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Dual formulation of constrained solutions of the multi-state Choquard equation

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The Choquard equation is a partial differential equation that has gained significant interest and attention in recent decades. It is a nonlinear equation that combines elements of both the Laplace and Schrödinger operators, and it arises frequently in the study of numerous physical phenomena, from condensed matter physics to nonlinear optics.

In particular, the steady states of the Choquard equation were thoroughly investigated using a variational functional acting on the wave functions.

In this article, we introduce a dual formulation for the variational functional in terms of the potential induced by the wave function, and use it to explore the existence of steady states of a multi-state version the Choquard equation in critical and sub-critical cases.

Keywords: Shrodinger-Poisson; Choquard equation; variational methods

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1. Introduction

1.1. Background

The Choquard equation

$$-\Delta\phi + \phi - \left(\int \frac{|\phi(y)|^2}{|x-y|}\,\mathrm{d}y\right)\phi(x) = 0$$

in $\mathbb{R}^3$ was originally proposed by Ph. Choquard, as an approximation to Hartree-Fock theory for a one component plasma. Equation of similar types also appear to be a prototype of the so-called nonlocal problems, which arise in many situations (see, e.g [17]) and as a model of self-gravitating matter [11].

A generalized version in $\mathbb{R}^n$ takes the form

$$-\Delta\phi + \phi = (I_\alpha * |\phi|^p)\,|\phi|^{p-2}\phi \tag{1.1}$$

where

$$I_\alpha = A(\alpha)|x|^{\alpha-n}; \quad A(\alpha) := \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{n/2}\Gamma\left(\alpha/2\right)} \tag{1.2}$$

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is the Rietz potential, $\alpha \in (0, n)$, $p \in (1, \infty)$ was considered by many authors in the last decades, using its variational structure as a critical point of the functional

$$E_{p,\alpha}(\phi) = \frac{1}{2} \int_{\mathbb{R}^n} \left(|\nabla u|^2 + |\phi|^2 - \frac{1}{2p} \left(I_\alpha * |\phi|^p\right) |\phi|^p \right) \tag{1.3}$$

on an appropriate space. In particular, existence of solutions the case $p = 2$ (and for more general singular interaction kernels) was studied by E.H. Lieb, P.L Lions and G. Menzala [**6**, **7**, **10**]. For existence, regularity and asymptotic behaviour of solutions in the general case see [**12**, **13**] and references therein.

The non-linear Schrödinger equation associated with $E_{p,\alpha}$ takes the form

$$-i\partial_t \psi - \Delta\psi - a(I_\alpha * |\psi|^p)|\psi|^{p-2}\psi = 0\ . \tag{1.4}$$

The number $a \in \mathbb{R}$ is the strength of interaction. The case $a > 0$ corresponds to the *attractive, gravitation-like* dynamics, and is related to Choquard's equation. The case $a < 0$ is the repulsive, electrostatic case and is related to the Hartree system (see, e.g. [**18**]). In this paper we deal with the attractive case.

Considering an eigenmode $\psi = \mathrm{e}^{-i\lambda t}\phi$ we get that ϕ satisfy the non-linear eigenvalue problems

$$-\Delta\phi - a\left(I_\alpha * |\phi|^p\right)|\phi|^{p-2}\phi - \lambda\phi = 0 \tag{1.5}$$

which can be reduced to (1.1) by a proper scaling[1] . However, the solutions of the nonlinear equation (1.4) preserve the $\mathbb{L}^2$ norm, so it is natural to look for stationary solutions (1.5) under a prescribed $\mathbb{L}^2$ norm (say, $\|\phi\|_2 = 1$). It is not difficult to see that, in general, one can find a scaling $\phi \mapsto \phi_\epsilon(x) = \epsilon^{-n/2}\phi(\epsilon/x)$ which preserves the $\mathbb{L}^2$ norm and transform the strength of interaction in (1.5) into $a = 1$, making this parameter mathematically insignificant. There is, however, an exceptional case $\alpha = n(p-1) - 2$. In that case the first two terms in (1.5) are transformed with equal coefficients under $\mathbb{L}^2$ preserving scaling, so the size of the interaction coefficient a is mathematically significant in that case.

In the case $p = 2$ and in the presence of a prescribed, confining potential W, the $\mathbb{L}^2-$ constraint version of (1.5) takes the form

$$-\Delta\phi + W\phi - a\left(\int_{\mathbb{R}^n} \frac{|\phi(y)|^2}{|x-y|^{n-\alpha}}\,\mathrm{d}y\right)\phi - \lambda\phi = 0, \quad \|\phi\|_2 = 1\ . \tag{1.6}$$

A solution of (1.6) is given by a minimizer of the functional

$$E_a^W(\phi) := \frac{1}{2}\int_{\mathbb{R}^n} \left(|\nabla\phi|^2 + W|\phi|^2\right)\,\mathrm{d}x - \frac{a}{4}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\phi(x)|^2|\phi(y)|^2}{|x-y|^{n-\alpha}}\mathrm{d}x\mathrm{d}y \tag{1.7}$$

restricted to the $\mathbb{L}^2$ unit ball $\|\phi\|_2 = 1$.

