Bi-determinants

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The important Binet-Cauchy theorem of determinants (3, p. 81) may be illustrated as follows. Consider the identity

$$\begin{vmatrix} a_1 & b_1 & -1 & . & . \\ a_2 & b_2 & . & -1 & . \\ a_3 & b_3 & . & . & -1 \\ . & . & x_1 & x_2 & x_3 \\ . & . & y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & -1 & . & . \\ a_2 & b_2 & . & -1 & . \\ a_3 & b_3 & . & . & -1 \\ a_x & b_x & . & . & . \\ a_y & b_y & . & . & . \end{vmatrix} \qquad (1)$$

where $a_x = a_1 x_1 + a_2 x_2 + a_3 x_3$, and similarly for a_y, b_x, b_y. This identity follows by adding $x_1 \text{row}_1 + x_2 \text{row}_2 + x_3 \text{row}_3$ to row_4 in the first determinant; and $y_1 \text{row}_1 +$ etc. to row_5. Then by expanding both determinants in a Laplace development of the first three rows and the last two we obtain the identity

$$(ab)_{23}(xy)_{23} + (ab)_{31}(xy)_{31} + (ab)_{12}(xy)_{12} = \begin{vmatrix} a_x & b_x \\ a_y & b_y \end{vmatrix}. \qquad (2)$$

This is a particular case of the Binet-Cauchy theorem; and it can be proved in exactly the same way for r columns $a, b, \ldots, c$, each of n elements, together with r rows $x, y, \ldots, z$, each of n elements, provided that $r \leqq n$. When $r = n$ it yields the ordinary multiplication theorem of determinants. If $r > n$ every term on the left side of (2) vanishes. In the general case we simply start with an $(n + r)$ rowed determinant—the negative unit determinant of n rows, bordered by r further rows and columns as in (1), and proceed in the same way with $a_x \equiv \sum_{i=1}^{n} a_i x_i$.

Inspection of relations (1) and (2) suggests the following useful abbreviation for the first determinant above:

$$B \equiv \left| \begin{array}{cc|cc} a_1 & b_1 & x_1 & y_1 \\ a_2 & b_2 & x_2 & y_2 \\ a_3 & b_3 & x_3 & y_3 \end{array} \right|. \qquad (3)$$

Let this be called a *bi-determinant*. In fact let the general B have n rows and $2r$ columns, written in the usual Cayleyan notation of determinants, but with a rule severing the columns into two equal sets of r. The Binet-Cauchy theorem then states that this bi-determinant is equal to an r rowed determinant Δ. Here $r=2$ and

$$\Delta = \begin{vmatrix} a_x & b_x \\ a_y & b_y \end{vmatrix}, \quad a_x = \sum_{i=1}^{3} a_i x_i:$$

and further that it is also equal to a series Σ of $\binom{n}{r}$ terms. Here $n=3$, $r=2$ and Σ is

$$\sum_{i,j} (ab)_{ij} (xy)_{ij} \qquad i \neq j, \qquad i, j = 1, 2, 3.$$

As I do not recollect ever seeing any account of bi-determinants, it seems worth while to give some of their properties, particularly as they have both theoretical and practical importance. Let the twin sets be called the *early* and the *late* columns; then the following properties are at once apparent from Δ.

(i) Early columns may be deranged, with change of sign, as in a determinant. Likewise late, but not early with late.

(ii) Rows may be deranged without change of sign.

(iii) B is unchanged by attaching or interpolating zero rows.

(iv) Multiplying B by λ is equivalent to multiplying any one column (early or late) throughout by λ.

(v) If $r > n$, B vanishes identically.

(vi) Early columns may be linearly combined in the usual way of determinants. Likewise late, but not early with late.

(vii) B is unchanged by the twin operations

$$\mathrm{row}_i + p\,\mathrm{row}_j, \quad \mathrm{row}_j - p\,\mathrm{row}_i$$

applied respectively to the early and the late *columns*.

