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REFINING RECURSIVELY THE HERMITE–HADAMARD INEQUALITY ON A SIMPLEX

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Abstract

In the present paper, a coupled algorithm refining recursively the Hermite–Hadamard inequality on a simplex is investigated. Our approach allows us to express the integral mean value M_f of a convex function f on a simplex as both the limit of sequences and sum of series involving iterative lower and upper bounds of M_f. Two examples of interest are discussed.

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1. Introduction

The following result is well known in the literature (see, for example, [1, 6–8]).

THEOREM 1.1. *Let D be an $(n+1)$-simplex of $\mathbb{R}^n$ and $f : D \longrightarrow \mathbb{R}$ be a convex function. If $p_1, p_2, \ldots, p_{n+1}$ denote the vertices of D then we have*

$$f\left(\sum_{i=1}^{n+1} \frac{p_i}{n+1}\right) \le \frac{1}{|D|} \int_D f(x)\,dx \le \frac{1}{n+1} \sum_{i=1}^{n+1} f(p_i), \tag{1.1}$$

where $|D| = \int_D dx$ stands for the Lebesgue volume of D in $\mathbb{R}^n$.

The double inequality (1.1), known in the literature as the Hermite–Hadamard inequality, and henceforth denoted by (HHI), has a large array of applications in many mathematical areas and has proved very useful from the theoretical point of view as well as for practical purposes. A refinement of (HHI) was discussed by Mitroi and Spiridon in [4]. A converse version of (HHI) was investigated by Mitroi and Symeonidis in [5]. An extension of (HHI), due to Choquet, for convex functions on a compact set can be found in [9].

For the sake of simplicity, the middle term of (1.1) will be denoted by $M_f(D)$, that is,

$$M_f(D) = \frac{1}{|D|} \int_D f(x)\,dx,$$

known in the literature as the (arithmetic) integral mean value of f on D. The left and right terms of (1.1) will be called the initial lower and upper bounds of $M_f(D)$, respectively. The computation of $M_f(D)$, when f and D are given, is in general hard. The initial lower and upper bounds of $M_f(D)$ in (1.1) can be considered as estimates of $M_f(D)$, but of course without good precision in general.

The fundamental goal of the present paper appears out of the following procedure: we will construct a coupled algorithm involving two recursive sequences, denoted by $(L_k(D))_k$ and $(U_k(D))_k$, such that

$$\begin{aligned} f\Big(\sum_{i=1}^{n+1}\frac{p_i}{n+1}\Big) &:= L_0(D) \le L_1(D) \le L_2(D) \le \cdots \le L_k(D) \\ &\le M_f(D) \le \cdots \le U_k(D) \le \cdots \le U_2(D) \le U_1(D) \le U_0(D) \\ &:= \frac{1}{n+1}\sum_{i=1}^{n+1} f(p_i), \end{aligned}$$

together with the property

$$\lim_{k\to\infty} L_k(D) = \lim_{k\to\infty} U_k(D) = M_f(D).$$

Our approach also allows us to give an expression for M_f in terms of series:

$$M_f(D) = U_0(D) - \frac{1}{n+1}\sum_{k=0}^{\infty}(U_k(D) - L_k(D)).$$

In the one-dimensional case, $n = 1$, (HHI) takes the form

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}, \tag{1.2}$$

provided that $f : [a, b] \longrightarrow \mathbb{R}$, $a < b$, is convex. It is well known (see [2]) that the left inequality of (1.2) gives a better estimate of the integral mean value than the inequality on the right, that is,

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f\Big(\frac{a+b}{2}\Big) \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.3}$$