In [**8**] the authors studied the equation (1.6) in the exceptional case $\alpha = n - 2$, for $n \geqslant 3$, $a > 0$ and W a prescribed function satisfying $\lim_{x\to\infty} W(x) = \infty$. In particular, they showed the existence of a critical strength $\bar{a}_c > 0$, depending on n but independent of W, such that E_a^W is bounded from below on the sphere $\|\phi\|_2 = 1$

[1]Note that $\lambda < 0$ is an eigenvalue below the essential spectrum of $-\Delta$

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iff $a \leqslant \bar{a}_c$. Moreover, a minimizer of E^W_a exists if $a < \bar{a}_c$, and is a solution of (1.6) (c.f. [8]). It was also shown that $a_c = \|\bar{\phi}\|_2$, where $\bar{\phi}$ is the unique, positive solution (c.f. [9]) of the equation of

$$-\Delta\bar{\phi} - \left(\int_{\mathbb{R}^n} \frac{|\bar{\phi}(y)|^2}{|x-y|^2}\,\mathrm{d}y\right)\bar{\phi} + \bar{\phi} = 0. \tag{1.8}$$

The object of the present paper is two-fold.

The first object is to extend the $\mathbb{L}^2$-constraint Choquard equation (1.6) into a $k-$ state system

$$\begin{aligned} &-\Delta\phi_j + W\phi_j - a\left(\sum_{i=1}^k \beta_i \int_{\mathbb{R}^n} \frac{|\phi_i|^2(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y\right)\phi_j - \lambda_j\phi_j \\ &= 0 \quad \|\phi_j\|_2 = 1, \quad ; \quad j = 1\ldots k \end{aligned} \tag{1.9}$$

where $(\phi_1, \ldots \phi_k)$ constitutes an orthonormal $k-$sequence in $\mathbb{L}^2(\mathbb{R}^n)$ and

$$\beta_j > 0, \qquad \sum_1^k \beta_j = 1 \tag{1.10}$$

are the *probabilities of occupation* of the states $j = 1\ldots k$,

In § 1.2 we introduce the time dependent Heisenberg system which leads naturally to (1.9), while the steady state (1.9) and its constraint variational formulation are introduced in § 1.3.

The second object is to introduce a dual approach to the $\mathbb{L}^2$ constraint Choquard problem in the case $p = 2$. For the case of single state $k = 1$, the dual formulation of E^W_a (1.7) for $\alpha = 2$ on the constraint $\mathbb{L}^2$ sphere takes the form of the functional $V \mapsto \mathcal{H}^{W,\alpha}_a(V)$

$$\mathcal{H}^{W,2}_a(V) = \frac{a}{2}\int_{\mathbb{R}^n} |\nabla V|^2 + \lambda_1(V)$$

over the *unconstrained* Beppo-Levi space $V \in \dot{\mathbb{H}}_1(\mathbb{R}^n)$ (c.f. §§ 1.4). Here the functional $\lambda_1 = \lambda_1(V)$ is the leading (minimal) eigenvalue of the Schrödinger operator $-\Delta + W - aV$ on $\mathbb{R}^n$.

The extension of this dual formulation to the $k-$system (1.9) for $\alpha \in (0, 2]$ is introduced in (1.28). In case $\alpha = 2$ it takes the form

$$\mathcal{H}^{W,2}_{\beta,a}(V) = \frac{a}{2}\int_{\mathbb{R}^n} |\nabla V|^2 + \sum_{j=1}^k \beta_j\lambda_j(V)$$

where $\lambda_1(V) < \lambda_2(V) \leqslant \ldots \lambda_k(V)$ are the leading k eigenvalues of the Schrödinger operator, while $\beta_1 > \beta_2 > \ldots \beta_k > 0$.

The main result of this paper is summarized below (§ 1.6):

Using the dual variational formulation we show the existence of a minimizer of $\mathcal{H}^{W,\alpha}_{\beta,a}$ corresponding to a solution of (1.9) in $\mathbb{R}^n$ for any $a > 0$ where $\alpha \in (0, 2]$, $3 \leqslant n < 2 + \alpha$. In the critical cases $\alpha = 2$, $n = 4$ and $\alpha = 1$, $n = 3$ we show the

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existence of a critical interaction level $a_c^{(n)}(\beta)$ for which there is a minimizer of $\mathcal{H}_{\beta,a}^{W,\alpha}$ if $a < a_c^{(n)}(\beta)$ corresponding to a solution of (1.9), while $\mathcal{H}_{\beta,a}^{W,\alpha}$ is unbounded from below for any $a > a_c^{(n)}(\beta)$ for $n = 3, 4$.

1.2. Mean-field Heisenberg system

Consider the Von Neumann-Heisenberg equation

$$i\frac{\partial R}{\partial t} = \left[L^W - aV, R\right], \quad t \in \mathbb{R} \tag{1.11}$$

on a Hilbert space $\mathbb{H}$. Here R is a density operator, namely a bounded linear operator on $\mathbb{H}$ which is self-adjoint, non-negative and of trace equal one. L^W is an Hermitian operator generating a norm preserving group e^{itL^W} on $\mathbb{H}$ and V is a non-linear operator.

