J. Aust. Math. Soc. **88** (2010), 353–361
doi:10.1017/S1446788710000212

NONDECREASING FUNCTIONS, EXCEPTIONAL SETS AND GENERALIZED BOREL LEMMAS

R. G. HALBURD and R. J. KORHONEN✉

(Received 11 June 2009; accepted 18 March 2010)

Communicated by A. M. Hassell

Abstract

According to the classical Borel lemma, any positive nondecreasing continuous function T satisfies $T(r+1/T(r)) \leq 2T(r)$ outside a possible exceptional set of finite linear measure. This lemma plays an important role in the theory of entire and meromorphic functions, where the increasing function T is either the logarithm of the maximum modulus function, or the Nevanlinna characteristic. As a result, exceptional sets appear throughout Nevanlinna theory, in particular in Nevanlinna's second main theorem. In this paper, we consider generalizations of Borel's lemma. Conversely, we consider ways in which certain inequalities can be modified so as to remove exceptional sets. All results discussed are presented from the point of view of real analysis.

2000 *Mathematics subject classification*: primary 26A48; secondary 26A12, 30D35.
Keywords and phrases: Borel, exceptional set, nondecreasing functions, Nevanlinna.

1. Introduction

Nondecreasing functions appear in many contexts in analysis; in particular, they appear naturally in the theory of entire and meromorphic functions. Much information about the value distribution of an entire function f is encoded in the asymptotic behaviour of the real-valued nondecreasing maximum modulus function $M_f(r) := \max_{|z|=r} |f(z)|$ as $r \to \infty$. In the case of a meromorphic function, the role of $\log M_f(r)$ is played by the Nevanlinna characteristic $T_f(r)$, which contains information about the distribution of poles of f in $|z| \leq r$, as well as information about how large $|f|$ is on the circle $|z| = r$. The asymptotic behaviour of the nondecreasing function $T_f(r)$ contains information regarding the number of asymptotic directions of f as well as the form of certain types of product representations (the Weierstrass and Hadamard factorizations).

The first author's research was partially supported by an EPSRC Advanced Research Fellowship and a project grant from the Leverhulme Trust. The second author's research was partially supported by the Academy of Finland (grant no. 118314).

Recall that for any nonrational meromorphic function f, Picard's great theorem says that f takes every value in $\mathbb{C} \cup \{\infty\}$ infinitely many times, with at most two exceptions. The centrepiece of Nevanlinna theory is Nevanlinna's second main theorem [6], which is a vast generalization and quantification of Picard's theorem for meromorphic functions. In 1943, Weyl referred to the appearance of [6] as 'one of the few great mathematical events in our century' [8]. Nevanlinna's second main theorem provides a useful bound on $T_f(r)$ in terms of quantities that are readily interpreted. However, this bound only holds for r outside some possible exceptional set E of finite linear measure (that is, $\int_E dr < \infty$). The origin of this exceptional set is in an estimate of the logarithmic derivative f'/f, which in turn uses the following lemma due to Borel [1] (see also Hayman [3]). Borel's lemma has been generalized by Nevanlinna [7] and Hinkkanen [5] (see also [2, Lemma 3.3.1]).

LEMMA 1.1 (Classical Borel lemma). *Let T be a continuous nondecreasing function on $[r_0, \infty)$ for some r_0 such that $T(r_0) \geq 1$. Then*

$$T\left(r + \frac{1}{T(r)}\right) \leq 2T(r)$$

for all r outside a possible exceptional set E whose (linear) measure is at most 2, that is, $\int_{t \in E \cap [r_0, \infty)} dt \leq 2$.

Exceptional sets appear throughout Nevanlinna theory. The purpose of the present paper is to explore generalized Borel lemmas and their associated exceptional sets in a purely real setting, independent of (but largely motivated by) Nevanlinna theory. We do so for two reasons. The first is to try to develop a unified approach to many of the results concerning exceptional sets in Nevanlinna theory. To this end we wish to emphasize the common elements of these results, which lie in real rather than complex analysis. The second reason is the authors' belief that these results, which are so important in Nevanlinna theory, should also be of value in other areas of mathematics in which nondecreasing functions naturally arise.

