AN OVERVIEW OF STELLAR PULSATION THEORY

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Abstract. In this paper I will give overview of the stellar pulsation theory. Starting with basic equations I will discuss modal properties of oscillations and excitation mechanisms. I also mention briefly the effects of rotation.

1. Introduction

The pulsation is a global oscillation of a star. If the oscillation is spherically symmetric, it is called radial pulsation. One the other hand, if the oscillation does not have spherical symmetry, it is called nonradial oscillation or non-radial pulsation. Observationally, giant stars tend to pulsate radially, while main-sequence stars and degenerate stars tend to pulsate nonradially. (But there are exceptions, of course.)

Since the stellar pulsations are eigen-oscillations of the star, the pulsation frequencies contain information about the interior structure of the star. One of the classical applications of this fact is the well known period-luminosity relation of the Cepheids. The relation comes from the fact that the fundamental period is inversely proportional to the mean density of the star. More recently, the frequencies of nonradial pulsations of the sun and white dwarfs were used to infer their interior structure.

Although pulsations with finite amplitudes are nonlinear phenomena, the pulsation periods are well approximated by those from the linear analysis. Furthermore, the linear analysis tells us which star should pulsate because stellar pulsations are, in most cases, self-excited by vibrational instability (or overstability).

We start the discussion presenting the general basic equations for stellar oscillations. Then we discuss general properties of adiabatic pulsations and possible excitation mechanisms for stellar pulsations. And finally we mention briefly the effect of rotation on the pulsation frequencies.

The presentation of the material is intended for non-experts in the field. Since only basic facts are to be discussed, readers who wish to learn more details are recommended to consult a textbook; *e.g.*, Cox (1980) or Unno *et al.* (1989).

Astrophysics and Space Science **210**: 61–72, 1993.

2. Basic Equations

2.1. Equations Governing a Self-gravitating Fluid

The basic equations for the stellar pulsations are the equations of hydrodynamics with self-gravity. The continuity equation may be written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{1}$$

where ρ is the density, and $\boldsymbol{u}$ is the fluid velocity caused by oscillation and rotation in general. The momentum equation may be written as

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p - \nabla \psi - \overline{\boldsymbol{V}\nabla \cdot \boldsymbol{V}} + \frac{1}{4\pi\rho}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}, \tag{2}$$

where p is the pressure, ψ the gravitational potential, $\boldsymbol{B}$ the magnetic field and $\boldsymbol{V}$ the velocity of turbulent convection. The overbar means a small scale average. Poisson's equation for the gravitational potential is written as

$$\nabla^2 \psi = 4\pi G \rho \tag{3}$$

with the gravitational constant G. The conservation of thermal energy may be written as

$$\frac{dE}{dt} - \frac{p}{\rho^2}\frac{d\rho}{dt} = \epsilon - \frac{1}{\rho}\nabla \cdot (\boldsymbol{F}_R + \boldsymbol{F}_C), \tag{4}$$

where E is the internal energy per unit mass, ϵ the nuclear energy generation rate per unit mass, $\boldsymbol{F}_C$ the energy flux carried by convection and $\boldsymbol{F}_R$ the radiative flux which may be written as

$$\boldsymbol{F}_R = -\frac{4ac}{3\kappa\rho}T^3 \nabla T \tag{5}$$

by using a diffusion approximation, where a is the radiation constant, c the speed of light and κ the radiative opacity.

For magnetic fields we may use the MHD approximation, in which the evolution of magnetic fields is governed by the equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}). \tag{6}$$

To close equations we need an equation of state which gives a relation among p, ρ and T, and the radiative opacity being expressed as a function of ρ, T and chemical composition. Also, quantities related to convection $\overline{\boldsymbol{V} \cdot \nabla \boldsymbol{V}}$ and $\boldsymbol{F}_C$ must be given, which is the most complicated part of the theory (see e.g., Unno and Xiong 1992 these proceedings). In most cases the effect of convection is neglected.