In the context of mean-field system we consider $(\mathbb{H}, \langle\cdot, \cdot\rangle)$ to be the Hilbert space $\mathbb{L}^2(\mathbb{R}^n)$ where $\langle\phi, \psi\rangle := \int_{\mathbb{R}^n} \phi\bar{\psi}$ the canonical inner product. A density operator can be represented by a kernel K_R acting on $\phi \in \mathbb{H}$ via $R(\phi) = \int_{\mathbb{R}^n} K_R(x, y)\phi(y)\,\mathrm{d}y$ and $Tr(R)(x) := K_R(x, x)$. In these terms we define $V(R)$ as the operator acting on $\phi \in \mathbb{H}$ by multiplication with

$$V(R) := I_\alpha * Tr(R) \tag{1.12}$$

Since $L^W - aV$ is hermitian for any prescribed potential V, all observables along the orbit $t \mapsto R(\cdot, t)$ are unitary equivalent:

$$R(\cdot, t) = \exp\left(-i\int_0^t (L^W - aV(\cdot, s))\mathrm{d}s\right) R(\cdot, 0) \exp\left(i\int_0^t (L^W - aV(\cdot, s))\mathrm{d}s\right). \tag{1.13}$$

We restrict ourselves to a class of observables of a finite rank $k \in \mathbb{N}$. Hence the kernel of R can be represented as

$$R(x, y, t) = \sum_1^k \beta_j \psi_j(x, t)\bar{\psi}_j(y, t) \tag{1.14}$$

where $\beta_j > 0$ are the eigenvalues of R, which are constant in time, and $\psi_j(\cdot, t) \in \mathbb{H}$ constitute an orthonormal sequence for any $t \in \mathbb{R}$. Under this representation (1.11) takes the form

$$i\frac{\partial \psi_j}{\partial t} = (L^W - aV)\psi_j \quad , j = 1, 2, \ldots k \tag{1.15}$$

The eigenvalues $\beta_j \in [0, 1]$ are interpreted as the probability of occupation of the $j-$ level satisfying $\sum_{j=1}^k \beta_j = 1$. For any $t \in \mathbb{R}$, the trace of R conditioned on

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$x \in \mathbb{R}^n$ is

$$Tr(R)(x,t) = \sum_{j=1}^{k} \beta_j |\psi_j(x,t)|^2, \tag{1.16}$$

and the potential V is determined in terms of the solution R by (1.12)

$$V = \sum_{j=1}^{k} \beta_j I_\alpha * |\psi_j|^2 \ .$$

Consider now the Hamiltonian

$$\mathcal{E}^{(\alpha)}_{\beta,a}(\vec{\psi}) := \frac{1}{2}\sum_{j=1}^{k}\beta_j\left[\left\langle L^W\psi_j,\psi_j\right\rangle - \frac{a}{2}\sum_{i=1}^{k}\beta_i\left\langle|\psi_j|^2, I_\alpha * |\psi_i|^2\right\rangle\right]$$

acting on $k-$ orthonormal frames $\vec{\psi} = (\psi_1, \ldots \psi_k)$. The system (1.11) (equivalently (1.15)) is, in fact, an Hamiltonian system in the canonical variables $\{\psi_i, \bar{\psi}_j\}$:

$$i\frac{\partial\psi_j}{\partial t} = -\frac{1}{\beta_j}\delta_{\bar{\psi}_j}\mathcal{E}^{(\alpha)}_{\beta,a}; \quad i\frac{\partial\bar{\psi}_j}{\partial t} = \frac{1}{\beta_j}\delta_{\psi_j}\mathcal{E}^{(\alpha)}_{\beta,a}. \tag{1.17}$$

In particular, $\mathcal{E}^{(\alpha)}_{\beta,a}$ is constant along the solution of (1.11).

1.3. Steady states

The steady states of this system are given by $\psi_j = \mathrm{e}^{-i\lambda_j}\phi_j$ where $\{\phi_j\}$ is an orthonormal sequence corresponding to eigenvalues λ_j of the operator $L^W - aV$, satisfying

$$L^W\phi_j - aI_\alpha * \left(\sum_{i=1}^{k}\beta_i|\phi_i|^2\right)\phi_j - \lambda_j\phi_j = 0. \tag{1.18}$$

Definition 1.1.

$$\mathbb{H}^1 := \{\phi \in \mathbb{L}^2(\mathbb{R}^n), \ ; \ \nabla\phi \in \mathbb{L}^2(\mathbb{R}^n); \ \|\phi\|_2 = 1, \quad \int_{\mathbb{R}^n} W|\phi|^2 < \infty \ \}$$

$$\oplus^k\mathbb{H}^1 := \left\{\vec{\phi} = (\phi_1, \ldots \phi_k), ; \phi_j \in \mathbb{H}^1; \ \langle\phi_j, \phi_i\rangle = \delta_i^j, \quad i, j \in \{1, \ldots k\}\right\}.$$

$\langle\langle\phi, \phi\rangle\rangle_W$ is the quadratic form on $\mathbb{H}^1$ defined by the completion of $\left\langle L^W\phi, \phi\right\rangle$:

$$\langle\langle\phi,\phi\rangle\rangle_W := \int_{\mathbb{R}^n}|\nabla\phi|^2 + W|\phi|^2.$$

Let

$$\mathcal{E}^{(\alpha)}_{\beta,a}(\vec{\phi}) := \frac{1}{2}\sum_{1}^{k}\beta_j\left[\langle\langle\phi_j,\phi_j\rangle\rangle_W - \frac{a}{2}\sum_{i=1}^{k}\beta_i\left\langle|\phi_j|^2, I_\alpha * |\phi_i|^2\right\rangle\right]$$

is defined over $\oplus^k\mathbb{H}^1$ (c.f. Corollary 2.4 below).

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We formally obtain from (1.17) that the steady states (1.18) are critical points of $\mathcal{E}^{(\alpha)}_{\beta,a}$ subject to the orthogonality constraints.