In [10], Wasowicz and Witkowski proved, via a counterexample, that (1.3) does not remain true for convex functions involving several variables. This means, using our previous notation, that the inequality

$$M_f(D) - L_0(D) \le U_0(D) - M_f(D)$$

does not hold for convex functions with several arguments. After checking a particular example of the two-dimensional case, Wasowicz and Witkowski conjectured that

$$M_f(D) - L_0(D) \le n(U_0(D) - M_f(D)), \tag{1.4}$$

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and proved it later by an elementary but lengthy argument. Our approach, described above, will allow us to deduce (1.4) as a simple consequence of our theoretical results. In fact, we show more: inequality (1.4) is conserved for all iterates $L_k(D)$ and $U_k(D)$ previously constructed. That is, the iterative inequality

$$M_f(D) - L_k(D) \leq n(U_k(D) - M_f(D))$$

remains true for each integer $k \geq 0$. In the one-dimensional case, $n = 1$ (that is $D = [a, b]$), the latter inequality means that the left iterate $L_k([a, b])$ gives a better estimate of the integral mean value than the right iterate $U_k([a, b])$. That is, for all $k \geq 0$, we have

$$\frac{1}{b-a}\int_a^b f(x)\,dx - L_k([a, b]) \leq U_k([a, b]) - \frac{1}{b-a}\int_a^b f(x)\,dx,$$

where the general iterates $L_k([a, b])$ and $U_k([a, b])$ can be explicitly computed in terms of a, b and f in a simple recursive manner; see [3] or Example 3.3 below.

2. Basic notions

In this section, we state some basic notions that will be needed later. Let D be an arbitrary $(n+1)$-simplex of $\mathbb{R}^n$ with vertices $p_1, p_2 \ldots, p_{n+1}$, in short $D = \overline{co}(p_1, p_2, \ldots, p_{n+1})$, where $\overline{co}$ refers to the closed convex hull. Let $b = \sum_{i=1}^{n+1} p_i/(n+1)$ be the barycenter of the Lebesgue measure of D and $\overline{co}(b, a_1, a_2, \ldots, a_n)$ be the sub-simplex of D, where the points $a_1, a_2, \ldots, a_n$ are distinct and belong to the set $\{p_1, p_2, \ldots, p_{n+1}\}$. There are $n+1$ choices of $(a_1,\ a_2,\ \ldots,\ a_n)$ and so we have $n+1$ sub-simplices of D which will be denoted by D_i, $1 \leq i \leq n+1$.

Following the above construction, the sub-simplices D_i, $1 \leq i \leq n+1$, form a quasi-partition of D in the sense that

$$D = \bigcup_{i=1}^{n+1} D_i \quad \text{and} \quad |D_i \cap D_j| = \emptyset \quad \forall i \neq j, \tag{2.1}$$

and

$$|D_i| = |D_j| = \frac{|D|}{n+1} \quad \forall i, j. \tag{2.2}$$

Recall that D_j, for fixed $j = 1, 2, \ldots, n+1$, has the same vertices as D except for one vertex which is the barycentre b of D. Explicitly, we can write

$$D_j = \overline{co}\{p_1, p_2, \ldots, p_{j-1}, b, p_{j+1}, \ldots, p_{n+1}\},$$

where b figures in the jth place, with $D = \overline{co}\{p_1, p_2, \ldots, p_{n+1}\}$.

For the sake of clarity, we state the following definition.

Definition 2.1. For an arbitrary $(n+1)$-simplex $D \subset \mathbb{R}^n$ with vertices $p_1, p_2, \ldots, p_{n+1}$ and a convex function $f : D \longrightarrow \mathbb{R}$ we set

$$L_0(D) = f\left(\sum_{i=1}^{n+1} \frac{p_i}{n+1}\right) \quad \text{and} \quad U_0(D) = \frac{1}{n+1}\sum_{i=1}^{n+1} f(p_i), \tag{2.3}$$

which are the lower and upper bounds of (HHI) for f on D. We also define, for $k = 1, 2, \ldots, n$,

$$L_{k+1}(D) = \frac{1}{n+1} \sum_{j=1}^{n+1} L_k(D_j), \quad U_{k+1}(D) = \frac{1}{n+1} \sum_{j=1}^{n+1} U_k(D_j). \tag{2.4}$$