In Section 2 we discuss a generalization of Lemma 1.1 and we show that the estimate for the size of the exceptional set is the best possible. We then consider a number of applications. In Section 3 we consider nondecreasing functions f and g that satisfy inequalities of the form $f(r) \leq g(r)$ outside some exceptional set E. We show how sufficiently small exceptional sets can be removed by modifying the argument of g so that it is larger than r. As applications we consider functions of finite order and functions of finite type. The order of a positive function T is defined to be

$$\rho(T) := \limsup_{r\to\infty} \frac{\log T(r)}{\log r}.$$

The order is always well defined but it may be infinite. If $\rho := \rho(T) \in (0, \infty)$, then the type of T is defined to be

$$\tau(T) := \limsup_{r\to\infty} \frac{T(r)}{r^{\rho}},$$

https://doi.org/10.1017/S1446788710000212 Published online by Cambridge University Press

which again may be infinite. We show that if T is a positive continuous nondecreasing function of finite order ρ and type τ, where $0 < \rho < \infty$ and $0 < \tau < \infty$, then for any $\epsilon > 0$,

$$(\tau - \epsilon)r^{\rho} \le T(r) \le (\tau + \epsilon)r^{\rho}$$

on a set of infinite linear measure.

We also consider other measures of growth and other ways of describing the size of exceptional sets. Let $\log^{\circ 1} x := \log x$ and for $n \ge 2$ define the iterated logarithm by $\log^{\circ n} x := \log(\log^{\circ\{n-1\}} x)$. For $n \ge 1$, the n-order of a (sufficiently large) function T is defined to be

$$\rho_n(T) := \limsup_{r\to\infty} \frac{\log^{\circ n} T(r)}{\log r}.$$

The case where $n = 1$ gives the usual order $\rho_1 = \rho$. The case where $n = 2$ is usually referred to as the *hyperorder* of T.

2. A generalized Borel lemma

We begin by presenting a generalization of Lemma 1.1 (and of [2, Lemma 3.3.1]). In the following, $F^{\circ k}$ means F composed with itself k times.

LEMMA 2.1 (The generalized Borel lemma). *Let T and μ be positive continuous functions of r for $r \in [r_0, \infty)$ for some r_0. Suppose further that T is nondecreasing and that μ is differentiable and strictly increasing. Let ψ and F be positive and continuous on $[T(r_0), \infty)$. Suppose that on $[T(r_0), \infty)$, ψ is nonincreasing, F is nondecreasing and $\lim_{k\to\infty} F^{\circ k}(T(r_0)) = \infty$. Let $s(r) = \mu^{-1}(\mu(r) + \psi(T(r)))$ and define*

$$E := \{r \ge r_0 : T(s(r)) \ge F(T(r))\}. \tag{2.1}$$

Then

$$\int_{t\in E\cap[r_0,r)} d\mu(t) \le \sum_{n=1}^{\nu_r} \psi(F^{\circ\{n-1\}}(T(r_0))), \tag{2.2}$$

where ν_r is the largest integer such that

$$F^{\circ\{\nu_r-1\}}(T(r_0)) \le T(r). \tag{2.3}$$

This lemma is presented in a very general form, but we will soon specialize to some important cases. The classical Borel lemma corresponds to the case $F(x) = 2x$, $\mu(r) = r$ and $\psi(x) = 1/x$. The set E in Lemma 2.1 corresponds to the exceptional set in Lemma 1.1. Other choices of ψ that give stronger estimates with a larger exceptional set include $\psi(x) = 1/x^{\epsilon}$ and

$$\psi(x) = 1/((\log x)(\log\log x)(\log\log\log x)\cdots(\log^{\circ\{n-1\}} x)(\log^{\circ n} x)^{1+\epsilon}),$$

where $\epsilon > 0$.

https://doi.org/10.1017/S1446788710000212 Published online by Cambridge University Press

PROOF. If E is empty there is nothing to prove, so we suppose that E is nonempty. We define two sequences (r_n) and (s_n), which may possibly be finite, by induction. Let $r_1 = \min(E \cap [r_0, \infty))$. Assuming that we have defined r_n for some integer n, we define $s_n = s(r_n)$. If $E \cap [s_n, \infty) \neq \emptyset$, then we let $r_{n+1} = \min(E \cap [s_n, \infty))$.