This last is the characteristic property of bi-determinants. In illustration we have

$$B = \left| \begin{array}{ll|ll} a_1 & b_1 & x_1 - px_2 & y_1 - py_2 \\ a_2 + pa_1, & b_2 + pb_1 & x_2 & y_2 \\ a_3 & b_3 & x_3 & y_3 \end{array} \right|.$$

To prove this we simply remark that

$$a_x = a_1 (x_1 - px_2) + (a_2 + pa_1) x_2 + a_3 x_3$$

for all values of p. Similarly for b_y. It applies to n rows and $2r$ columns in just the same way.

The bi-determinant, written as Σ, may be regarded as the r^{th} compound of a bilinear form in two compound sets of cogredient variables, having a unit matrix of coefficients. Each single set of variables has n components. When the early and the late sets are contragredient to one another, then, as is well known, each element of the determinant Δ is unaltered by a linear transformation of the variables. If x, y, a, b each denote a column of B, and x', y', a', b' the corresponding rows of a transposed matrix, then B can be written in matrix notation,

$$B = \left| \begin{bmatrix} x' \\ y' \end{bmatrix} \begin{bmatrix} a, b \end{bmatrix} \right| = \left| X' A \right| :$$

that is, the determinant of the matrix $\begin{bmatrix} a_x & b_x \\ a_y & b_y \end{bmatrix}$ is here expressed as two factors X', A. The original array (3) is then $B = |A|X|$. If now we perform operation (vii) upon B it is equivalent to an elementary collineatory transformation (*cf.* **4**, p. 12) of contragredient variables by means of the matrix

$$H = \begin{bmatrix} 1 & . & . \\ p & 1 & . \\ . & . & 1 \end{bmatrix} \quad \text{where } H^{-1} = \begin{bmatrix} 1 & . & . \\ -p & 1 & . \\ . & . & 1 \end{bmatrix}.$$

Thus (vii) may be written

$$H\,[a, b], \quad [x', y']\,H^{-1}.$$

Hence (vii) replaces $X'A$ by $X'H^{-1}HA$ which has the same product, and therefore leaves B unchanged.

Geometrically the bi-determinant is a convenient way of dealing with such a problem as the following: to find the homogeneous equation of the line through the point (3, 2, 4) and the point of intersection of the lines $x + 2y - 3z = 0$ and $3x - 4y + 5z = 0$.

It is given at once by Δ but numerically by

$$\left| \begin{array}{rr|rc} 1 & 3 & 3 & x \\ 2 & -4 & 2 & y \\ -3 & 5 & 4 & z \end{array} \right| = 0,$$

where the early columns denote the two given lines and the late columns the given point and a moving point (x, y, z).

By $\text{row}_2 - 2\,\text{row}_1$, $\text{row}_3 + 3\,\text{row}_1$, $\text{row}_3 + \text{row}_2$, $\text{row}_2 + 3\,\text{row}_3$ applied to the early columns, with similar operations on the later, we obtain

$$\left| \begin{array}{cccl} 1 & 3 & -5 & x + 2y - 3z \\ . & 2 & -2 & y - z \\ . & 4 & 10 & 4z - 3y \end{array} \right|.$$

Whence, by $\text{row}_3 - 2\,\text{row}_2$, the early columns yield but one two-rowed minor, and the result after dividing by 2 is

$$-18x - 11y + 19z = 0.$$

The method applies to more elaborate examples in this and higher dimensions.

Bi-determinants throw light on the general theory of determinants in at least two ways, (i) by providing a new expansion theorem connected with the 1825 theorem of Schweins (*cf.* **1**, p. 169), and (ii) by giving the natural approach to the dual form of the Sylvester identities. These will now be briefly explained.

(i) Consider the identity

$$\left|\begin{array}{cccccc|cccccc} a & b & c & d & e & f & \xi & \eta & \zeta & x & y & z \\ . & . & . & . & e & f & . & . & . & -x & -y & -z \end{array}\right|$$

$$= \left|\begin{array}{cccccc|cccccc} a & b & c & d & . & . & \xi & \eta & \zeta & x & y & z \\ . & . & . & . & e & f & \xi & \eta & \zeta & . & . & . \end{array}\right| \qquad (4)$$

where the right hand member is the result of the operation

$$\text{row}_1 - \text{row}_2, \quad \text{row}_2 + \text{row}_1.$$

Interpret each letter $a, \ldots, z$ as a column of six arbitrary elements. We are thus dealing with bi-determinants of 12 rows and $6+6$ columns. The Binet-Cauchy determinant Δ of either side is at once