2.2. Linearized Perturbation Equations

Let us consider a small perturbation around the equilibrium state. The displacement vector $\boldsymbol{\xi}$ is defined as $\boldsymbol{\xi} \equiv \boldsymbol{r} - \boldsymbol{r}_0$, where the subscript 0 denotes a quantity at its equilibrium position. Then, the Lagrangian perturbation of velocity, $\delta \boldsymbol{u}$ may be written as

$$\delta \boldsymbol{u} \equiv \boldsymbol{u}(\boldsymbol{r}_0 + \boldsymbol{\xi}) - \boldsymbol{u}(\boldsymbol{r}_0) = \frac{d\boldsymbol{\xi}}{dt} = \frac{\partial \boldsymbol{\xi}}{\partial t} + \boldsymbol{u}_0 \cdot \nabla \boldsymbol{\xi}. \tag{7}$$

On the other hand, the Eulerian perturbation of velocity, $\boldsymbol{u}'$, is expressed as

$$\boldsymbol{u}' \equiv \boldsymbol{u}(\boldsymbol{r}) - \boldsymbol{u}_0(\boldsymbol{r}) = \delta \boldsymbol{u} - \boldsymbol{\xi} \cdot \nabla \boldsymbol{u}_0. \tag{8}$$

In the following part of this paper, we disregard magnetic fields for the sake of simplicity, and use spherical coordinates (r, θ, ϕ), in which the axis of $\theta = 0$ is equal to the axis of rotation. Furthermore, we assume that a possible velocity field in the equilibrium state is only due to a uniform rotation (*i.e.*, $\boldsymbol{u}_0 = \boldsymbol{\Omega} \times \boldsymbol{r}$) and that the equilibrium state is axisymmetric. The temporal and azimuthal dependence of the perturbed quantities may then be written as $\exp[i(\sigma t + m\phi)]$. The perturbation equations for eqs. (1)–(5) can be expressed as follows:

$$\rho' + \nabla \cdot (\rho \boldsymbol{\xi}) = 0, \tag{9}$$

$$-(\sigma + m\Omega)^2 \boldsymbol{\xi} + 2i(\sigma + m\Omega)(\boldsymbol{\Omega} \times \boldsymbol{\xi}) = -\frac{1}{\rho}\nabla p' + \frac{\rho'}{\rho^2}\nabla p - \nabla \psi' + (\overline{\boldsymbol{V} \cdot \nabla \boldsymbol{V}})', \tag{10}$$

$$\nabla^2 \psi' = 4\pi G \rho', \tag{11}$$

$$i\sigma\rho T \delta S = \rho\epsilon \left(\frac{\delta\rho}{\rho} + \frac{\delta\epsilon}{\epsilon}\right) - \delta(\nabla \cdot \boldsymbol{F}_R + \nabla \cdot \boldsymbol{F}_C), \tag{12}$$

and

$$\boldsymbol{F}_R' = \boldsymbol{F}_R \left(3\frac{T'}{T} - \frac{\kappa'}{\kappa} - \frac{\rho'}{\rho}\right) - \frac{4acT^3}{3\kappa\rho}\nabla T', \tag{13}$$

where δ and the prime ($'$) indicate Lagrangian and Eulerian perturbations, respectively. The entropy perturbation is denoted by S.

3. Adiabatic Oscillations of Nonrotating Spherical Stars

3.1. General Discussion

In many cases, the pulsation period of a star is much shorter than the thermal timescale of the envelope and therefore the nonadiabatic effects are small. In the adiabatic approximation, the governing equations become considerably simple because we can use the adiabatic relation

$$\frac{\delta\rho}{\rho} = \frac{1}{\Gamma_1}\frac{\delta p}{p} \tag{14}$$

instead of using the energy conservation and flux equations [Eqs. (12), (13)], where the adiabatic exponent Γ_1 is defined by $\Gamma_1 \equiv (\partial \ln p/\partial \ln \rho)_S$. [We also disregard the effect of convection in Eq. (10).] The governing equations for the perturbations in a nonrotating spherically symmetric star may then be written symbolically as

$$-\sigma^2\boldsymbol{\xi} + \mathcal{L}(\boldsymbol{\xi}) = 0, \tag{15}$$

where $\mathcal{L}$ is a Hermitian operator if the pressure vanishes at the stellar surface. (This is a good approximation for real and hence Hermiticity remains approximately true.) The explicit form of $\mathcal{L}$ is given in e.g., §15.2 of Cox (1980). Because $\mathcal{L}$ is Hermitian, all eigenvalues σ^2 are real, and eigenfunctions associated with different eigenvalues are orthogonal to one another; *i.e.*,

$$\int \boldsymbol{\xi}_i^* \cdot \boldsymbol{\xi}_j \rho d^3\mathbf{x} = 0 \qquad \text{if} \quad \sigma_i^2 \neq \sigma_j^2. \tag{16}$$

Since σ^2 is real, the temporal behavior of the adiabatic perturbations is purely oscillatory when $\sigma^2 > 0$ or monotonic when $\sigma^2 < 0$ (dynamical instability).