Proposition 1.2. *Suppose $\beta_j \neq \beta_i$ for any $1 \leqslant i \neq j \leqslant k$. Then any critical point of $\mathcal{E}^{(\alpha)}_{\beta,a}$ restricted to orthonormal frames $\overrightarrow{\phi} = (\bar{\phi}_1 \dots \bar{\phi}_k)$ is composed of k normalized eigenstates of the operator $L^W - a\bar{V}$ where $\bar{V} = I_\alpha * (\sum_1^k \beta_j |\bar{\phi}_j|^2)$.*

For the proof of proposition 1.2 see the beginning of §2.

From now on we assume

$$\beta_1 > \beta_2 > \dots > \beta_k > 0. \tag{1.19}$$

Formulation of the problem : Consider the multi-state Choquard system satisfying the equivalent of (1.18):

$$(L^W - aV)\phi_j - \lambda_j \phi_j = 0 \;\; ; j = 1, \dots \; k \tag{1.20}$$

on $\mathbb{R}^n$. Here:

i) $L^W = -\Delta + W$, $\Delta := \sum_{i=1}^n \partial^2_{x_i}$ is the Laplacian on $\mathbb{R}^n$ and

$$W \in \mathbb{L}^\infty_{loc}(\mathbb{R}^n), \quad \lim_{|x|\to\infty} W(x) = \infty, \quad \inf_{x \in \mathbb{R}^n} W(x) = W(0) = 0 \tag{1.21}$$

ii) $\vec{\phi} = (\phi_1, \dots \phi_k) \in \oplus^k \mathbb{H}^1$ are normalized eigenfunctions of $L^W - aV$ and $\lambda_j \in \mathbb{R}$ are the corresponding eigenvalues.

iii)

$$V = \sum_{i=1}^k \beta_i I_\alpha * |\phi_i|^2 \tag{1.22}$$

where β_j are the *probabilities of occupation* of the states j, thus $\beta_j > 0$ and $\sum_1^k \beta_j = 1$. iv) $a > 0$.

1.4. A crush review on Rietz kernels and its dual

Let us recall some definitions and theorems we use later (for more details see [4]): For $V_1, V_2 \in C_0^\infty(\mathbb{R}^n)$ and $\alpha \in (0, n)$, consider the quadratic form

$$\langle V_1, V_2 \rangle_{\alpha/2} := A(-\alpha) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(V_1(x) - V_1(y))(V_2(x) - V_2(y))}{|x-y|^{n+\alpha}} \mathrm{d}x\mathrm{d}y$$

where the constant $A(-\alpha)$ is defined as in (1.2). If $\alpha = 2$

$$\langle V_1, V_2 \rangle_{(1)} := \int_{\mathbb{R}^n} \nabla V_1 \cdot \nabla V_2 \, \mathrm{d}x \; .$$

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The closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm induced by the inner product $\langle \cdot, \cdot \rangle_{\alpha/2}$ is denoted by $\dot{\mathbb{H}}^{\alpha/2}$. We denote the associated norm by $|||\cdot|||_{\alpha/2}$.[2] Recall that $\dot{\mathbb{H}}^{\alpha/2}$ is a Hilbert space so, in particular, is weakly locally compact.

LEMMA 1.3 [**1**]. *For $\alpha \in (0, 2]$, $n > 2$, the space $\dot{\mathbb{H}}^{\alpha/2}$ is continuously embedded in $\mathbb{L}^{2n/(n-\alpha)}(\mathbb{R}^n)$, so there exists $S = S_{n,\alpha} > 0$ such that*

$$\|V\|_{2n/(n-\alpha)} \leqslant S_{n,\alpha} |||V|||_{\alpha/2}.$$

The fractional Laplacian $(-\Delta)^{\alpha/2}$, $0 < \alpha < 2$ is defined as a distribution by

$$\langle V, \phi \rangle_{\alpha/2} = \left\langle (-\Delta)^{\alpha/2} V, \phi \right\rangle \ \forall \ \ \phi \in C_0^\infty(\mathbb{R}^n).$$

and the pointwise definition of the fractional Laplacian for $0 < \alpha < 2$ is given in terms of the singular integral

$$(-\Delta)^{\alpha/2} V(x) = A(-\alpha) \int_{\mathbb{R}^n} \frac{V(x+y) - V(x)}{|y|^{n+\alpha}} \, \mathrm{d}y \ .$$

For $\alpha = 2$, the above definition is reduced to the classical, local Laplacian $-\Delta = \sum_{j=1}^n \partial_{x_j}^2$.

The Rietz potential I_α is defined as a distribution via the quadratic form induced by the dual of the $\langle \cdot, \cdot \rangle_{\alpha/2}$ inner product:

$$\frac{1}{2} \langle I_\alpha * \rho, \rho \rangle := \sup_{V \in C_0^\infty(\mathbb{R}^n)} \langle \rho, V \rangle - \frac{1}{2} \langle V, V \rangle_{\alpha/2} . \tag{1.23}$$

The Euler-Lagrange equation corresponding to the right hand side of (1.23) takes the form

$$(-\Delta)^{\alpha/2} V = \rho. \tag{1.24}$$

In particular, $I_\alpha \equiv (-\Delta)^{-\alpha/2}$ corresponds to the right inverse of the fractional Laplacian

$$I_\alpha * (-\Delta)^{\alpha/2} V = V \ . \tag{1.25}$$

The pointwise representation of the kernel I_α is given by (1.2). Moreover

LEMMA 1.4 [**16**]. *For any $0 < \alpha < n$, the Rietz potential is a bounded operator from $\mathbb{L}^p(\mathbb{R}^n)$ to $\mathbb{L}^q(\mathbb{R}^n)$ iff $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$.*

Our main results, described below, concern the Choquard problem on $\mathbb{R}^n$. However, in order to overcome problems of lack of compactness, we shell need to introduce a version of this problem in bounded domain $\Omega \subset \mathbb{R}^n$. In order to handle this we need to define the Green function corresponding to the fractional Laplacian

[2]Note that $\dot{\mathbb{H}}^{\alpha/2}(\mathbb{R}^n)$ does not contain $\mathbb{L}^2(\mathbb{R}^n)$. In case $\alpha = 2$ it is sometimes called Beppo-Levi space.