For example, for every fixed $j = 1, 2, \ldots, n+1$,

$$L_0(D_j) = f\left(\sum_{i=1, i \neq j}^{n+1} \frac{p_i}{n+1} + \frac{b}{n+1} \right), \quad U_0(D_j) = \frac{1}{n+1} \sum_{i=1, i \neq j}^{n+1} f(p_i) + \frac{1}{n+1} f(b). \tag{2.5}$$

3. Refinement of (HHI): the main results

Let D and f be fixed as in Theorem 1.1. As previously defined, the D_j, for $j = 1, 2, \ldots, n+1$, are the $n+1$ sub-simplices of D. The barycentre of D will be denoted here by b. Applying (HHI) for f in $D_j \subset D$, for fixed $j = 1, 2, \ldots, n+1$, we obtain

$$L_0(D_j) \leq \frac{1}{|D_j|} \int_{D_j} f(x)\, dx \leq U_0(D_j), \tag{3.1}$$

where $L_0(D_j)$ and $U_0(D_j)$ are defined as in (2.5).

Replacing $|D_j|$ by $|D|/(n+1)$ (following (2.2)) and summing (3.1) over $j = 1, 2, \ldots, n+1$, with (2.5), we obtain

$$\begin{aligned} \frac{1}{n+1} \sum_{j=1}^{n+1} f\left(\sum_{i=1, i \neq j}^{n+1} \frac{p_i}{n+1} + \frac{b}{n+1} \right) &\leq \frac{1}{|D|} \sum_{j=1}^{n+1} \int_{D_j} f(x)\, dx \\ &\leq \frac{1}{(n+1)^2} \sum_{j=1}^{n+1} \sum_{i=1, i \neq j}^{n+1} f(p_i) + \frac{1}{n+1} f(b), \end{aligned}$$

which, with (2.1) and

$$\sum_{j=1}^{n+1} \int_{D_j} f(x)\, dx = \int_{\bigcup_{j=1}^{n+1} D_j} f(x)\, dx = \int_D f(x)\, dx = |D| M_f(D),$$

yields

$$\begin{aligned} \frac{1}{n+1} \sum_{j=1}^{n+1} f\left(\sum_{i=1, i \neq j}^{n+1} \frac{p_i}{n+1} + \frac{b}{n+1} \right) &\leq M_f(D) \\ &\leq \frac{1}{(n+1)^2} \sum_{j=1}^{n+1} \sum_{i=1, i \neq j}^{n+1} f(p_i) + \frac{1}{n+1} f(b). \end{aligned} \tag{3.2}$$

Summarising, we have started from initial lower and upper bounds of (HHI), named $L_0(D)$ and $U_0(D)$, and obtained new lower and upper bounds, respectively given by

$$L_1(D) = \frac{1}{n+1} \sum_{j=1}^{n+1} L_0(D_j) = \frac{1}{n+1} \sum_{j=1}^{n+1} f\left(\sum_{i=1, i \neq j}^{n+1} \frac{p_i}{n+1} + \frac{b}{n+1} \right) \tag{3.3}$$

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and

$$U_1(D) = \frac{1}{n+1}\sum_{j=1}^{n+1} U_0(D_j) = \frac{1}{(n+1)^2}\sum_{j=1}^{n+1}\sum_{i=1,i\neq j}^{n+1} f(p_i) + \frac{1}{n+1}f(b). \tag{3.4}$$

This leads to the next result.