We may write the term $\mathcal{L}(\boldsymbol{\xi})$ in Eq. (15) in the form

$$\mathcal{L}(\boldsymbol{\xi}) = f_1\hat{\mathbf{e}}_r + \nabla f_2, \tag{17}$$

where $\hat{\mathbf{e}}_r$ is the unit vector in the radial direction, and f_1 and f_2 consist of terms proportional to ξ_r or $\nabla_\perp \cdot \boldsymbol{\xi}_\perp$. Here $\boldsymbol{\xi}_\perp$ and $\nabla_\perp$ are defined as

$$\boldsymbol{\xi}_\perp \equiv \xi_\theta\hat{\mathbf{e}}_\theta + \xi_\phi\hat{\mathbf{e}}_\phi \quad \text{and} \quad \nabla_\perp \equiv \hat{\mathbf{e}}_\theta\frac{\partial}{\partial\theta} + \frac{\hat{\mathbf{e}}_\phi}{\sin\theta}\frac{\partial}{\partial\phi}.$$

Combining Eq. (17) with Eq. (15), we see that the angular dependence of ξ_r and $\boldsymbol{\xi}_\perp$ can be specified by using a single spherical harmonic $Y_l^m(\theta,\phi)$ as

$$\xi_r \propto Y_l^m(\theta,\phi) \quad \text{and} \quad \boldsymbol{\xi}_\perp \propto \nabla_\perp Y_l^m(\theta,\phi),$$

because a spherical harmonic $Y_l^m(\theta,\phi)$ is an eigenfunction of the operator $\nabla_\perp^2$, *i.e.*,

$$\nabla_\perp^2 Y_l^m(\theta,\phi) = -l(l+1)Y_l^m(\theta,\phi), \tag{18}$$

and hence

$$\nabla_\perp \cdot \boldsymbol{\xi}_\perp \propto l(l+1)Y_l^m.$$

Because of the above property of the spherical harmonics, the angular dependence of perturbed quantities can be expressed by a *single* $Y_l^m(\theta,\phi)$. Perturbed scalar variables are proportional to Y_l^m and the displacement vector is written as

$$\boldsymbol{\xi} = \left[\xi_r \hat{\mathrm{e}}_r + \xi_h \left(\hat{\mathrm{e}}_\theta \frac{\partial}{\partial\theta} + \hat{\mathrm{e}}_\phi \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)\right] Y_l^m(\theta,\phi) e^{i\sigma t}. \tag{19}$$

The governing equations are then reduced to differential equations of the radial coordinate only. (This property holds even for nonadiabatic oscillations; see e.g., Unno *et al.* 1989 §13.)

The effects of the angular dependence of eigenfunctions enter into the governing equations only through the terms proportional to $l(l+1)$, which is related to the horizontal wave number of the oscillation, $\sqrt{l(l+1)}/r$. Since the azimuthal order $m(-l < m < l)$ does not appear in the governing equations, the eigenvalues (*i.e.*, oscillation frequencies) are $(2l+1)$-fold degenerate.

3.2. Toroidal Displacements

Inspecting Eqs. (15) and (17), we notice that these equations are also satisfied if

$$\sigma = 0, \quad \xi_r = 0, \quad \text{and} \quad \nabla_\perp \cdot \boldsymbol{\xi}_\perp = 0.$$

The last two relations are satisfied if we assume that displacements are toroidal; *i.e.*,

$$\boldsymbol{\xi} \propto \nabla_\perp \times (\hat{\mathrm{e}}_r Y_l^m). \tag{20}$$

The eigenfrequency is zero, because such toroidal displacements of a fluid do not cause any effect on the spherical equilibrium structure if there is no rotation or no magnetic field. Such modes of displacement are called trivial modes. [The oscillation modes whose displacement vectors have a form of Eq. (19) are sometimes called spheroidal modes.] If the star rotates, toroidal displacements describe Rossby waves with finite oscillation frequencies.