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$(-\Delta)^\alpha$ in a bounded domain under homogeneous Dirichlet condition. This is the motivation to define the *local Rietz potential* I^Ω_α on $\mathbb{L}^p(\Omega)$ by

$$\frac{1}{2}\left\langle I^\Omega_\alpha(\rho),\rho\right\rangle := \sup_{\phi\in C_0^\infty(\Omega)}\langle\rho,\phi\rangle-\frac{1}{2}\langle\phi,\phi\rangle_{\alpha/2}\,. \tag{1.26}$$

In the case $\alpha=2$ this definition induces the Green function of the Dirichlet problem $I^\Omega_2\equiv(-\Delta_\Omega)^{-1}$, that is, $V(x)=\int_\Omega I^\Omega_2(x,\,y)\rho(y)\,\mathrm{d}y$ is the solution of the Poisson problem

$$\Delta V+\rho=0\quad x\in\Omega;\quad V=0\ \text{ on }\ \partial\Omega\,. \tag{1.27}$$

Not much is known[3] on the Green function I^Ω_α for $\alpha<2$. In case $\alpha=2$ the maximum principle implies immediately that for any $x,\,y\in\Omega$ the inequality $I_2(x-y)\geqslant I^\Omega_2(x,\,y)$ holds, and that $I^\Omega_2(x,\,y)=0$ if $x\in\Omega,\ y\in\partial\Omega$. In the general case we obtain from (1.26) :

Lemma 1.5. *For any $0<\alpha<2$, $\Omega_2\supset\Omega_1$ and supp$(\rho)\subset\Omega_1$ then $I^{\Omega_1}_\alpha(\rho)\leqslant I^{\Omega_2}_\alpha(\rho)\leqslant I_\alpha*\rho$.*

In addition: Let $\Omega_j\subset\mathbb{R}^n$, $\Omega_j\to\mathbb{R}^n$ is a monotone sequence of domains in $\mathbb{R}^n$. If ρ_j converges to ρ in $\mathbb{L}^p(\mathbb{R}^n)$, $p\in(1,\,n/\alpha)$, and ρ_j are supported in Ω_j then

$$\lim_{j\to\infty}I^{\Omega_j}_\alpha(\rho_j)=I_\alpha*\rho\ \ in\ \ \mathbb{L}^{\frac{pn}{n-p\alpha}}(\mathbb{R}^n)\ .$$

1.5. Spectrum of the Schrödinger operator

One of the most celebrated results on the discreteness of spectrum for the Schrödinger operator $-\Delta+W$ in $\mathbb{L}^2(\mathbb{R}^n)$ with a locally integrable potential is a result of K. Friedrichs [**2**] which ensures the discreteness of spectrum if the potential W grows at infinity at arbitrary rate. This and (1.21) implies, in particular

Proposition 1.6. *Let $V\in C_0^\infty(\mathbb{R}^n)$ and W satisfies (1.21). Then the spectrum of the operator L^W-aV is composed of an infinite set of eigenvalues $\lambda_j\to\infty$ and the corresponding normalized eigenfunctions ϕ_j constitute a complete orthonormal base of $\mathbb{L}^2(\mathbb{R}^n)$.*

For the proof of proposition 1.7 see Lemma 2.7 in § 2.

Proposition 1.7. *If $3\leqslant n\leqslant 5$, $0<\alpha\leqslant 2$, then proposition 1.6 can be extended for $V\in\dot{\mathbb{H}}^{\alpha/2}(\mathbb{R}^n)$, and $V\mapsto\lambda_j(V)$ is a continuous functional in the $|||\cdot|||_{\alpha/2}$ norm.*

Proposition 1.7 allows us to define the dual functional on $\dot{\mathbb{H}}^{\alpha/2}(\mathbb{R}^n)$:

$$\mathcal{H}^{W,\alpha}_{\beta,a}(V):=\frac{a}{2}\langle V,V\rangle_{\alpha/2}+\sum_{j=1}^k\beta_j\lambda_j(V) \tag{1.28}$$

where $\lambda_1(V)<\lambda_2(V)\leqslant\lambda_3(V)\ldots\leqslant\lambda_k(V)$ are the lowest k eigenvalues of the operator L^W-aV.

[3] But see § 3-c.

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1.6. Main theorem

For any $V \in \dot{\mathbb{H}}_{\alpha/2}(\mathbb{R}^n)$ let $\phi_j(V)$ be a normalized eigenstate corresponding to $\lambda_j(V)$[4]

[i] V is a minimizer of $\mathcal{H}^{W,\alpha}_{\beta,a}$ on $\dot{\mathbb{H}}_{\alpha/2}(\mathbb{R}^n)$ if and only if $(\phi_1(V), \ldots \phi_k(V))$ is a minimizer of $\mathcal{E}^{(\alpha)}_{\beta,a}$ on $\oplus^k \mathbb{H}^1$. If this is the case then $\{\lambda_j(V), \phi_j(V)\}_{1\leqslant j\leqslant k}$ is a solution of (1.20) .

