

## On product varieties of groups

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An example is given of a finitely based variety of groups  $\underline{V}$  such that  $\underline{VA}$  is not finitely based.

Let  $\underline{V}$  be the variety of groups determined by the laws

$$(1) \quad [[x_1, x_2], [x_3, x_4], [x_5, x_6]] ,$$

$$(2) \quad [[x_1, x_2, x_3], [x_4, x_5]] [[x_1, x_2], [x_4, x_5, x_3]]^{-1} , \\ [[x_1, x_2, x_3], [x_1, x_2]] .$$

Then  $\underline{VA}$  is not finitely based.

Only an indication of the proof is included as it is very similar to the proofs of the existence of non-finitely based varieties given in [3] and [2]. The crucial step is the following lemma.

LEMMA. *If  $a, b, c, d$  are elements of a group which satisfies the laws (1) and (2) then*

$$[[ab, c, d], [ab, c]] = [[a, c, d], [a, c]]^b [[b, c, d], [b, c]]$$

and

$$[[a, bc, d], [a, bc]] = [[a, b, d], [a, b]]^c [[a, c, d], [a, c]] .$$

Using this lemma it can be shown that  $\underline{VA}$  is determined by the laws

$$(3) \quad \left[ [[x_1, x_2], [x_3, x_4]], [[x_5, x_6], [x_7, x_8]], [[x_9, x_{10}], [x_{11}, x_{12}]] \right] ,$$

$$(4) \quad \left[ [x_1, x_2], [x_3, x_4], [x_5, x_6], [x_7, x_8], [x_9, x_{10}] \right] \cdot \left[ [x_1, x_2], [x_3, x_4], [x_7, x_8], [x_9, x_{10}], [x_5, x_6] \right]^{-1},$$

and the laws

$$w_k = \left[ [x_1, x_2], [x_3, x_4], [y_1, y_2], [y_3, y_4], \dots, [y_{2k-1}, y_{2k}], \right. \\ \left. [x_1, x_2], [x_3, x_4] \right]$$

for  $k = 1, 2, \dots$

For each  $n \geq 1$  a group is constructed which satisfies the laws (3) and (4), and the laws  $w_k$  for  $k \neq n$ , but which does not satisfy the law  $w_n$ . This proves that  $\underline{VA}$  is not finitely based.

Strongly finitely based varieties were introduced in [1], and it was proved in [1] that if  $\underline{V}$  is strongly finitely based and  $\underline{W}$  is finitely based then  $\underline{VW}$  is finitely based. Thus the variety  $\underline{V}$  defined above cannot be strongly finitely based, although it is finitely based.

### References

- [1] M.S. Brooks, L.G. Kovács and M.F. Newman, "A finite basis theorem for product varieties of groups", *Bull. Austral. Math. Soc.* 2 (1970), 39-44.
- [2] M.F. Newman, "Just non-finitely-based varieties of groups", *Bull. Austral. Math. Soc.* 4 (1971), 343-348.
- [3] M.R. Vaughan-Lee, "Uncountably many varieties of groups", *Bull. London Math. Soc.* 2 (1970), 280-286.

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