$$\Delta = \left|\begin{array}{cccccc} a_\xi & b_\xi & c_\xi & d_\xi & e_\xi & f_\xi \\ a_\eta & b_\eta & c_\eta & d_\eta & e_\eta & f_\eta \\ a_\zeta & b_\zeta & c_\zeta & d_\zeta & e_\zeta & f_\zeta \\ a_x & b_x & c_x & d_x & . & . \\ a_y & b_y & c_y & d_y & . & . \\ a_z & b_z & c_z & d_z & . & . \end{array}\right| \qquad (5)$$

where $a_\xi = \sum_{i=1}^{6} a_i\,\xi_i$. The Laplace development of four columns with two gives at once

$$(abcd\,|\,\dot\zeta xyz)\,(ef\,|\,\dot\xi\dot\eta) \qquad (6)$$

where the dots ., .. indicate a determinantal permutation series of three terms (ζ, $\xi\eta$; ξ, $\eta\zeta$; η, $\zeta\xi$). Here $(ef\,|\,\xi\eta)$ denotes the two-rowed minor $e_\xi f_\eta - e_\eta f_\xi$, and the other factor the four-rowed minor. The analogous development of Δ by three rows with three gives

$$-(\dot a b \dot c\,|\,xyz)\,(\dot d ef\,|\,\xi\eta\zeta) \qquad (7)$$

a series of four terms, obtained by deranging abc, d. Hence these two series must be identically equal, a result which is virtually one of Schweins' Theorems of 1825. This last follows by taking $[\xi\eta\zeta xyz]$ to be the unit matrix (so that $a_\xi = a_1$, $a_\eta = a_2$, etc,), and taking Δ with n rows and columns, and with rs zeros in the last r rows and s columns. Here, for example

$$(abcd)_{\dot{3}456}\,(ef)_{\dot{1}\dot{2}} = -(\dot{a}\dot{b}\dot{c})_{456}\,(\dot{d}ef)_{123}. \tag{8}$$

This result in general, derived in this manner, is well known (*cf.* 2, p. 125). But the original bi-determinants may also be expanded by the Binet-Cauchy summation Σ, in products of six rowed determinants. When this is done the result is

$$\Sigma = (abcdef)\,(\xi\eta\zeta xyz) - (abcd\dot{e} \mid \xi\eta\zeta\bar{x}\bar{y})\,(\dot{f} \mid z) + (abcd \mid \xi\eta\zeta\bar{x})\,(ef \mid \bar{y}\bar{z}). \tag{9}$$

Here are three terms, the second denoting a series of six (2×3) terms, and the third, three. The letters a, b, c, d, ξ, η, ζ are fixed, while e, f are permuted and so are x, y, z (unless either ef or xyz completely stands in one or other factor of a term). These two sets of permuted letters refer to such columns and rows of (5) as contain zeros, and likewise in general. If ν denote the smaller of r and s (the general numbers of zero rows and columns), then the above series has $(\nu + 1)$ terms, with alternate signs starting from a positive. The factors of the successive terms are obtained by decreasing the order of the first and increasing that of the second factor by one step at a time, without disturbing the original row and column arrangement of Δ.

In proof of this result, when

$$B = \left| \begin{array}{cccccc|cccccc} a & b & c & d & e & f & \xi & \eta & \zeta & x & y & z \\ . & . & . & . & e & f & . & . & . & -x & -y & -z \end{array} \right|$$

is expanded systematically, the upper matrix of B yields the first term of Σ at once: five of the six rows in $ab \ldots . f$, with one from the lower matrix of B, taken in every way, yield the second term. The negative sign is due to the $-x_i$, $-y_i$, $-z_i$. Next, four of the upper and two of the lower yield the third term, now with positive sign. And so on.

