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EXTREME n-POSITIVE LINEAR MAPS

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In this article we prove that if a completely positive linear map Φ of a unital C^*-algebra A into another B with only finite dimensional irreducible representations is pure, then we have $N_\Phi = {}_\Phi\ker + \ker_\Phi$, where $N_\Phi = \{x \in A \mid \Phi(x) = 0\}$, ${}_\Phi\ker = \{x \in A \mid \Phi(x^*x) = 0\}$, and $\ker_\Phi = \{x \in A \mid \Phi(xx^*) = 0\}$. We also prove that for every unital strongly positive and n-positive linear map Φ of a C^*-algebra A onto another B with $n \geqq 2$, if $N_\Phi = {}_\Phi\ker + \ker_\Phi$, then Φ is extreme in $P_n(A, B, I_B)$. By this null-kernel condition, many new extreme n-positive linear maps are identified. A general procedure for constructing extreme n-positive linear maps is suggested and discussed.

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1. Introduction

In 1947, Segal [**7**] proved that a state ϕ of a C^*-algebra is pure if and only if the *GNS* representation induced by ϕ is irreducible. This characterization later was further generalized to completely positive linear maps Φ of a C^*-algebra A into $B(\mathscr{H})$, the C^*-algebra of all bounded linear operators on a Hilbert space $B(\mathscr{H})$, by Arveson [**2**], via the Stinespring representation induced by Φ. In 1957 Kadison, using his celebrated transitivity theorem for irreducible representation, derived another characterization of pure states as follows. For a state ϕ on a C^*-algebra A, with $N_\phi = \{x \in A \mid \phi(x) = 0\}$, ${}_\phi\ker = \{x \in A \mid \phi(x^*x) = 0\}$, $\ker_\phi = \{x \in A \mid \phi(xx^*) = 0\}$, ϕ is a pure state if and only if $N_\phi = {}_\phi\ker + \ker_\phi$. In this article we start out by studying the relation between an n-positive linear map Φ of a C^*-algebra A into another being pure or extreme and the condition $N_\Phi = {}_\Phi\ker + \ker_\Phi$, where $N_\Phi = \{x \in A \mid \Phi(x) = 0\}$, ${}_\Phi\ker = \{x \in A \mid \Phi(x^*x) = 0\}$ and $\ker_\Phi = \{x \in A \mid \Phi(xx^*) = 0\}$, for $n \geqq 2$. Earlier work on the similar subjects can be found in Størmer [**9**], Arveson [**2**], Choi [**4**], Paschke [**6**], and more recently in Anantharaman-Delaroche [**1**].

1.1. Notation. In this paper, all C^*-algrbra are unital, and they are denoted by A, B, D, etc. The set of all n-positive linear maps of A into B is denoted by $P_n(A, B)$, $n = 1, 2, \ldots, \infty, P_\infty(A, B) = \bigcap_{n=1}^\infty P_n(A, B)$. In case $n = \infty$, $P_\infty(A, B)$ is the set of all completely positive linear maps of A into B, which can also be denoted by $CP(A, B)$. An element Φ in $P_n(A, B)$ is called *pure*, if for every ψ in $P_n(A, B)$ with $\psi \leqq \Phi$ (i.e., $\Phi - \psi \in P_n(A, B)$) ψ must be of the form $\alpha\Phi$ for some positive scalar α. An element Φ is said to be *extreme* in $P_n(A, B, p) = \{\theta \in P_n(A, B) \mid \theta(I_A) = p = \Phi(I_A)\}$, if we have $\Phi = \Phi_1 = \Phi_2$,

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whenever $\Phi=\frac{1}{2}(\Phi_1+\Phi_2)$ for Φ_1, Φ_2 in $P_n(A,B,p)$. For Φ in $P_n(A,B)$, ${}_\Phi\text{ker}$ is a left ideal and ker_Φ is a right ideal in A.

