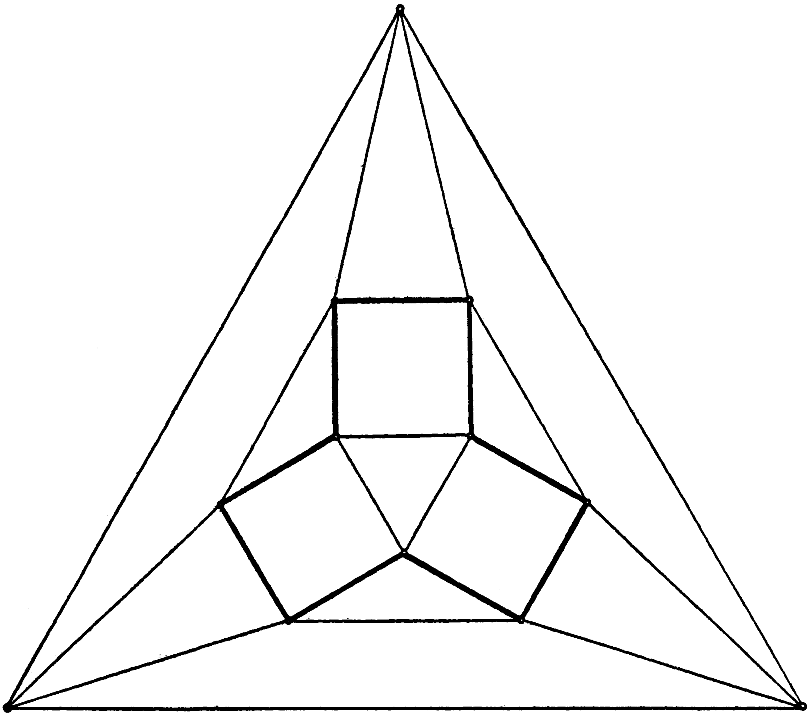

FIGURE 3

joined by an internal face (the octahedra are of type B, as defined in § 3), and on each internal face of each octahedron is a tetrahedron. The external faces of the octahedra are the faces of a 3^9.

The vertex figure of the polyhedron consists of nine sides of a cuboctahedron, as indicated in Fig. 3. It is clear that all the vertex figures are congruent; and so, as all the faces are equilateral triangles, the polyhedron is facially-regular. (See Fig. 5.)

Referred to rectangular Cartesian coordinates, the vertices of the polyhedron may be taken as the aggregate of the points $P + A_r$; where P is the set of points

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$$\{2\xi + \eta + \zeta, \tfrac{1}{3}\sqrt{3}(3\eta + \zeta), \tfrac{2}{3}\sqrt{6}\zeta\}$$

where ξ, η, ζ take all integer values, and A_r is the set of points

$$(\pm \tfrac{1}{2}, \tfrac{1}{6}\sqrt{3}, \tfrac{1}{12}\sqrt{6}) \quad (0, 0, -\tfrac{1}{4}\sqrt{6}) \quad (0, -\tfrac{1}{3}\sqrt{3}, \tfrac{1}{12}\sqrt{6})$$
$$(\tfrac{1}{2}, \pm \tfrac{1}{2}\sqrt{3}, -\tfrac{1}{4}\sqrt{6}) \quad (1, 0, -\tfrac{1}{4}\sqrt{6}) \quad (1, -\tfrac{1}{3}\sqrt{3}, \tfrac{1}{12}\sqrt{6}).$$

A practical method of construction has already been indicated; onto each face of each of a set of tetrahedra build a B octahedron by an internal face, and onto each internal face of each of the B octahedra build a tetrahedron; the aggregate of the external faces of the B octahedra gives the faces of a 3^9.

5. The polyhedron 3^8. The existence of a facially-regular polyhedron 3^8 is more easily demonstrated by its construction.

We replace each cube of the ordinary space-filling $[4^4]$ by an inscribed snub cube, with the condition that any two adjacent snub cubes are images of each other by reflection in their common face, so that laevo and dextro varieties occur alternately. Then the vertices of the whole set of snub cubes coincide in pairs, and the removal of the square faces of the snub cubes leaves a polyhedron having eight triangles meeting at each vertex, and whose vertex figures are congruent, which is therefore facially regular. (See Fig. 6.)

The vertices of the polyhedron can be classified by referring it to rectangular Cartesian coordinates; if we take as the points P all the points $(2\xi, 2\eta, 2\zeta)$, where ξ, η, ζ take all integer values, but with the condition that $\xi + \eta + \zeta$ is even; and in the cubes of side 2, centres P, inscribe laevo snub cubes, the vertices of all such laevo snub cubes will be the vertices of a 3^8.

Consequently the aggregate of points $P + A_r (r = 1, \ldots, 24)$ are the vertices of a 3^8, where

A_1 is $(1, a, b)$,	A_5 is $(-1, b, a)$,
A_2 is $(1, b, -a)$,	A_6 is $(-1, a, -b)$,
A_3 is $(1, -a, -b)$,	A_7 is $(-1, -b, -a)$,
A_4 is $(1, -b, a)$,	A_8 is $(-1, -a, b)$,

and $A_9, \ldots, A_{24}$ are cyclic permutations of these; and where a and b are defined by

$$a + b + ab = 1, \qquad 2a = b^2 + 1;$$

i.e., a is the real root of the equation

$$a^3 + a^2 + a - 1 = 0,$$

and

$$b = a^2.$$

As a typical vertex figure, take that of the vertex $(1, a, b)$; the adjacent vertices are, in order:

$$(2 - a, b, 1), (2 - b, 1, a), (1, b, -a), (b, 1, a),$$
$$(a, b, 1), (b, -a, 1), (1, -b, a), (2 - b, -a, 1).$$

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