[ii] If $\alpha \in (0, 2]$ $3 \leqslant n < 2+\alpha$ then $\mathcal{H}^{W,\alpha}_{\beta,a}$ is bounded from below on $\dot{\mathbb{H}}_{\alpha/2}(\mathbb{R}^n)$ for any β satisfying (1.19) and any $a > 0$, and there is a minimizer of $\mathcal{H}^{W,\alpha}_{\beta,a}$ in $\dot{\mathbb{H}}_{\alpha/2}(\mathbb{R}^n)$.

[iii] If $\alpha = 1$, $n = 3$ or $\alpha = 2$, $n = 4$ then there exists $a_c(\beta) > 0$ such that $\mathcal{H}^{W,\alpha}_{\beta,a}$ is bounded from below on $\dot{\mathbb{H}}_{\alpha/2}(\mathbb{R}^n)$ if $a < a_c(\beta)$ and unbounded from below if $a > a_c(\beta)$. If $a < a_c(\beta)$ there exists a minimizer of $\mathcal{H}^{W,\alpha}_{\beta,a}$ in $\dot{\mathbb{H}}_{\alpha/2}(\mathbb{R}^n)$.

2. Proofs

We start by proving proposition 1.2

Proof. Let $\gamma_{i,j}$ be the Lagrange multiplier for the constraints $\langle \phi_i, \phi_j\rangle = \delta_{i,j}$. Then $\overrightarrow{\phi}$ is an *unconstraint* critical point of

$$\mathcal{E}^{(\alpha)}_{\beta,a}(\vec{\phi}) + \sum_1^k \gamma_{j,j}\|\phi_j\|^2_{\mathbb{H}^1} + \sum_{i\neq j}\gamma_{i,j}\left\langle \phi_i, \phi_j\right\rangle \ .$$

This implies

$$\frac{\delta\mathcal{E}^{(\alpha)}_{\beta,a}}{\delta\bar{\phi}_j} + 2\gamma_{j,j}\bar{\phi}_j + \sum_{i\neq j}\gamma_{i,j}\bar{\phi}_i = \beta_j(L^W - a\bar{V})\bar{\phi}_j + 2\gamma_{j,j}\bar{\phi}_j + \sum_{i\neq j}\gamma_{i,j}\bar{\phi}_i = 0 \ .$$

In particular, $Sp(\bar{\phi}_1 \ldots \bar{\phi}_k)$ is an invariant subspace of $L^W - a\bar{V}$. Since $(\bar{\phi}_1 \ldots \bar{\phi}_k)$ is an orthonormal sequence we get, after multiplying the above line by $\bar{\phi}_i$ and taking the inner product:

$$\left\langle \beta_j \left[L^W - a\bar{V}\right]\bar{\phi}_j, \bar{\phi}_i\right\rangle + \gamma_{i,j} = 0$$

for any $i \neq j$. switching i with j and taking into consideration that $L + \bar{V}$ is self-adjoint, we also get

$$\left\langle \beta_i \left[L^W - a\bar{V}\right]\bar{\phi}_j, \bar{\phi}_i\right\rangle + \gamma_{i,j} = 0$$

Subtracting the two inequalities we obtain $\left\langle(\beta_j - \beta_i)(L^W - a\bar{V})\bar{\phi}_j, \bar{\phi}_i\right\rangle = 0$ thus $\left\langle(L^W - a\bar{V})\bar{\phi}_j, \bar{\phi}_i\right\rangle = 0$ for any $i \neq j$. Since $Sp(\bar{\phi}_1 \ldots \bar{\phi}_k)$ is an invariant subspace of $L^W - a\bar{V}$, this implies that $\bar{\phi}_j$ is are eigenstates of $L^W - a\bar{V}$. □

[4]If $\lambda_j(V)$ is degenerate, so $\lambda_{j-1}(V) > \lambda_j(V) = \ldots = \lambda_{j+l}(V) > \lambda_{j+l+1}(V)$, then $\{\phi_j(V), \ldots \phi_{j+l}(V)\}$ is any orthonormal base of the eigenspace of λ_j.

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The first part of the following Lemma follows from a compactness embedding Theorem (c.f. Theorem XIII.67 in [**15**]). The second part from Sobolev and HLS inequalities (see, e.g. [**13**], sec. 3.1.1)

LEMMA 2.1. *For any* $n \geqslant 3$, $\mathbb{H}^1$ *is compactly embedded in* $\mathbb{L}^r$ *for* $2 < r < 2n/n-2$. *If* $n-2/n+\alpha \leqslant 1/2 \leqslant n/n+\alpha$ *and* $\phi \in \mathbb{H}^1$ *then* $|\phi|^2 \in \mathbb{L}^{2n/(n+\alpha)}(\mathbb{R}^n) \cap \mathbb{L}^1(\mathbb{R}^n)$ *and*

$$\int_{\mathbb{R}^n} (I_\alpha * |\phi|^2)|\phi|^2 \leqslant C_{n,\alpha} \left(\int |\phi|^{4n/(n+\alpha)} \right)^{1+\alpha/n}$$

In particular we obtain:

COROLLARY 2.2. *If* $\max(0, n-4) \leqslant \alpha \leqslant n$ *then the functional* $\mathcal{E}^{(\alpha)}_{\beta,a}$ *is defined on* $\oplus^k \mathbb{H}^1$.