1.2. In Section 2, we show that, for those C^*-algebras B having only finite dimensional irreducible representations, every pure element in $CP(A,B)$ satisfies $N_\Phi={}_\Phi\text{ker}+\text{ker}_\Phi$ (see Theorem 2.4). We believe that the theorems remain valid for any C^*-algebra B in general. In Section 3, we show that every strongly positive and n-positive linear map Φ of A onto B satisfying $N_\Phi={}_\Phi\text{ker}+\text{ker}_\Phi$ is extreme in $P_n(A,B,\Phi(I))$. This comes as a consequence of a more general theorem (see Theorem 3.5). We are also able to get a description of a set of extreme completely positive linear maps in $P_n(A,B,p)$, for $n\geqq 2$, analogous to Arveson's characterization mentioned in the beginning of this section, for $n=\infty$ (see Theorem 2.1); however, our constructive procedure is considerably simpler than Arveson's (for details, see Example 3.9).

Finally in Section 4 we direct our attention to a general procedure for constructing extreme completely positive linear maps in some special cases. Further investigations in this direction will appear elsewhere.

2. Pure completely positive linear maps

Let Φ be in $CP(A,B(\mathscr{H}))$. For $x,y\in A\odot B(\mathscr{H})$, the algebraic tensor product of A and $B(\mathscr{H})$, $x=\sum_{i=1}^n a_i\otimes\eta_i, y=\sum_{j=1}^m b_j\otimes\xi_j$, a semidefinite inner product $\langle\cdot,\cdot\rangle$ can be defined by $\langle x,y\rangle\triangleq\sum_{i,j}\langle\Phi(b_j^*a_i)\eta_i,\xi_j\rangle$, due to Stinespring [**8**]. Let $I=\{x\in A\odot\mathscr{H}\mid \langle x,x\rangle=0\}$, which is a subspace of $A\odot\mathscr{H}$ invariant under the A-action defined by $a(\sum a_i\otimes\eta_i)=\sum aa_i\otimes\eta_i$, for $a\in A$, $\sum a_i\otimes\eta_i\in A\odot\mathscr{H}$. The Stinespring representation space $\mathscr{K}$ is defined as the completion of $A\odot\mathscr{H}/I$. The Stinespring representation π of A on $\mathscr{K}$ is defined as $\pi(A)([\sum a_i\otimes\eta_i])=[\sum aa_i\otimes\eta_i]$ for $a\in A$ where $[x]$ is the coset in $\mathscr{K}$ determined by x in $A\odot\mathscr{H}$. Indeed, $\pi(a)([x])\in I$ for $[x]\in I$ and $\pi(a)$ is well-defined for all a in A, and $\Phi(a)=v^*\pi(a)v$ where v is a bounded operator of $B(\mathscr{H})$ into $\mathscr{K}$ defined as $v(\eta)=[a\otimes\eta]$. (We call $\{\pi,\mathscr{K},v\}$ the Stinespring representation triple induced by Φ.) In 1969, Arveson characterized the pure elements in $CP(A,B(\mathscr{H}))$ and the extreme elements in $CP(A,B(\mathscr{H}),p)$ as follows [**2**].

Theorem 2.1 (Arveson). (1) *Let $\Phi\in CP(A,B(\mathscr{H}))$. Φ is pure if and only if the Stinespring representation π induced by Φ is irreducible.*

(2) *Let $\Phi\in CP(A,B(\mathscr{H}),p)$. Φ is extreme if and only if the following map ι from the commutant $\pi(A')$ of $\pi(A)$ onto $P_{[v\mathscr{H}]^-}\pi(A)'|_{[v\mathscr{H}]^-}$ is injective, where $P_{[v\mathscr{H}]^-}$ is the orthogonal projection of $\mathscr{K}$ onto $[v\mathscr{H}]^-$, $\iota(T)\triangleq P_{[v\mathscr{H}]^-}T|_{[v\mathscr{H}]^-}$.*

Two natural questions arise. (1) How is the condition $N_\Phi={}_\Phi\text{ker}+\text{ker}_\Phi$ related to Φ being pure in $CP(A,B(\mathscr{H}))$? (2) What are the characterizations of pureness and extremeness when the range is an arbitrary C^*-algebra?

Paschke answered question 2 in case of normal completely positive maps between von Neumann algebras. Here we answer question 1 for C^*-algebra B with only finite-dimensional irreducible representations, and the discussion of question 2 in the context of C^*-algebra will be included in the later sections.