Proposition 3.1. *With the notation as above, the following relationships hold:*

$$L_1(D) = \frac{1}{n+1}\sum_{j=1}^{n+1} f\left(\frac{p_j + (n+2)\sum_{i=1,i\neq j}^{n+1} p_i}{(n+1)^2}\right), \tag{3.5}$$

$$U_1(D) = \frac{n}{(n+1)^2}\sum_{i=1}^{n+1} f(p_i) + \frac{1}{n+1} f\left(\frac{\sum_{i=1}^{n+1} p_i}{n+1}\right). \tag{3.6}$$

Proof. Use (3.3) and (3.4) with $b = \sum_{i=1}^{n+1} p_i/(n+1)$ and a classical manipulation of the summation. The details are straightforward. □

We are now in a position to state the following result.

Proposition 3.2. *With the notation as above, (3.2) is a refinement of (1.1), that is,*

$$L_0(D) \le L_1(D) \le M_f(D) \le U_1(D) \le U_0(D). \tag{3.7}$$

Proof. Inequalities (3.7) can be proved by using (3.5) and (3.6) with the help of the generalised Jensen inequality applied to the convex function f. See also [4] for a similar argument. □

Example 3.3. Let $n = 1$ and $D = [a, b]$ with $a < b$. Then we have

$$L_0(D) = f\left(\frac{a+b}{2}\right), \quad L_1(D) = \frac{1}{2}\left(f\left(\frac{a+3b}{4}\right) + f\left(\frac{3a+b}{4}\right)\right),$$
$$U_0(D) = \frac{f(a)+f(b)}{2}, \quad U_1(D) = \frac{1}{2}\left(\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right).$$

Substituting these expressions in (3.7) we obtain a well-known refinement of (HHI) for a convex function $f : [a, b] \longrightarrow \mathbb{R}$ (see, for example, [2, 3]).

Example 3.4. Let $n = 2$ and D be the triangle of sides a, b and c. We have

$$L_0(D) = f\left(\frac{a+b+c}{3}\right), \quad U_0(D) = \frac{f(a)+f(b)+f(c)}{3},$$

and, using (3.5) and (3.6) respectively, we obtain

$$L_1(D) = \frac{1}{3}\left(f\left(\frac{a+4b+4c}{9}\right) + f\left(\frac{4a+b+4c}{9}\right) + f\left(\frac{4a+4b+c}{9}\right)\right),$$
$$U_1(D) = \frac{2}{3}\left(\frac{f(a)+f(b)+f(c)}{3}\right) + \frac{1}{3} f\left(\frac{a+b+c}{3}\right).$$

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We can repeat the same procedure as above by starting from the new lower and upper bounds, $L_1(D)$ and $U_1(D)$ respectively, of $M_f(D)$. We then construct, by a mathematical induction, two recursive sequences $(L_k(D))_k$ and $(U_k(D))_k$ such that

$$L_{k+1}(D) = \frac{1}{n+1}\sum_{j=1}^{n+1} L_k(D_j), \quad U_{k+1}(D) = \frac{1}{n+1}\sum_{j=1}^{n+1} U_k(D_j), \tag{3.8}$$

where the initial data $L_0(D_j)$ and $U_0(D_j)$ are given in Definition 2.1. The following result is clear.

Proposition 3.5. *With the notation as above, $(L_k(D))_k$ is an increasing sequence while $(U_k(D))_k$ is a decreasing one. Further, the chain of refinements for (HHI) given by*

$$\begin{aligned} L_0(D) &\le L_1(D) \le L_2(D) \le \cdots \le L_{k-1}(D) \le L_k(D) \\ &\le M_f(D) \le U_k(D) \le U_{k-1}(D) \le \cdots \le U_1(D) \le U_0(D) \end{aligned} \tag{3.9}$$

holds true for every integer $k \ge 0$.

We need the following result which will be a good tool for ensuring our claim.

Theorem 3.6. *With the notation as above, we have for every integer $k \ge 0$,*

$$U_{k+1}(D) = \frac{n}{n+1}U_k(D) + \frac{1}{n+1}L_k(D), \tag{3.10}$$

where $L_k(D)$ and $U_k(D)$ are defined recursively as in (2.3) and (2.4).