The consequent identity

$$\Sigma = \Delta \tag{10}$$

may be regarded as a process for removing from a complete determinant $|a_\xi b_\eta c_\zeta d_x e_y f_z|$ an arbitrary rectangle of elements, so obtaining Δ. On replacing the unit matrix, as before, it gives

$$(abcdef)_{123456} - (abcd\dot{e})_{1234\dot{5}}\dot{f}_{\dot{6}} + (abcd)_{123\dot{4}}(\dot{e}f)_{\dot{5}\dot{6}}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \\ a_4 & b_4 & c_4 & d_4 & . & . \\ a_5 & b_5 & c_5 & d_5 & . & . \\ a_6 & b_6 & c_6 & d_6 & . & . \end{vmatrix}. \tag{11}$$

No further information is obtained by expanding the right hand member of (4) in this way: it simply yields (6).

(ii) Let an n-rowed determinant be partitioned into i columns A, j columns B, and k columns C; and written

$$(ABC), \qquad i + j + k = n.$$

With the same partitions let (DEF) be another arbitrary determinant. Then the three Sylvester identities (*cf.* **3**, p. 45) can be written

$$\text{I} \quad (A\dot{B}C)(\dot{D}EF) = (DB\dot{C})(\dot{A}EF), \qquad i + j < n,$$

$$\text{II} \quad (A\dot{B})(\dot{D}E) = (DB)(AE), \qquad i + j = n,$$

$$\text{III} \quad (A\dot{B})(\dot{D}E) = 0, \qquad i + j > n.$$

Each has $\binom{n}{j}$ terms on the left: the first has $\binom{n}{i}$ terms on the right. The first of these follows from the determinantal identity

$$\begin{vmatrix} A & B & C & D & E & F \\ . & B & . & D & . & . \end{vmatrix} = \begin{vmatrix} A & B & C & D & E & F \\ -A & . & -C & . & -E & -F \end{vmatrix}$$

when each side is expanded by a Laplace development of the upper and lower matrices (each having n rows).

Bi-determinants lead to corresponding dual identities (where $i + j + k \leqq n$, and X, Y, Z also have i, j, k columns respectively):

$$\text{IV} \quad (\dot{A}\,|\,X)(\dot{B}C\,|\,YZ) = (AB\,|\,X\dot{Y})(C\,|\,\dot{Z}), \qquad i + j < n,$$

$$\text{V} \quad (\dot{A}\,|\,X)(\dot{B}\,|\,Y) = (AB)(XY), \qquad i + j = n,$$

$$\text{VI} \quad (\dot{A}\,|\,X)(\dot{B}\,|\,Y) = 0, \qquad i + j > n.$$

Here V and VI are trivial: IV follows from the bi-determinantal identity

$$\left|\begin{array}{ccc|ccc} A & B & . & X & . & . \\ A & B & C & . & Y & Z \end{array}\right| = \left|\begin{array}{ccc|ccc} A & B & . & X & Y & Z \\ . & . & C & . & Y & Z \end{array}\right|.$$

The Σ development of such identities at once gives rise to the dual type of Sylvester identities given in 3, p. 95. With $i = k = 2$, from

$$\Delta = \left|\begin{array}{cccc|cccc} a & b & c & d & x & y & z & w \\ . & . & c & d & . & . & -z & -w \end{array}\right| = \left|\begin{array}{cccccccc} a & b & . & . & x & y & z & w \\ . & . & c & d & x & y & . & . \end{array}\right|$$

we obtain as respective Σ expansions

$$(abcd \,|\, xyzw) - (ab\dot{c} \,|\, xy\bar{z})\,(\dot{d} \,|\, \bar{w}) + (ab \,|\, xy)\,(cd \,|\, zw) = (ab \,|\, zw)\,(cd \,|\, xy)$$

which is identity (29) of p. 95.

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Pictorial relativity

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There are some people who find it easier to absorb abstruse theories when encouraged by picturesque analogies. There are also some people, mathematicians, who hold such assistance in austere contempt. Pictorialism, however, is no new thing in expositions of relativity, and I will not apologise for the following attempt to introduce a little more. The object of this note is to suggest ways of visualising some of the metrics which are of importance in the general theory of relativity. For each metric two pictures are supplied; they may be called (A) geometrical and (B) dynamical. The first is got by taking a section of the four-dimensional continuum corresponding to a given metric, and immersing this section in ordinary space. The second shows graphically the magnitude of the