Proposition 2.2. *Let M_n denote the C^*-algebra of $n\times n$ matrices. If Φ is pure in $CP(A,M_n)$, then $N_\Phi = {}_\Phi\text{ker} + \text{ker}_\Phi$.*

Proof. By a property of Φ due to Choi [3], that $\Phi(a)\Phi(a)^* \leqq \Phi(aa^*)$ for all $a\in A$, it follows that if $a\in {}_\Phi\text{ker}$ (or $a\in \text{ker}_\Phi$ respectively) then $\Phi(a)^*\Phi(a)\leqq\Phi(a^*a)=0$ (or $\Phi(a)\Phi(a)^* \leqq\Phi(aa^*)=0$ respectively), namely $a\in N_\Phi$. Thus, we need only to show $N_\Phi\subseteq {}_\Phi\text{ker}+\text{ker}_\Phi$. Let $\{\pi,\mathscr{K},v\}$ be the Stinespring representation triple induced by Φ. By 2.1, π is irreducible. Suppose $b\in N_\Phi$ and that $\{\eta_1,\dots,\eta_n\}$ is an orthonormal basis for C^n. Then $\{[b\otimes\eta_i]\,|\,i=1,\dots,n\}$ is orthogonal to $\{[1\otimes\eta_i]\,|\,i=1,\dots,n\}$ in $\mathscr{K}$. By Theorem 2.1, π is irreducible. Then it follows from Kadison's transitivity for irreducible representations that there exists a self-adjoint element a in A such that

$$\begin{cases}\pi(a)([b\otimes\eta_i])=[b\otimes\eta_i]\\ \pi(a)([I\otimes\eta_i])=0 \quad \text{for} \quad i=1,\dots,n.\end{cases} \tag{1}$$

(1) is equivalent to the following:

$$\begin{cases}\|[ab\otimes\eta_i-b\otimes\eta_i]\|=0\\ \|[a\otimes\eta_i]\|=0 \quad i=1,\dots,n.\end{cases} \tag{2}$$

It follows from (2) that

$$\begin{cases}\langle\Phi((ab-b)^*(ab-b)\eta_i,\eta_i\rangle=0\\ \langle\Phi(a^2)\eta_i,\eta_i\rangle=0 \quad i=1,2,\dots,n,\end{cases}$$

and hence $\Phi((ab-b)^*(ab-b))=0$ and $\Phi(a^2)=0$ i.e., $b-ab\in{}_\Phi\text{ker}$ and $a\in\text{ker}_\Phi$. Thus, we have $ab\in\text{ker}_\Phi$ and $b=(b-ab)+ab\in{}_\Phi\text{ker}+\text{ker}_\Phi$. □

Proposition 2.3. *Let $B=\sum_{i\in\mathscr{I}}\oplus B_i$ be the C^*-direct sum of C^*-algebras B_i, $i\in\mathscr{I}$, and I_i be the unit element of B_i. If Φ is a pure element in $CP(A,B)$, then $\Phi(\cdot)I_i=0$ for all i in $\mathscr{I}$ except one.*

Proof. Suppose that there exists $a\in A$ and $i_0\neq j_0$ in $\mathscr{I}$ such that $\Phi(a)I_{i_0}\neq 0$ and $\Phi(a)I_{j_0}\neq 0$. Since $\Phi(a)^*\Phi(a)I_i\leqq\Phi(a^*a)I_i$ and $a^*a\leqq\|a\|^2I$, it follows that $\Phi(a^*a)I_{i_0}\neq 0$, $\Phi(a^*a)I_{j_0}\neq 0$, and $\Phi(I)I_{i_0}\neq 0$, $\Phi(I)I_{j_0}\neq 0$. This $\psi(a)$, defined by $\psi(a)=\Phi(A)I_{i_0}$, is an element in $CP(A,B)$ with is majorized by Φ, and not equal to a scalar multiple of Φ. This is a contradiction to the assumption of Φ being pure, and this completes the proof.