Proof. We use a mathematical induction on $k \ge 0$. For $k = 0$, the assertion follows from (3.6) with (2.3). Assume that (3.10) is true for $k = p$. We have, with (3.8),

$$\begin{aligned} (n+1)U_{p+1}(D) &= \sum_{j=1}^{n+1} U_p(D_j) = \sum_{j=1}^{n+1}\Big(\frac{n}{n+1}U_{p-1}(D_j) + \frac{1}{n+1}L_{p-1}(D_j)\Big) \\ &= \frac{n}{n+1}\sum_{j=1}^{n+1} U_{p-1}(D_j) + \frac{1}{n+1}\sum_{j=1}^{n+1} L_{p-1}(D_j). \end{aligned}$$

By (3.8) again we deduce

$$U_{p+1}(D) = \frac{n}{n+1}U_p(D) + \frac{1}{n+1}L_p(D),$$

that is, (3.10) is true for $p+1$. This concludes the proof. □

We are now in a position to state the following result which ensures our claim.

Theorem 3.7. *The sequences $(L_k(D))_k$ and $(U_k(D))_k$ both converge with the same limit $M_f(D)$:*

$$\lim_{k\to\infty} L_k(D) = \sup_{k\ge 0} L_k(D) = M_f(D) = \inf_{k\ge 0} U_k(D) = \lim_{k\to\infty} U_k(D). \tag{3.11}$$

Further, the following estimate holds:

$$\forall k \ge 0, \quad 0 \le U_k(D) - M_f(D) \le \Big(\frac{n}{n+1}\Big)^k (U_0(D) - L_0(D)). \tag{3.12}$$

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PROOF. Following Proposition 3.5, the sequence $(L_k(D))_k$ is monotonic increasing and bounded above by $U_0(D)$ while the sequence $(U_k(D))_k$ is monotonic decreasing and bounded below by $L_0(D)$, so they both converge. According to (3.10) we obtain, by letting $k \to \infty$,

$$\lim_k U_{k+1}(D) = \frac{n}{n+1} \lim_k U_k(D) + \frac{1}{n+1} \lim_k L_k(D).$$

Simplifying the latter equality, we get $\lim_k L_k(D) = \lim_k U_k(D) := L(D)$. Now, letting $k \to \infty$ in (3.9), we immediately deduce that $L(D) = M_f(D)$.

We now prove (3.12). According to (3.10) again, we can write

$$U_{k+1}(D) - M_f(D) = \frac{n}{n+1}(U_k(D) - M_f(D)) + \frac{1}{n+1}(L_k(D) - M_f(D)),$$

which, with the help of Proposition 3.5, yields

$$0 \le U_{k+1}(D) - M_f(D) \le \frac{n}{n+1}(U_k(D) - M_f(D)).$$

The desired estimation follows by a simple mathematical induction on k, using the fact that $L_0(D) \le M_f(D)$. The proof of the theorem is complete. □

The relation (3.10) is very useful: it can be used for showing again that the limits of $(L_k(D))_k$ and $(U_k(D))_k$ coincide. Such a relation is also interesting in a practical sense for computing recursively the terms of $(U_k(D))_k$. See the next section for some examples. Further, (3.10) will be a good tool for deducing more interesting results as discussed below.

COROLLARY 3.8. *The inequalities*

$$0 \le M_f(D) - L_k(D) \le n\,(U_k(D) - M_f(D)) \tag{3.13}$$

hold true for every integer $k \ge 0$.

In what follows and for the sake of simplicity, we will omit the D in the iterative lower and upper bounds of $M_f(D)$ and write L_k, M_f, U_k.

PROOF. By Proposition 3.5 with (3.10), we have (for all $k \ge 0$)

$$M_f \le U_{k+1} = \frac{nU_k + L_k}{n+1},$$

or again,

$$0 \le M_f - L_k \le nU_k - nM_f = n(U_k - M_f),$$

which is the desired result. □

As already pointed out in the introduction, the particular case $k = 0$ in the above corollary was proved (in a different and longer way) in [10, pages 595–596].