Next we show that if the sequence (r_n) has infinitely many terms, then $\lim_{n\to\infty} r_n = \infty$. Suppose that this is not the case. Then since $r_{n+1} \geq s_n \geq r_n$, it follows that (r_n) has a finite limit r_∞. Then for all n,

$$\mu(r_{n+1}) - \mu(r_n) \geq \mu(s_n) - \mu(r_n) = \psi(T(r_n)) \geq \psi(T(r_\infty)).$$

Since $\psi(T(r_\infty)) > 0$ and independent of n, it follows that $\lim_{n\to\infty} \mu(r_n) = \infty$. But the continuity of μ implies that $\lim_{n\to\infty} \mu(r_n) = \mu(r_\infty) < \infty$. So we have shown that either r_n is defined for only finitely many n or $\lim_{n\to\infty} r_n = \infty$. It follows that

$$E \cap [r_0, r) \subseteq \bigcup_{n=1}^{N} [r_n, s_n],$$

where N is the largest integer such that $r_N \leq r$. Therefore

$$\int_{t\in E\cap[r_0,r)} d\mu(t) \leq \sum_{n=1}^{N} \int_{r_n}^{s_n} \mu'(t)\, dt \leq \sum_{n=1}^{N} \psi(T(r_n)). \tag{2.4}$$

Now

$$\begin{aligned} T(r_n) &\geq T(s_{n-1}) \geq F(T(r_{n-1})) \geq F \circ F(T(r_{n-2})) \\ &\geq \cdots \geq F^{\circ\{n-1\}}(T(r_1)) \geq F^{\circ\{n-1\}}(T(r_0)). \end{aligned} \tag{2.5}$$

In particular,

$$T(r) \geq T(r_N) \geq F^{\circ\{N-1\}}(T(r_0)).$$

Hence $N \leq \nu_r$. The proposition is proved on substituting inequality (2.5) into (2.4). □

Lemma 2.1 shows that if

$$\sum_{n=1}^{\infty} \psi(F^{\circ\{n-1\}}(T(r_0))) = L < \infty, \tag{2.6}$$

then $T(s(r)) < F(T(r))$ for all r outside a possible exceptional set E of μ-measure no greater than L. The following example shows that this is optimal.

EXAMPLE 1. Let $r_0 \leq 1$, and let μ be a strictly increasing positive continuous function of r on $[r_0, \infty)$ such that $\lim_{r\to\infty} \mu(r) = \infty$. Moreover, let ψ be a nondecreasing positive continuous function of r on $[r_0, \infty)$, and let F be nondecreasing and continuous on $[1, \infty)$ such that $F(x) > x$ for all $x \geq 1$. Furthermore, let $\varepsilon > 0$ and let $(r_n)_{n=1}^{\infty}$ be a sequence of points such that

$$\mu(r_j) - \mu(r_{j-1}) \geq \psi(F^{\circ\{j-1\}}(1)) + \varepsilon/2^j$$

for all $j \in \mathbb{N}$, and $r_j \to \infty$ as $j \to \infty$. Denote $\hat{r}_j := \mu^{-1}(\mu(r_j) - (\varepsilon/2^j))$, and define T as follows:

$$T(x) := \begin{cases} F^{\circ\{j-1\}}(1) & \text{if } x \in [r_{j-1}, \hat{r}_j] \\ \dfrac{(F^{\circ j}(1) - F^{\circ\{j-1\}}(1))(x - r_j)}{r_j - \hat{r}_j} + F^{\circ j}(1) & \text{if } x \in [\hat{r}_j, r_j], \end{cases}$$

where $j \in \mathbb{N}$. If $s(r) = \mu^{-1}(\mu(r) + \psi(T(r)))$ for $r \in [r_0, \infty)$, then it follows by the definition of T that the set E of points r such that

$$T(s(r)) \geq F(T(r))$$

contains all $r \in [\mu^{-1}(\mu(r_j) - \psi(F^{\circ\{j-1\}}(1))), \mu^{-1}(\mu(r_j) - \varepsilon/2^j)]$, where $j \in \mathbb{N}$. Since $T(r_0) = 1$, we have that the μ-measure of E is at least

$$\begin{aligned} \int_{t \in E \cap [r_0, \infty)} d\mu(t) &\geq \sum_{j=1}^{\infty} \mu(r_j) - \frac{\varepsilon}{2^j} - (\mu(r_j) - \psi(F^{\circ\{j-1\}}(1))) \\ &= \sum_{j=1}^{\infty} \psi(F^{\circ\{j-1\}}(T(r_0))) - \varepsilon. \end{aligned}$$

Therefore the constant L in (2.6) cannot be replaced by $L - \epsilon$ for any $\epsilon > 0$.