□

Theorem 2.4. *Suppose that B has only finite-dimensional irreducible representations. For every pure element Φ in $CP(A,B)$, we have $N_\Phi = {}_\Phi\ker + \ker_\Phi$.*

Proof. Let the centre of B be isomorphic to $C(X)$. We decompose X as the disjoint union of the atomic part X_a and the diffused part X_d. Let P_d be the central projection of B onto $C(X_d)$. It is clear that $CP(A, BP_d)$ has no pure element. Since B is isomorphic onto $\sum_{i\in X_a} \oplus BP_i \oplus BP_d$, where P_i is the central projection corresponding to $i \in X_a$. BP_i is isomorphic to M_{n_i} for some n_i. By Proposition 2.3 Φ is pure in $CP(A,B)$ if and only if $\Phi(\ )P_d = 0$, $\Phi(\ )P_i = 0$ for all $i \in X_a$ except for $i = i_0$. By Proposition 2.2, if Φ is pure, then $N_\Phi = {}_\Phi\ker + \ker_\Phi$. □

Example 2.5. The converse of Theorem 2.4 does not hold in general. This can be explained in the following example.

Let Φ be a completely positive linear map of M_2 into itself defined as

$$\Phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}.$$

$$\text{Then } N_\Phi = \left\{\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \middle| \, b, c, d \text{ arbitrary}\right\}, \; {}_\Phi\ker = \left\{\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} \middle| \, b, d, \text{ arbitrary}\right\}$$

$$\ker_\Phi = \left\{\begin{bmatrix} 0 & 0 \\ c & \mathrm{d} \end{bmatrix} \middle| \, c, d \text{ arbitrary}\right\}.$$

Hence $N_\Phi = {}_\Phi\ker + \ker_\Phi$. However, Φ is not pure in $P_\infty(M_2, M_2)$. In fact, let E_{ij} be the elementary matrix unit with all entries zero, except the ijth entry equal to 1, for $i, j = 1, 2$. Define $\Phi_1(X) = (\frac{1}{2}E_{11} + \frac{1}{4}E_{21})X(\frac{1}{2}E_{11} + \frac{1}{4}E_{12}) + \frac{3}{16}E_{21}XE_{12}$ for all $X \in M_2$. It is obvious that Φ_1 is completely positive and

$$\Phi_1\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{4}a & \frac{1}{8}a \\ \frac{1}{8}a & \frac{1}{4}a \end{bmatrix}.$$

Since $\Phi(X) = E_{11}XE_{11} + E_{21}XE_{12}$, it follows that

$$(\Phi - \Phi_1)(X) = \left(\frac{\sqrt{3}}{2}E_{11} + \frac{1}{4\sqrt{3}}E_{21}\right)X\left(\frac{\sqrt{3}}{2}E_{11} + \frac{1}{4\sqrt{3}}E_{12}\right) + \frac{35}{48}E_{21}XE_{12}$$

is completely positive and $\Phi_1 \neq \alpha\Phi$ for any scalar α.

3. Extreme *n*-positive linear maps

In this section, we study the relation between $N_\Phi = {}_\Phi\ker + \ker_\Phi$ and the extremeness

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for a n-positive linear map Φ, for $n \geqq 2$. Let Φ be a projection map of A onto a C^*-algebra B of B of norm one. According to Tomiyama [10], Φ is completely positive and satisfies $\Phi(xy) = x\Phi(y)$, $\Phi(yx) = \Phi(y)x$ for all $y \in A$, $x \in B$.

Theorem 3.1. *Let Φ be a projection map of norm one from A onto B. If $N_\Phi = {}_\Phi\ker + \ker_\Phi$, then Φ is extreme in $P_n(A, B, I_B)$, for $n \geqq 2$.*

We need the following lemmas in proving Theorem 3.1. We assume Φ is a projection map of norm one and $N_\Phi = {}_\Phi\ker + \ker_\Phi$ in the following lemmas.