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Corollary 3.9. *The two numerical series* $\sum_{k=0}^{\infty}(U_k - M_f)$ *and* $\sum_{k=0}^{\infty}(M_f - L_k)$ *both converge with the estimates*

$$\sum_{k=0}^{\infty}(M_f - L_k) \le n\sum_{k=0}^{\infty}(U_k - M_f) \le n(n+1)(U_0 - L_0), \tag{3.14}$$

and the relations

$$\sum_{k=0}^{\infty}(M_f - L_k) + \sum_{k=0}^{\infty}(U_k - M_f) = (n+1)(U_0 - M_f), \tag{3.15}$$

$$\sum_{k=0}^{\infty}(U_k - L_k) = (n+1)(U_0 - M_f). \tag{3.16}$$

Proof. Since $0 < n/(n+1) < 1$ we deduce from (3.12) that the series $\sum_{k=0}^{\infty}(U_k - M_f)$ converges with

$$\sum_{k=0}^{\infty}(U_k - M_f) \le (U_0 - L_0)\sum_{k=0}^{\infty}\Big(\frac{n}{n+1}\Big)^k = (n+1)(U_0 - L_0).$$

This, with (3.13), implies that the series $\sum_{k=0}^{\infty}(M_f - L_k)$ converges with

$$\sum_{k=0}^{\infty}(M_f - L_k) \le n\sum_{k=0}^{\infty}(U_k - M_f).$$

Summarising the above, inequalities (3.14) are completely proved.

Now, by virtue of (3.10), we can write

$$\sum_{k=0}^{\infty}(U_{k+1} - M_f) = \frac{n}{n+1}\sum_{k=0}^{\infty}(U_k - M_f) - \frac{1}{n+1}\sum_{k=0}^{\infty}(M_f - L_k),$$

or equivalently,

$$\sum_{k=0}^{\infty}(U_k - M_f) - (U_0 - M_f) = \frac{n}{n+1}\sum_{k=0}^{\infty}(U_k - M_f) - \frac{1}{n+1}\sum_{k=0}^{\infty}(M_f - L_k).$$

The desired relationship (3.15) follows from the latter equality by a simple reduction.

Since the two series above converge, we can write

$$\sum_{k=0}^{\infty}(M_f - L_k) + \sum_{k=0}^{\infty}(U_k - M_f) = \sum_{k=0}^{\infty}((M_f - L_k) + (U_k - M_f)) = \sum_{k=0}^{\infty}(U_k - L_k).$$

Relation (3.16) follows by combining the latter equality with (3.15). The proof of the corollary is complete. □

Remark 3.10. Relation (3.11) states that M_f can be expressed as the common limit of the sequences $(L_k)_k$ and $(U_k)_k$ of iterates, while (3.16) states that

$$M_f = U_0 - \frac{1}{n+1}\sum_{k=0}^{\infty}(U_k - L_k),$$

that is, M_f is determined from the series $\sum_{k=0}^{\infty}(U_k - L_k)$ whose general term is the difference between the above iterated estimates of M_f.

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4. Two examples

Let E be the canonical $(n+1)$-simplex of $\mathbb{R}^n$ with vertices $0, e_1, e_2, \ldots, e_n$ where $(e_1, e_2, \ldots, e_n)$ denotes the canonical basis of $\mathbb{R}^n$, that is,

$$E = \Big\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i \leq 1 \text{ and } x_i \geq 0 \ \forall i\Big\}.$$

Recalling that $|E| = (n!)^{-1}$, (HHI) yields

$$f\Big(\frac{e}{n+1}\Big) \leq n! \int_E f(x)\, dx \leq \frac{1}{n+1}\Big(f(0) + \sum_{i=1}^{n} f(e_i)\Big), \tag{4.1}$$

where $e := \sum_{i=1}^{n} e_i = (1, 1, 1, \ldots, 1)$. We will consider some typical situations by choosing appropriate convex functions.