3.3. Radial Pulsations

Oscillations for $l = 0$ are spherically symmetric, *i.e.* radial pulsation. In this case Eq. (15) is reduced to

$$-\sigma^2 \xi_r - \frac{1}{r^4\rho}\left(\Gamma_1 p r^4 \frac{d\xi}{dr}\right) - \frac{1}{\rho r}\left\{\frac{d}{dr}[(3\Gamma_1 - 4)p]\right\}\xi_r = 0.$$

This equation with boundary condition ($\xi_r = 0$ at the center and $\delta p = 0$ at the surface) forms a Strum-Liouville type eigenvalue problem with the eigenvalue σ^2. Let the eigenvalue associated with the eigenfunction with n

https://doi.org/10.1017/S0252921100013932 Published online by Cambridge University Press

nodes be represented by σ_n^2. The eigenvalues are ordered as $\sigma_0^2 < \sigma_1^2 < \sigma_2^2 < \ldots$. Thus the period of oscillation decreases as the number nodes increases, which is interpreted that the period is approximately the sound travel time between two adjacent nodes (Hansen 1972). From the above equation and the variational property of the eigenvalues the inequalities

$$(3\bar{\Gamma}_1 - 4)(-E_{grav}/I) > \sigma_0^2 > (3\bar{\Gamma}_1 - 4)4\pi G\bar{\rho}/3$$

can be derived, where $\bar{\rho}$ is the mean density, $\bar{\Gamma}_1$ an average of the adiabatic exponent Γ_1, E_{grav} the gravitational potential energy and I the moment of inertia of the star. (see §8.9, 8.10 in Cox 1980 for a derivation and discussion). If $\bar{\Gamma}_1 < 4/3$, at least the fundamental mode is dynamically unstable ($\sigma_0^2 < 0$). We note that since $(-E_{grav}/I)$ is proportional to the mean density of the star, the period of the fundamental mode $(2\pi/\sigma_0)$ is inversely proportional to the square of mean density, which states the period-mean density relation of pulsation.

3.4. p-MODES AND g-MODES

Let us now discuss the properties of nonradial pulsations. In order to make the discussion simple, let us disregard the Eulerian perturbations ψ' in this subsection. This approximation is called the Cowling approximation, and provides a good description of higher order modes. Using Eq. (19), and neglecting the Eulerian perturbation of gravitational potential, we obtain

$$\frac{1}{r^2}\frac{d}{dr}(r^2\xi_r) - \frac{g}{c_s^2}\xi_r + \left(1 - \frac{L_l^2}{\sigma^2}\right)\frac{p'}{\rho c_s^2} = 0, \tag{21}$$

$$\frac{1}{\rho}\frac{dp'}{dr} + \frac{g}{\rho c_s^2}p' + (N^2 - \sigma^2)\xi_r = 0, \tag{22}$$

where L_l and N are, respectively, the Lamb frequency and the Brunt-Väisälä frequency defined as

$$L_l^2 \equiv \frac{l(l+1)c_s^2}{r^2} \quad \text{and} \quad N^2 \equiv g\left(\frac{1}{\Gamma_1}\frac{d\ln p}{dr} - \frac{d\ln\rho}{dr}\right), \tag{23}$$

and g is the local gravitational acceleration and c_s the sound speed.

The two frequencies given in Eq. (23) play essential roles charaterizing noradial oscillations. This can be seen in a local analysis. Let us assume that

$$\xi_r, \quad p' \propto \exp(ik_r r),$$

where $|k_r| \gg 1$. Substituting these expressions into Eqs. (21) and (22), we obtain the dispersion relation

$$k_r^2 \simeq \frac{(\sigma^2 - L_l^2)(\sigma^2 - N^2)}{\sigma^2 c_s^2}. \tag{24}$$

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The oscillation propagates in the radial direction when k_r is real. The dispersion relation shows that two types of oscillations are possible: one is called p-mode which is a propagating wave when $\sigma^2 > L_l^2, N^2$, and the other type is called g-mode which is a propagating when $\sigma^2 < L_l^2, N^2$. For extreme cases k_r^2 of p-modes increases as σ^2/c_s^2 increases, while k_r^2 of g-modes increases as $l(l+1)N^2/(\sigma^2 r^2)$ increases. The restoring force for the p-modes is the pressure force just as for the radial pulsations. Therefore the p-modes are the relatives to radial pulsations and are sound waves influenced by the gravitational field of the star. On the other hand, the restoring force for the g-modes is the buoyancy force, which works only for non-spherical symmetric perturbations.