Lemma 3.2. *If $\Phi = \frac{1}{2}(\Phi_1 + \Phi_2)$ for $\Phi_i \in P_n(A, B, I_B)$, $i = 1, 2$, then $N_{\Phi_1} = N_{\Phi_2} = N_\Phi$.*

Proof. Since $\frac{1}{2}\Phi_i \leqq \Phi$, $i = 1, 2$, it follows that ${}_\Phi\ker \subseteq {}_{\Phi_i}\ker$ and $\ker_\Phi \subseteq \ker_{\Phi_i}$, $i = 1, 2$. Hence $N_\Phi = {}_\Phi\ker + \ker_\Phi \subseteq {}_{\Phi_i}\ker + \ker_{\Phi_i}$, $i = 1, 2$. For $\Phi_i \in P_n(A, B, I_B)$, by a theorem due to Choi [3], we have $\Phi_i(a)^*\Phi_i(a) \leqq \Phi_i(a^*a)$ for $a \in A$, $i = 1, 2$, and hence ${}_{\Phi_i}\ker + \ker_{\Phi_i} \subseteq N_{\Phi_i}$ $i = 1, 2$. Thus, $N_\Phi \subseteq N_{\Phi_i}$ $i = 1, 2$. We need only show $N_\Phi \subseteq N_{\Phi_i} = 1, 2$. For all elements a in A, a can be decomposed as $a = \Phi(a) + \bar{a}$, where $\bar{a} = a - \Phi(a)$. It is obvious that $\bar{a} \in N_\Phi$. Suppose that a is in $N\Phi_1$, but not in N_Φ. Then $\Phi(a) = \frac{1}{2}(\Phi_1(a) + \Phi_2(a)) = \frac{1}{2}\Phi_2(a) = \frac{1}{2}\Phi_2(\Phi(a) + \bar{a}) = \frac{1}{2}\Phi_2(\Phi(a))$, with $\Phi(a) \neq 0$; namely $\Phi_2(\Phi(a)) = 2\Phi(a)$. That implies $\|\Phi_2\| \geqq 2$ which is a contradiction to the fact that Φ_2 is in $P_n(A, B, I_B)$. Therefore $N_{\Phi_1} = N_\Phi$. Similarly $N_{\Phi_2} = N_\Phi$. □

Lemma 3.3. *If $\Phi = \frac{1}{2}(\Phi_1 + \Phi_2)$ for $\Phi_i \in P_n(A, B, I_B)$ $i = 1, 2$, then $\Phi_i\Phi = \Phi_i$ and $\Phi_i^2 = \Phi_i$, $i = 1, 2$.*

Proof. For $a \in A$, $\Phi_i(a) = \Phi_i(\Phi(a) + \bar{a})$, where $\bar{a} \in N_\Phi \subseteq N_{\Phi_i}$ $i = 1, 2$. Thus $\Phi_i(a) = \Phi_i\Phi(a)$ $i = 1, 2$. Since $\Phi|_B = id$ is extreme in $P(B, I_B)$, and

$$\Phi(a) = \Phi(\Phi(a)) = \tfrac{1}{2}\{\Phi\Phi_1(a) + \Phi\Phi_2(a)\}$$

$$= \tfrac{1}{2}\{\Phi\Phi_1(\Phi(a)) + \Phi\Phi_2(\Phi(a))\},$$

we have $\Phi(a) = \Phi(\Phi(a)) = \Phi\Phi_i(\Phi(a))$ for all a in A, $i = 1, 2$. Hence $\Phi_i^2 = (\Phi_i\Phi)^2 = \Phi_i(\Phi\Phi_i\Phi) = \Phi_i\Phi = \Phi_i$. □

3.4. Proof of Theorem 3.1.

$$\Phi(a) = \tfrac{1}{2}\{\Phi_1(a) + \Phi_2(a)\}$$

$$= \tfrac{1}{2}\{\Phi_1\Phi(a) + \Phi_2\Phi(a)\} \quad \text{(by Lemma 3.3)}.$$

Since the identity map is extreme in $P(B, I_B)$, it follows that $\Phi(a) = \Phi_i\Phi^2(a) = \Phi_i\Phi(a)$ for all a in A. Thus $\Phi = \Phi_i\Phi = \Phi_i$ for $i = 1, 2$. □

The requirement for Φ to be a projection map of norm one in Theorem 3.1 seems somewhat restrictive. In fact, Theorem 3.1 can be extended to include other maps in the following theorem. Even though the following theorem will supersede Theorem 3.1, I think the difference of approaches in these two proofs warrants the appearance of both. A positive map Φ of A into B is said to be *strongly positive*, if $\Phi^{-1}(x)$ contains a positive element in A for every positive element x in B.