4.1. The case where f is a power-norm. Let $p \geq 1$ be a real number and take $f(x) = \|x\|^p$ in (4.1):

$$L_0(E) := \frac{\|e\|^p}{(n+1)^p} \leq n! \int_E \|x\|^p\, dx \leq \frac{1}{n+1} \sum_{i=1}^{n} \|e_i\|^p := U_0(E).$$

Following (3.11) or (3.7) (after a simple reduction),

$$U_1(E) = \frac{n}{(n+1)^2} \sum_{i=1}^{n} \|e_i\|^p + \frac{\|e\|^p}{(n+1)^{p+1}},$$

and by (3.6) (after a long but elementary computation and reduction),

$$L_1(E) = \frac{(n+2)^p}{(n+1)^{2p+1}} \|e\|^p + \frac{1}{(n+1)^{2p+1}} \sum_{i=1}^{n} \|(n+2)e - (n+1)e_i\|^p.$$

If the norm $\|.\|$ is symmetric in $x_1, x_2, \ldots, x_n$, as the three classical norms of $\mathbb{R}^n$ are, then the above expressions reduce to

$$U_0(E) = \frac{n}{n+1}\|e_1\|^p, \quad U_1(E) = \frac{n^2}{(n+1)^2}\|e_1\|^p + \frac{\|e\|^p}{(n+1)^{p+1}},$$
$$L_1(E) = \frac{(n+2)^p}{(n+1)^{2p+1}}\|e\|^p + \frac{n}{(n+1)^{2p+1}}\|(n+2)e - (n+1)e_1\|^p.$$

4.2. The case where f is power-quadratic. Let $A = (a_{ij})$ be a real or complex (self-adjoint) positive matrix of size n. For a fixed real number $\alpha > 0$, we set

$$f_\alpha(x) = (\langle Ax, x\rangle)^\alpha = \Big(\sum_{i,j=1}^{n} a_{ij} x_i x_j\Big)^\alpha$$

for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, where $\langle x, y\rangle$ is the standard inner product of $\mathbb{R}^n$.

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PROPOSITION 4.1. *If $\alpha \geq 1/2$ then f_α is convex on $\mathbb{R}^n$.*

PROOF. Since A is (self-adjoint) positive, it follows that $\langle Ax, x\rangle = \|A^{1/2}x\|^2$, where $A^{1/2}$ denotes the matrix root of A and $\|\cdot\|$ the Euclidian norm of $\mathbb{R}^n$. The desired result follows after elementary manipulation. □

Assuming that $\alpha \geq 1/2$ in what follows, we can apply (HHI) for f_α on $E \subset \mathbb{R}^n$ and (4.1) yields

$$0 \leq \frac{(\langle Ae, e\rangle)^\alpha}{(n+1)^{2\alpha}} \leq (n!) \int_E (\langle Ax, x\rangle)^\alpha \, dx \leq \frac{\sum_{i=1}^n (\langle Ae_i, e_i\rangle)^\alpha}{n+1}.$$

It is easy to see that $\langle Ae_i, e_i\rangle = a_{ii}$ and $\langle Ae, e\rangle = \sum_{i,j=1}^n a_{ij}$, and so the above double inequality becomes

$$0 \leq L_0(E) := \frac{(\sum_{i,j=1}^n a_{ij})^\alpha}{(n+1)^{2\alpha}} \leq (n!) \int_E \bigg(\sum_{i,j=1}^n a_{ij}x_i x_j\bigg)^\alpha dx \leq \frac{\sum_{i=1}^n (a_{ii})^\alpha}{n+1} := U_0(E).$$

We leave to the reader the routine task of computing the corresponding $L_1(E)$ and $U_1(E)$ via (3.5) and (3.6), respectively.

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