The most common applications of Lemma 2.1 involve the choice $F(x) = Cx$ for some constant $C > 1$. In this case (2.3) gives

$$\nu_r = 1 + \left\lfloor \log_C \frac{T(r)}{T(r_0)} \right\rfloor,$$

where $\lfloor \lambda \rfloor$ denotes the largest integer not exceeding λ. The inequality (2.2) then becomes

$$\begin{aligned} \int_{t \in E \cap [r_0, r)} d\mu(t) &\leq \sum_{n=1}^{\nu_r} \psi(C^{n-1} T(r_0)) \\ &\leq \psi(T(r_0)) + \int_0^{\nu_r - 1} \psi(C^x T(r_0))\, dx \\ &\leq \psi(T(r_0)) + \frac{1}{\log C} \int_{T(r_0)}^{T(r)} \psi(u) \frac{du}{u}. \end{aligned} \tag{2.7}$$

The next theorem follows immediately.

THEOREM 2.2. *Let T and μ be positive continuous functions of r for $r \in [r_0, \infty)$ for some r_0. Suppose further that T is nondecreasing and that μ is differentiable and strictly increasing. Let ψ be a positive, continuous and nonincreasing function on $[T(r_0), \infty)$. Let $s(r) = \mu^{-1}(\mu(r) + \psi(T(r)))$ and let $C > 1$. If*

$$\int_k^{\infty} \psi(u) \frac{du}{u} < \infty$$

https://doi.org/10.1017/S1446788710000212 Published online by Cambridge University Press

for some k, then

$$T(s(r)) \leq CT(r)$$

outside a possible exceptional set of finite μ-measure.

The upper and lower logarithmic densities of a subset $E \subset \mathbb{R}$ are given by

$$\limsup_{r\to\infty} \frac{1}{\log r} \int_{t\in E\cap[r_0,r)} \frac{dt}{t} \quad \text{and} \quad \liminf_{r\to\infty} \frac{1}{\log r} \int_{t\in E\cap[r_0,r)} \frac{dt}{t}$$

respectively.

COROLLARY 2.3. *Let n be a positive integer, and let T be a positive continuous nondecreasing function on $[r_0, \infty)$ for some r_0 such that $T(r_0) > \exp^{\circ\{n-1\}}(0)$. Let $A > 0$ and $C > 1$ be constants. Define*

$$\begin{aligned}\sigma(u) &:= \exp\left(Au\frac{d}{du}\log^{\circ n} u\right) \\ &= \begin{cases} \exp(A) & \text{if } n = 1, \\ \exp(A((\log u)(\log\log u)\cdots(\log^{\circ\{n-1\}} u))^{-1}) & \text{if } n \geq 2. \end{cases}\end{aligned}$$

Let

$$E := \{r \geq r_0 : T(r\sigma(T(r))) \geq CT(r)\}. \tag{2.8}$$

Then:

(1) *if T has finite n-order ρ, that is,*

$$\limsup_{r\to\infty} \frac{\log^{\circ n} T(r)}{\log r} = \rho,$$

then the upper logarithmic density of E is at most $A\rho/\log C$;

(2) *if T has finite lower n-order λ, that is,*

$$\liminf_{r\to\infty} \frac{\log^{\circ n} T(r)}{\log r} = \lambda,$$

then the lower logarithmic density of E is at most $A\lambda/\log C$.

The $n = 1$ case of Corollary 2.3 is essentially the same as Lemma 4 in Hayman [4]. Hayman's result is expressed in terms of the derivative of a meromorphic function, but his proof shows that the result is real analytic in nature.