Theorem 3.5. *Let Φ be a strongly positive and n-positive linear map of A onto B with $\Phi(1)=1$, for $n\geqq 2$, and ψ be n-positive and extreme in $P(B,D,p)$ where $p=\psi\circ\Phi(I)$. If $N_\Phi={}_\Phi\mathrm{ker}+\mathrm{ker}_\Phi$, then $\psi\circ\Phi$ is extreme in $P_n(A,D,p)$.*

Proof. Suppose $\psi\circ\Phi=\frac{1}{2}(\theta_1+\theta_2)$ for $\theta_i\in P_2(A,D,p)$ $i=1,2$. We will "factor θ_i through Φ", i.e., find $\psi_i\in P(B,D,p)$ such that $\theta_i=\psi_i\circ\Phi$ $i=1,2$. Define ψ_i: $B\to D$ as follows: for x in B, $\psi_i(x)=\theta_i(y)$ for some y in A with $\Phi(y)=x$, $i=1,2$. We first show that the ψ_i's are well-defined. For, if $\Phi(y)=0$ and $\psi\circ\Phi(y)=0$, then by the assumption that $N_\Phi={}_\Phi\mathrm{ker}+\mathrm{ker}_\Phi$, we have $y=y_1+y_2$ with $\Phi(y_1^*y_1)=0$ and $\Phi(y_2y_2^*)=0$.

Since $\theta_i\leqq 2(\psi\circ\Phi)$, we have $\theta_i(y_1^*y_1)\leqq 2(\psi\circ\Phi)(y_1^*y_1)=0$ and $\theta_i(y_2y_2^*)\leqq 2(\psi\circ\Phi)(y_2y_2^*)=0$, i.e., $y_1\in{}_{\theta_i}\mathrm{ker}$ and $y_2\in\mathrm{ker}_{\theta_i}$ $i=1,2$. This implies that $y=y_1+y_2\in{}_{\theta_i}\mathrm{ker}+\mathrm{ker}_{\theta_i}\subseteq N_{\theta_i}$ $i=1,2$ and hence $\theta_i(y)=0$. Thus ψ_i is well-defined.

Since Φ is strongly positive, it follows that ψ_i is positive, $i=1,2$. Now, it is easy to verify that $\theta_i=\psi_i\circ\Phi$, $\psi_i(I)=p$, $i=1,2$, and $\frac{1}{2}(\psi_1+\psi_2)(x)=\frac{1}{2}(\theta_1+\theta_2)(y)=\psi\circ\Phi(y)=\psi(x)$ for all x in B.

Since ψ is assumed to be extreme in $P(A,D,p)$, it follows that $\psi_1=\psi_2=\psi$ and hence $\theta_1=\theta_2=\psi\circ\Phi$. □

Corollary 3.6. *Let Φ be a $*$-homomorphism of A onto B and let ψ be n-positive and extreme in $P(B,D,p)$ where $\psi(I)=p$ and $n\geqq 2$. Then $\psi\circ\Phi$ is extreme in $P_n(A,D,p)$.*

Proof. $N_\phi={}_\phi\mathrm{ker}=\mathrm{ker}_\phi$ holds for any $*$-homomorphism Φ, and any $*$-homomorphism is a strongly positive map. Thus, by Theorem 3.5, $\psi\circ\Phi$ is extreme in $P_n(A,D,p)$. □

In Theorem 3.5, when $B=D$ and $\psi=$identity map, we get a general version of Theorem 3.1 as follows.