The global oscillations of a star have a discrete spectrum of frequencies. In a simple stellar model such as a zero-age main-sequence model, the frequency range for p-modes is well separated from and higher than the frequency range for g-modes. The Lamb frequency for a given value of l monotonically decrease outward, while the Brunt-Väisälä frequency increases outward. At some zone in the star, these two frequencies have a same value, which approximately divides between the p-mode and g-mode frequency ranges (see Fig. 15.2 in Unno *et al.* 1989). The propagation zone of p-modes is in the envelope where the amplitude is large, while the propagation zone of g-modes is in the core. (However, the loci of propagation zones are opposite in white dwarfs.) The frequencies of p-modes increases as the number of nodes increases, while the frequencies of g-modes decreases as the number of nodes increases. Between the lowest order g- and p-modes ($n = 1$) for a given degree l (larger than one) there exists a mode called f-mode which has no node in the radial direction in the amplitude distribution. (See Fig. 17.2 in Cox 1980, or Fig. 14.1 in Unno *et al.* 1989. But nodes appear in a centrally concentrated model.)

As the central concentration of the star gradually increases with evolution, the Brunt-Väisälä frequency in the core and hence the frequencies of g-modes increase. When the frequency of a g-mode approaches and exceeds the frequency of the f- or a p-mode, the two frequencies undergo an 'avoided crossing' (see Fig. 15.7 of Unno *et al.*), because two different modes with a same l cannot have the same frequency. In an evolved star, the frequency range of g-modes overlaps the frequency range of p-modes. Any mode whose frequency is in the overlapping range has two propagation zones in the star; p-mode propagation zone in the envelope and g-mode propagation zone in the core. Although such a mode has dual character, the overall character may be distinguished by inspecting which propagation zone traps more pulsation energy.

In a highly evolved star, the Brunt-Väisälä frequency is extremely high in the core so that for any nonradial pulsation with a moderate frequency $N^2 \gg \sigma^2$ in the core. Then, as discussed above, the radial wave number is extremely high in the core, which means that the eigenfunction shows a rapid spatial oscillation in the core. Since the short wavelength of the spatial oscillation of the eigenfunction causes thermal dissipation of pulsation energy, it is difficult to excite a nonradial oscillation in a giant star.

For oscillations with a large radial order n, there exist asymptotic formulations for the frequencies (Tassoul 1980, see also Unno *et al.* 1989);

$$\sigma \simeq \pi(n + l/2)\left(\int_0^R \frac{1}{c} dr\right)^{-1} \tag{25}$$

for p-modes and

$$\sigma \simeq \frac{[l(l+1)]^{1/2}}{n\pi} \int_{r_a}^{r_b} \frac{N}{r} dr \tag{26}$$

for g-modes, where $N^2 > 0$ in a zone of $r_a < r < r_b$. We note that the separations of *frequencies* of high order p-modes is equidistant, while the separation of *periods* ($= 2\pi/\sigma$) of high order g-modes is equidistant.

4. Excitation Mechanism

4.1. Energy Equation

In discussing the excitation of pulsations, we will use, for the sake of simplicity, the quasi-adiabatic approximation, in which the entropy perturbation is evaluated from Eq. (12) (energy conservation) using the adiabatic relation and adiabatic eigenfunctions in the right hand side of the equation. This approximation gives reasonable results only when nonadiabaticity is small.

After some manipulations using Eqs. (9)–(12) and (14) we obtain

$$\frac{dE_W}{dt} = \int_0^M \delta T \frac{d\delta S}{dt} = \int_0^M \delta T \delta \left[\epsilon - \frac{1}{\rho} \nabla \cdot (\boldsymbol{F}_R + \boldsymbol{F}_C)\right] dM_r, \tag{27}$$

where

$$E_W = \frac{1}{2} \int \left[(\boldsymbol{u}')^2 + \left(\frac{p'}{\rho c_s}\right)^2 + \frac{g^2}{N^2}\left(\frac{p'}{\Gamma_1 \rho} - \frac{\rho'}{\rho}\right)^2 + \frac{\rho'}{\rho}\psi' \right] dM_r, \tag{28}$$

which represents the kinetic potential energies of the pulsation. [In deriving the above equations we disregarded the effect of rotation and convection in Eq. (10).] Equation (27) shows the change in pulsation energy caused by the interactions with nuclear energy generation rate and with the energy flux.