Let $\alpha \in (0, 2]$, $\vec{\phi} \in \oplus^k \mathbb{H}^1$ and $V \in C_0^\infty(\mathbb{R}^n)$. Define

$$\mathbf{H}^{(\alpha)}_\beta(\vec{\phi}, V) = \sum_1^k \beta_j \left\langle L^W \phi_j, \phi_j \right\rangle + a \left[\langle V, V \rangle_{\alpha/2} - \left\langle V, \sum \beta_j |\phi_j|^2 \right\rangle \right] . \tag{2.1}$$

By (1.23) we get (c.f definition 1.1)

$$\inf_{V \in C_0^\infty(\mathbb{R}^n)} \mathbf{H}^{(\alpha)}_\beta(\vec{\phi}, V) = 2\mathcal{E}^{(\alpha)}_{\beta,a}(\vec{\phi}) .$$

Thus

$$\inf_{V \in C_0^\infty} \inf_{\vec{\phi} \in \oplus^k \mathbb{H}^1} \mathbf{H}^{(\alpha)}_\beta(\vec{\phi}, V) \equiv \inf_{\vec{\phi} \in \oplus^k \mathbb{H}^1} \inf_{V \in C_0^\infty} \mathbf{H}^{(\alpha)}_\beta(\vec{\phi}, V) = \inf_{\vec{\phi} \in \oplus^k \mathbb{H}^1} \mathcal{E}^{(\alpha)}_{\beta,a}(\vec{\phi}) .$$

Let

$$\mathcal{H}^{W,\alpha}_{\beta,a}(V) = \inf_{\vec{\phi} \in \oplus^k \mathbb{H}^1} \mathbf{H}^{(\alpha)}_\beta(\vec{\phi}, V) .$$

From (1.23 , 2.1) we observe that

$$\mathcal{H}^{W,\alpha}_{\beta,a}(V) = \frac{a}{2} \langle V, V \rangle_{\alpha/2} + \inf_{\vec{\phi} \in \oplus^k \mathbb{H}^1} \sum_{j=1}^k \beta_j \left\langle (L^W - aV)\phi_j, \phi_j \right\rangle .$$

Let

$$\inf_{\vec{\phi} \in \oplus^k \mathbb{H}^1} \sum_{j=1}^k \beta_j \left\langle (L^W - aV)\phi_j, \phi_j \right\rangle := G_{\beta,a}(V) . \tag{2.2}$$

As the infimum over linear functionals, $V \mapsto G_{\beta,a}(V)$ is a concave functional, so

$$\mathcal{H}^{W,\alpha}_{\beta,a}(V) = \frac{a}{2} \langle V, V \rangle_{\alpha/2} + G_{\beta,a}(V) \tag{2.3}$$

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is the sum of convex and concave functionals. In the case $k = 1$ ($\vec{\beta} = \beta_1 = 1$) we observe, by the Rayleigh-Ritz principle

$$G_{1,a}(V) = \inf_{\|\phi\|=1} \left\langle (L^W - aV)\phi, \phi \right\rangle = \lambda_1(V)$$

and the supremum is obtained at the normalized ground state $\bar{\phi}_1$ satisfying $(L^W - aV - \lambda_1)\bar{\phi}_1 = 0$. In particular we reassure that $G_{1,a}(V) = \lambda_1(V)$ is a concave functional. In general, higher eigenvalues $\lambda_j = \lambda_j(V)$ are *not* concave functions if $j > 1$. However, if $\vec{\beta} := (\beta_1, \ldots \beta_k)$ satisfies (1.19) then we claim that $V \mapsto \sum_{j=1}^k \beta_j \lambda_j(V)$ is concave. Indeed:

LEMMA 2.3.

$$G_{\beta,a}(V) = \sum_{j=1}^{k} \beta_j \lambda_j(V) \tag{2.4}$$

where $\lambda_j(V)$ are the k lowest eigenvalues of the operator $L^W - aV$ arranged by order

$$\lambda_1(V) < \lambda_2(V) \leqslant \ldots \leqslant \lambda_k(V) \ .$$

Moreover, the minimum in (2.2) is obtained at the eigenfunction $\bar{\phi}_j$ of $L^W - aV$ corresponding to λ_j.

Recall the definition of sup-gradient of a concave functional G on a vector space $C_0^\infty(\mathbb{R}^n)$ at V:

$$\partial_V G := \left\{ \zeta \in (C_0^\infty)^{'}; \quad G(Z) \leqslant G(V) + \langle Z - V, \zeta \rangle \ \ \forall Z \in C_0^\infty \right\}$$

while G is differentiable at V if $\partial_V G$ is a singleton.

COROLLARY 2.4. *The sup-gradient of the functional $G_{\beta,a}$ on $C_0^\infty(\mathbb{R}^n)$ is contained in $\mathbb{L}^1$. In fact $\partial_V G_{\beta,a} = a \sum_{j=1}^k \beta_j |\bar{\phi}_j|^2$ where $\bar{\phi}_j \in \mathbb{L}^2$ is a normalized eigenstate of $L^W - aV$ corresponding to the $j-$ eigenvalue. So, in particular, $\|\partial_V G_{\beta,a}\|_1 = a$. If all eigenvalues of $L^W - aV$ are simple then $G_{\beta,a}$ is differentiable at V.*

Proof. Since $V \mapsto \sum_{j=1}^k \beta_j \left\langle (L^W - aV)\phi_j,\, \phi_j \right\rangle$ is a linear functional, (2.2) would imply, in particular, that the functional $G_{\beta,a}$ is, indeed, a concave one.