Corollary 3.7. *Every unital strongly positive and n-positive linear map Φ of A onto B with $n\geqq 2$ satisfying $N_\Phi={}_\Phi\mathrm{ker}+\mathrm{ker}_\Phi$ is extreme in $P_n(A,B,I_B)$.*

Remarks 3.8. Furthermore, in case of nonunital Φ, Corollary 3.7 is still true. For, if $\Phi(1)=p\neq I_B$, then consider $\psi(x)=p^{-1/2}\Phi(x)p^{-1/2}$, where $p^{-1/2}$ is the densely defined positive operator and $\psi(x)$ is strongly positive and n-positive unital with $N_\psi=N_\Phi$, ${}_\psi\mathrm{ker}={}_\Phi\mathrm{ker}$, $\mathrm{ker}_\psi=\mathrm{ker}_\Phi$. ψ is extreme in $P_n(A,B,I_B)$ if and only if Φ is extreme in $P_n(A,B,p)$. Thus, every strongly positive and n-positive linear map Φ of A onto B satisfying $N_\Phi={}_\Phi\mathrm{ker}+\mathrm{ker}_\Phi$ is extreme in $P_n(A,B,\Phi(I))$.

3.9. Eamples

3.9.1. Let A be a unital C^*-algebra and p a projection in A. Define a completely positive map Φ of A onto the reduced algebra pAp by $\Phi(x)=pxp$ for x in A. It is clear that ${}_{\Phi}\ker=A(1-p)$ and $\ker_{\Phi}=(1-p)A$ and $N_{\Phi}=\{a\in A \mid pap=0\}$. For $x\in N_{\Phi}$, $x=[(1-p)+p]x[(1+p)+p] = (1-p)x(1-p)+(1-p)xp+px(1-p)+pxp = (1-p)x+px(1-p)\in {}_{\Phi}\ker+\ker_{\Phi}$. Hence $N_{\Phi}={}_{\Phi}\ker+\ker_{\Phi}$. By Theorem 3.1 Φ is extreme in $P_n(A,B,1)$, for $n\geqq 2$.

3.9.2. This example can be viewed as analogous to Arveson's characterization of extreme completely positive linear maps of A into $B(\mathscr{H})$. Let π be a $*$-homomorphism of A onto B and B act on a Hilbert space $\mathscr{H}$. Let v be a bounded operator from Hilbert space $\mathscr{K}$ into $\mathscr{H}$ such that the support of $vv^*(=q)\in B$ and $v^*v=p$. Then $\Phi(x)$ defined by $\Phi(x)=v^*\pi(x)v$ for $x\in A$ is extreme in $P_n(A,D,p)$, where $D=v^*\pi(A)v$. For Φ is strongly positive and n-positive, and

$$N_{\Phi}=\pi^{-1}\{x\in B \mid v^*xv=0\}=\pi^{-1}\{x\in B \mid qxq=0\},$$

$${}_{\Phi}\ker=\pi^{-1}\{x\in B \mid v^*x^*xv=0\}=\pi^{-1}\{x\in B \mid xq=0\}=\pi^{-1}(B(1-q)),$$

$$\ker_{\Phi}=\pi^{-1}\{x\in B \mid v^*xx^*v=0\}=\pi^{-1}\{x\in B \mid qx=0\}=\pi^{-1}((1-q)B).$$

From the above calculation, we have $N_{\Phi}={}_{\Phi}\ker+\ker_{\Phi}$, for π is a $*$-homomorphism onto B. It is clear that Φ is strongly positive. Then, by Remark 3.8, Φ is extreme in $P_n(A,B,p)$, for $n\geqq 2$.

4. Constructions of extreme n-positive maps

The condition $N_{\Phi}={}_{\Phi}\ker+\ker_{\Phi}$ is not necessary for Φ to be extreme in $P_n(A,B,\Phi(I))$, even for $n=\infty$. This will be made clear in the following example.

Example 4.1. Let Φ be a completely positive linear map of M_2, the 2×2 matrix algebra, into $C\oplus C$ defined by

$$\Phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)=a\oplus d.$$

It is easy to see that

$$N_{\Phi}=\left\{\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \middle| b,c,\in \mathbf{C}\right\}$$

and ${}_{\Phi}\ker=\ker_{\Phi}=0$. Indeed, if $\Phi(x^*x)=0$ for

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$$x=\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then $|a|^2+|c|^2=0+|b|^2+|d|^2=0$ and $x=0$. Then fact that such a map Φ is extreme in $CP(M_2, \mathbf{C}^2, I_{\mathbf{C}_2})$ can be calculated directly or seen as an easy consequence of the following theorem.