Integrating over one cycle of pulsation, we obtain an expression for the work integral of pulsation;

$$\begin{aligned} W &= \oint dt \frac{dE_W}{dt} \\ &= \frac{\pi}{\sigma} \int_0^M dM_r \left[\frac{\delta T_r}{T} \delta \epsilon_r - \frac{\delta T_r}{T} \delta \left(\frac{1}{\rho} \nabla \cdot \boldsymbol{F}_R\right)_r - \frac{\delta T_r}{T} \delta \left(\frac{1}{\rho} \nabla \cdot \boldsymbol{F}_C\right)_r \right], \end{aligned} \tag{29}$$

https://doi.org/10.1017/S0252921100013932 Published online by Cambridge University Press

where the temporal and angular dependence of the perturbed quantities are assumed as

$$\delta T = Re[\delta T_r Y_l^m e^{i\sigma t}], \quad \delta\epsilon = Re[\delta\epsilon_r Y_l^m e^{i\sigma t}],$$
$$\delta\left(\frac{1}{\rho}\nabla\cdot\boldsymbol{F}\right) = Re\left[\delta\left(\frac{1}{\rho}\nabla\cdot\boldsymbol{F}\right)_r Y_l^m e^{i\sigma t}\right], \ldots$$

with $Re(\ldots)$ stands for the real part of the indicated quantity. The subscript r is attached to indicate that the quantity is a function of r only. All the quantities with subscript r are real since we are using the quasi-adiabatic approximation. When the work W is positive, the pulsation energy grows over one pulsation cycle, *i.e.* the pulsation is excited (overstable). Let us examine each term in the integrand of Eq. (29).

4.2. *Epsilon*-Mechanism

The first term of the right hand side of Eq. (29) corresponds to the driving by the *epsilon*-mechanism. After some manipulations we may write

$$\int_0^M dM_r \frac{\delta T_r}{T}\delta\epsilon_r = \int_0^M dM_r \epsilon\left(\epsilon_R + \frac{\epsilon_\rho}{\Gamma_3 - 1}\right)\left(\frac{\delta T_r}{T}\right)^2, \tag{30}$$

where $\epsilon_T \equiv (\partial\ln\epsilon/\partial\ln T)_\rho$ ($\sim 4 - 30$), $\epsilon_\rho \equiv (\partial\ln\epsilon/\partial\ln\rho)_T$ ($\sim 1 - 2$), and $\Gamma_3 - 1 \equiv (\partial\ln T/\partial\ln\rho)_S$ ($\sim 2/3$). Thus this term always gives a positive contribution to the work integral W. Physically, this term may be understood as follows: In the compression phase, temperature and hence nuclear energy generation rate are higher than in the equilibrium condition so that matter gains thermal energy. Therefore, the amplitude in the next expansion phase is larger than the previous one. In the expansion phase the nuclear energy generation rate is lower than the equilibrium value and hence the matter loses its thermal energy. Therefore, the amplitude in the next compression phase is higer than the previous one. In this way, the amplitude of pulsation gradually increases as pulsation goes on.

However, the amplitude of pulsation is so small in the nuclear burning region that the *epsilon*-mechanism is too weak to excite pulsation in most stars except very massive ($M \gtrsim 100M_\odot$) main-sequence stars.

4.3. *Kappa*-Mechanism

The second term on the right hand side of Eq. (29) may be written as

$$-\int_0^M \frac{\delta T_r}{T}\delta\left(\frac{1}{\rho}\nabla\cdot\boldsymbol{F}_R\right)_r \sim \int_0^R \left(\frac{\delta T_r}{T}\right)^2 \frac{d}{dr}\left[\left(\kappa_T + \frac{\kappa_\rho}{\Gamma_3 - 1}\right) L_R\right] + \ldots (31)$$

where $\kappa_T = (\partial\ln\kappa/\partial\ln T)_\rho$ and $\kappa_\rho = (\partial\ln\kappa/\partial\ln\rho)_T$. This integral corresponds to the *kappa*-mechanism of driving. If radiative luminosity L_R is constant as in a radiative envelope, a region in which

$$\frac{d}{dr}\left(\kappa_T + \frac{\kappa_\rho}{\Gamma_3 - 1}\right) > 0 \tag{32}$$

helps to drive pulsation.

Let us discuss the reason for the *kappa* mechanism driving. When the above condition is satisfied, the opacity perturbation in the compressed phase of pulsation increases outward so that the radiative luminosity is blocked. Then, the zone gains thermal energy in the compressed phase. On the other hand the zone loses thermal energy in the expanding phase. As discussed previously for the *epsilon* mechanism, the pulsation energy tends to grow in this zone.