PROOF. Apply (2.7) with $\mu(r) = \log r$ and $\psi(u) = Au(d/du)\log^{\circ n} u$. □

COROLLARY 2.4. *Let T be a continuous nondecreasing function and let $\alpha > 0$ and $C > 1$ be constants. If*

$$T(r + \alpha) \geq CT(r),$$

on a set of infinite logarithmic measure, then the hyperorder of T is at least one, that is,

$$\limsup_{r\to\infty} \frac{\log\log T(r)}{\log r} \geq 1.$$

PROOF. Suppose to the contrary that the hyperorder of T is less than one. Then for sufficiently small $\epsilon > 0$ and sufficiently large r,

$$(\log T(r))^{1+\epsilon} \leq r.$$

Let

$$E := \{r : T(r+\alpha) \geq CT(r)\}.$$

Then

$$E \subseteq \widetilde{E} := \left\{r : T\left(r\left[1 + \frac{\alpha}{(\log T(r))^{1+\epsilon}}\right]\right) \geq CT(r)\right\}.$$

So, by applying Theorem 2.2 with $\mu(r) = \log r$ and

$$\psi(u) = \log\left(1 + \frac{\alpha}{(\log u)^{1+\epsilon}}\right),$$

it follows that

$$\int_{t\in E\cap[1,\infty)} \frac{dt}{t} \leq \int_{t\in\widetilde{E}\cap[1,\infty)} \frac{dt}{t} < \infty,$$

which is a contradiction. □

Another interesting choice for F in Lemma 2.1 is $F(x) = x^C$, for some constant $C > 1$. In this case, if $T(r_0) > e$,

$$\nu_r = 1 + \left\lfloor \frac{\log\log T(r) - \log\log T(r_0)}{\log C} \right\rfloor,$$

and

$$\int_{t\in E\cap[r_0,r)} \mu'(t)\,dt \leq \psi(T(r_0)) + \frac{1}{\log C}\int_{T(r_0)}^{T(r)} \psi(u)\frac{du}{u\log u}.$$

So, for example, we get the following analogue of Hayman's result. If T has hyperorder ρ then, by taking $\psi(u) = \log\alpha$ and $\mu(r) = \log r$, it follows that

$$T(\alpha r) \leq T(r)^C,$$

outside a set of upper logarithmic density at most $\rho\log\alpha/\log C$. Similarly, the results that we have described in this paper for $F(x) = Cx$ are easily extended to the obvious analogues for $F(x) = x^C$.

3. Removing exceptional sets

LEMMA 3.1. *Let μ be a positive strictly increasing differentiable function of r for all r greater than some r_0 and let f and g be nondecreasing functions for all $r > r_0$. Furthermore, suppose that $f(r) \leq g(r)$ for all $r \in (r_0, \infty) \setminus E$, where the exceptional set $E \subset (r_0, \infty)$ satisfies*

$$\int_{t\in E\cap[r_0,\infty)} d\mu(t) = \int_{t\in E\cap[r_0,\infty)} \mu'(t)\,dt < \infty. \tag{3.1}$$

Then, given $\epsilon > 0$, there is an $\hat{r} \geq r_0$ such that $f(r) \leq g(s(r))$ for all $r > \hat{r}$, where $s(r) = \mu^{-1}(\mu(r) + \epsilon)$.

https://doi.org/10.1017/S1446788710000212 Published online by Cambridge University Press

PROOF. Suppose that there is an infinite sequence $(r_n)_{n=1}^{\infty} \subset (r_0, \infty)$ that satisfies $r_{n+1} \geq s_n := s(r_n)$ and $(r_n, s_n) \subset E$, for all $n \in \mathbb{N}$. Then

$$\int_{t \in E \cap [r_0, \infty)} \mu'(t)\, dt \geq \sum_{m=1}^{\infty} \int_{r_n}^{s_n} \mu'(t)\, dt = \sum_{n=1}^{\infty} \epsilon = \infty,$$

which contradicts the finite measure condition (3.1). Therefore, there must be a number $\hat{r} \geq r_0$ such that for any $r > \hat{r}$, there exists $t \in (r, s(r)) \setminus E$. Since f and g are nondecreasing, it follows that

$$f(r) \leq f(t) \leq g(t) \leq g(s(r)).$$

This concludes the proof. □

THEOREM 3.2. *Let f and g be positive nondecreasing functions of r for all r greater than some r_0. Let μ be a positive differentiable strictly increasing function, fix $\epsilon > 0$ and set $s(r) = \mu^{-1}(\mu(r) + \epsilon)$ for all $r > r_0$. Suppose that*

$$\limsup_{r \to \infty} \frac{g(s(r))}{g(r)} = 1$$

and that

$$\limsup_{r \to \infty} \frac{f(r)}{g(r)} = \lambda,$$

for some nonzero finite λ. Then for any $\delta > 0$,

$$\left| \frac{f(r)}{g(r)} - \lambda \right| < \delta$$

on a set F of infinite μ-measure (that is, such that $\int_{t \in F \cap [r_0, \infty)} d\mu(t) = \infty$).