Theorem 4.2. *Let $\Phi_i \in P_n(A, B, p_i)$, $i \in \mathscr{I}$, and B_i be the hereditary C^*-subalgebra generated by p_i in B, $i \in \mathscr{I}$. Suppose that $B_i \cap B_j = \{0\}$, for $i \neq j$ in $\mathscr{I}$, and Φ_i is extreme in $P_n(A, B_i, p_i)$, $i \in \mathscr{I}$. Let Φ be defined by $\sum_{i\in\mathscr{I}} \Phi_i$ in point-norm topology as the limit of the net f finite sums of Φ_i's. Then Φ is extreme in $P_n(A, B, p)$ where $p=\sum_{i\in\mathscr{I}} p_i$ in norm topology. The value of n can be $1, 2, \ldots, \infty$.*

Proof. Suppose $\Phi=\frac{1}{2}(\theta+\psi)$ for $\theta, \psi \in P_n(A, B, p)$. The hereditary C^*-subalgebra generated $\{B_i, i\in\mathscr{I}\}$ *is* $\sum_{i\in\mathscr{I}} \oplus B_i$ the C^*-direct sum of B_i's, for $B_i \cap B_j=\{0\}$, $i\neq j$ in $\mathscr{I}$. For $i\in\mathscr{I}$, $\Phi_i(A)$ is contained in B_i, and hence $\Phi(A)$ is contained in $\sum_{i\in\mathscr{I}} \oplus B$. So, $\theta(A)$ and $\psi(A)$ are contained in $\sum_{i\in\mathscr{I}} \oplus B_i$, for $\theta \leqq 2\Phi$, $\psi \leqq 2\Phi$. For each $i\in\mathscr{I}$, define $\theta_i(x)=p_i\theta(x)p_i$, $\psi_i(x)=p_i\psi(x)p_i$. It is easy to see that $\theta(x)=\sum_{i\in\mathscr{I}}\theta_i(x)$, $\psi(x)=\sum_{i\in\mathscr{I}}\psi_i(x)$, both of these infinite sums converging in norm, and $\phi_i=\frac{1}{2}(\theta_i+\psi_i)$ with $\theta_i(I)=\psi_i(I)=P_i$, $i\in\mathscr{I}$. Since Φ_i is assumed to be extreme in $P_n(A, B, p_i)$, it follows that $\theta_i=\psi_i=\Phi_i$ and hence $\theta=\psi=\Phi$. □

4.3. Concluding remark. In finite dimensional cases when $A=M_n$ and $B=M_m$, M_k the $k\times k$ full matrix algebra, every completely positive linear map Φ of A onto B is of the form $\Phi(x)=\sum_{i=1}^k v_i^* x v_i$ for $x\in A$ where v_i is a $n\times m$ matrix and $\{v_i^*\}_{i=1}^k$ is linearly independent. Φ is extreme in $P_\infty(A, B, \phi(I))$ if and only if $\{v_j^* v_i\}_{i,j=1}^k$ is linearly independent (see [4] for details). In this case each map $\Phi_i(x)=v_i^* x v_i$, for $i=1,\ldots,k$, satisfies $N_{\Phi_i}={}_{\Phi_i}\ker+\ker_{\Phi_i}$ for $i=1,\ldots,k$. In [4] Choi completely determined the form of extreme completely positive linear maps of M_n into M_m. Recently, Anantharaman-Delraoche [1] described a similar procedure of constructing some extreme normal completely positive linear maps of a von Neumann algebra M to another von Neumann algebra N, given a fixed irreducible correspondence H from M to N. Now, it is interesting to find out if his procedure is general enough to include all normal extreme completely positive linear maps. On the other hand, Theorem 4.2 suggests a procedure to construct extreme n-positive linear maps by considering C^*-direct sums of extreme n-positive linear maps. Combining example 4.1 and Theorem 4.2 one cannot help but ask whether all extreme n-positive linear maps can be written as C^*-direct sums of n-positive linear maps ϕ_i, $i\in\mathscr{I}$, which satisfy the condition $N_{\Phi_i}={}_{\Phi_i}\ker+\ker_{\Phi_i}$ for $i\in\mathscr{I}$.

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