The *kappa* mechanism is responsible for pulsations of the stars in the cepheid instability strip. Also, it is recently found that the pulsation of the β-Cephei variables is excited by the *kappa* mechanism corresponding to the opacity peak caused by metal ions in a zone with a temperature of around 2×10^5K (e.g., Moskalik and Dziembowski 1992).

We note that when nonadiabaticity is very large the above excitation must be changed and complex phenomena occur (see Gautschy and Glatzel 1990).

4.4. Effect of Convection

The third term on the right hand side of Eq. (29) represents the effect of the perturbation of the convective flux. Since convection is turbulent, it is very difficult to evaluate this term correctly (see e.g. Unno and Xiong 1992). Moreover, the effect of convection also appears in the momentum equation (10), where the same difficulty applies. In most calculations the influence of convection on the stability of the stars is completely neglected. They give reasonable results for blue stars, where convection is weak, for example, the blue edge of the Cepheid instability strip and the β-Cephei variables. However, it is necessary to include the effect of convection in order to obtain the red edges of instability regions and to study the pulsations of red variables such as the Mira variables, where convection play an important role.

5. Effects of Rotation

For a rotating star the governing equations for adiabatic oscillations [Eqs. (9)–(11)] may be reduced to

$$-(\sigma + m\Omega)^2\boldsymbol{\xi} + 2i(\sigma + m\Omega)\boldsymbol{\Omega} \times \boldsymbol{\xi} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{\xi}) + \mathcal{L}(\boldsymbol{\xi}) = 0. \tag{33}$$

Although this equation is also correct for differentially rotating stars, we only consider the case of uniform rotation. Taking the scalar product with $\boldsymbol{\xi}^*$ and integrating over the whole volume we obtain

$$-(\sigma + m\Omega)^2 a + (\sigma + m\Omega)b + c = 0, \tag{34}$$

where

$$a \equiv \int_V \boldsymbol{\xi}^* \cdot \boldsymbol{\xi}\rho d^3\mathbf{x}, \qquad b \equiv 2i \int_V \boldsymbol{\xi}^* \cdot (\boldsymbol{\Omega} \times \boldsymbol{\xi})\rho d^3\mathbf{x},$$

and

$$c \equiv \int_V \boldsymbol{\xi}^* \cdot [\mathcal{L}(\boldsymbol{\xi}) + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{\xi})]\rho d^3\mathbf{x}.$$

The quantities, a, b, and c can be proved to be real (Lynden-Bell and Ostriker 1967). Solving the above equation we obtain

$$\sigma + m\Omega = \frac{1}{2a}\left(b \pm \sqrt{b^2 + 4ac}\right). \tag{35}$$

For a non-rotating star ($\Omega = 0$), Eq. (35) gives

$$\sigma = \pm\sqrt{\frac{c}{a}} \equiv \pm\sigma_0.$$

We note that the sign of the frequency is physically not important because it changes only the phase of pulsation. When rotation is slow ($\sigma_0 \gg \Omega$), $4ac \gg b^2$. Then

$$\sigma = \pm\sqrt{\frac{c}{a}} + \frac{1}{2}\frac{b}{a} - m\Omega + O(\Omega^2) = \pm\sigma_0 - m\Omega(1 - C_{nl}) + O(\Omega^2).$$

Thus rotation completely lifts the $(2l + 1)$ fold degeneracy.

When the rotation frequency Ω is comparable to or larger than σ_0, which is expected for high order g-modes, eigenfunctions and eigenvalues are considerably modified from those for non-rotating stars and new features arise (see e.g., Ch. 6 in Unno *et al.* 1989, or Saio and Lee 1991).

A special case is the toroidal displacements given in Eq. (20), for which $c = O(\Omega^3)$ and hence $\sigma_0 = 0$. For this case Eq. (34) leads to

$$\sigma + m\Omega = 0 \quad \text{or} \quad 2m\Omega/[l(l+1)].$$

(The next terms are of the order Ω^3.) The latter frequency corresponds to the global Rossby waves (or planetary waves).

Acknowledgements

The author is very grateful to A. Gautschy for helpful discussions and his hospitality.

https://doi.org/10.1017/S0252921100013932 Published online by Cambridge University Press

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