PROOF. It follows from the definition of lim sup that there is an $r_1 \geq r_0$ such that $f(r) \leq (\lambda + \delta) g(r)$ for all $r > r_1$. Now suppose that $f(r) \leq (\lambda - \delta) g(r)$ outside a set of finite μ-measure. From Lemma 3.1 with $\tilde{f} = f$ and $\tilde{g} = (\lambda - \delta) g$, we deduce that $f(r) \leq (\lambda - \delta) g(s(r))$ for all sufficiently large r. Hence

$$\limsup_{r \to \infty} \frac{f(r)}{g(r)} \leq (\lambda - \delta) \limsup_{r \to \infty} \frac{g(s(r))}{g(r)} = \lambda - \delta < \lambda,$$

which contradicts the definition of λ. So $f(r) > (\lambda - \delta) g(r)$ on a set of infinite μ-measure. □

COROLLARY 3.3. *Let T be a positive nondecreasing function of order ρ, where $0 < \rho < \infty$. Then for any $\epsilon > 0$,*

$$r^{\rho - \epsilon} \leq T(r) \leq r^{\rho + \epsilon}$$

on a set of infinite logarithmic measure.

PROOF. Apply Theorem 3.2 using $f(r) = \log T(r)$, $g(r) = \log r$, $\mu(r) = \log r$, $\delta = \epsilon$ and $\lambda = \rho$. □

COROLLARY 3.4. *Let T be a positive nondecreasing function of finite order ρ and type τ, where $0 < \rho < \infty$ and $0 < \tau < \infty$. Then for any $\epsilon > 0$,*

$$(\tau - \epsilon)r^{\rho} \leq T(r) \leq (\tau + \epsilon)r^{\rho}$$

on a set of infinite linear measure.

PROOF. Apply Theorem 3.2 using $f(r) = T(r)$, $g(r) = r^{\rho}$, $\mu(r) = r$, $\delta = \epsilon$ and $\lambda = \tau$. □

Acknowledgements

This paper was completed while both authors were visiting fellows in the programme on Discrete Integrable Systems at the Isaac Newton Institute for Mathematical Sciences in Cambridge. We also acknowledge the support of the European Commission's Framework 6 ENIGMA Network and the European Science Foundation's MISGAM Network.

References

[1] E. Borel, 'Sur les zéros des fonctions entières', *Acta Math.* **20** (1897), 357–396.
[2] W. Cherry and Z. Ye, *Nevanlinna's Theory of Value Distribution* (Springer, Berlin, 2001).
[3] W. K. Hayman, *Meromorphic Functions* (Clarendon Press, Oxford, 1964).
[4] W. K. Hayman, 'On the characteristic of functions meromorphic in the plane and of their integrals', *Proc. London Math. Soc.* **14a** (1965), 93–128.
[5] A. Hinkkanen, 'A sharp form of Nevanlinna's second fundamental theorem', *Invent. Math.* **108** (1992), 549–574.
[6] R. Nevanlinna, 'Zur Theorie der meromorphen Funktionen', *Acta Math.* **46** (1925), 1–99.
[7] R. Nevanlinna, 'Remarques sur les fonctions monotones', *Bull. Sci. Math., II. Sér.* **55** (1931), 140–144.
[8] H. Weyl, *Meromorphic Functions and Analytic Curves*, Annals of Mathematics Studies, 12 (Princeton University Press, Princeton, NJ, 1943).

R. G. HALBURD, Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK
e-mail: r.halburd@ucl.ac.uk

R. J. KORHONEN, Department of Physics and Mathematics, University of Eastern Finland, Joensuu Campus, PO Box 111, FI-80101 Joensuu, Finland
e-mail: risto.korhonen@helsinki.fi