Let $\bar{\phi}_j$ be the normalized eigenvalues of $L^W - aV$ corresponding to $\lambda_j(V)$. Fix some $m \geqslant j$ and let $\mathbb{H}_m = Sp(\bar{\phi}_1, \ldots \bar{\phi}_m)$. Let us restrict the supremum (2.2) to $\mathbb{H}_m^k := \{\vec{\phi} := (\phi_1, \ldots \phi_k),\, \phi_j \in \mathbb{H}_m\} \subset \mathbb{H}^k$.

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Then

$$\phi_j = \sum_{i=1}^{m} \langle \phi_j, \bar\phi_i \rangle \bar\phi_i, \quad (L^W - aV)\phi_j = \sum_{i=1}^{m} \lambda_i \langle \phi_j, \bar\phi_i \rangle \bar\phi_i.$$

Define $\beta_{k+1} = \ldots = \beta_m = 0$. Then we can write, for any $\vec\phi \in \mathbb{H}^k_m$

$$\sum_{j=1}^{k} \beta_j \langle (L^W - aV)\phi_j, \phi_j \rangle = \sum_{i=1}^{m}\sum_{j=1}^{m} \beta_j \lambda_i |\langle \phi_j, \bar\phi_i \rangle|^2. \tag{2.5}$$

Denote now $\gamma_{i,j} := |\langle \phi_j, \bar\phi_i \rangle|^2$. Then $\{\gamma_{i,j}\}$ is $m\times m$, bi-stochastic matrix, i.e $\sum_{i=1}^m \gamma_{i,j} = \sum_{j=1}^m \gamma_{i,j} = 1$ for all $i, j = 1\ldots m$. Consider now the infimum of $\sum_{i=1}^m \sum_{j=1}^m \tilde\gamma_{i,j}\lambda_i\beta_j$ over all bi-stochastic martices $\{\tilde\gamma_{i,j}\}$. By Krain-Milman theorem, the minimum is obtained on an extreme point in the convex set of bi-stochastic matrices. By Birkhoff theorem, the extreme points are permutations so, from(2.5)

$$\forall \vec\phi \in \mathbb{H}^k_m, \quad \sum_{j=1}^{k} \beta_j \langle (L^W - aV)\phi_j, \phi_j \rangle \geqslant \sum_{j=1}^{m} \beta_{\pi(j)}\lambda_j$$

for some permutation $\pi : \{1, \ldots m\} \mapsto \{1, \ldots m\}$. Now, recall that β_j are assumed to be in descending order while λ_j are in ascending order by definition. By the discrete rearangment theorem of Hardy, Littelwood and Polya [**3**] we obtain that the maximum on the right above is obtained at the identity permutation $\pi(i) = i$, that is, at the identity matrix $\tilde\gamma_{i,j} := \langle \phi_j, \bar\phi_i \rangle = \delta_{i,j}$. This implies that the eigenbasis $\bar\phi_1, \ldots \bar\phi_k$ of the k leading eigenvalues is the minimizer of (2.2) on $\mathbb{H}^k_m$ for any $m \geqslant k$.

In particular, the minimizer of (2.2) in $\mathbb{H}^k_m$ is independent of m, as long as $m \geqslant k$. Suppose there exists some $\vec\psi \in \mathbb{H}^k$ which is not contained in and finite dimensional subspace generated by eigenstates, for which (2.2) is strictly smaller than its value on the first $k-$ leading eigenspace. Since the eigenstates of the Schrödinger operator under assumption (1.21) generate the whole space we can find, for a sufficiently large m, an orthonormal base in $\mathbb{H}^k_m$ for on which the left side of (2.2) is strictly larger than $\sum_{j=1}^k \beta_j\lambda_j(V)$, and we get a contradiction for this value of m. □

From Corollary 2.4 and (1.24, 1.25) It follows that the Euler-Lagrange equation corresponding to $\mathcal{H}^{W,\alpha}_{\beta,a}$ is

$$(-\Delta)^{\alpha/2}V - \sum_{j=1}^{k} \beta_j|\phi_j|^2 = 0 \quad \Longleftrightarrow V = I_\alpha * \left(\sum_{j=1}^{k} \beta_j|\phi_j|^2 \right)$$

where ϕ_j are the normalized eigenfunction corresponding to $\lambda_j(V)$. In particular we obtain the proof of theorem 1.6-(i):

Corollary 2.5. *If $\bar V$ is a minimizer of $\mathcal{H}^{W,\alpha}_{\beta,a}$ then $\bar V = \sum_{j=1}^k \beta_j I_\alpha * |\bar\phi_j|^2$ where $\bar\phi_j$ are the normalized eigenfunction corresponding to $\lambda_j(\bar V)$. In particular, $\{\lambda_j(\bar V), \bar\phi_j\}$ is a solution of the Choquard system (1.20, 1.22).*

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Lemma 2.6. *Suppose* $\alpha \in (0, 2]$, $3 \leqslant n \leqslant 4+\alpha$ *and* $V \in C_0^\infty(\mathbb{R}^n)$ *is bounded in* $\dot{\mathbb{H}}^{\alpha/2}$. *If* ϕ *is a normalized eigenfunction of* $a^{-1}L^W - V$ *then* $\|\nabla\phi\|_2$, $\int W|\phi|^2\,\mathrm{d}x$ *and* $\|\phi\|_{2n/(n-2)}$ *are bounded in terms of* $|||V|||_{\alpha/2}$ *and the corresponding eigenvalue* λ.

Proof. By assumption, $\|\phi\|_2 = 1$ and satisfy

$$(-\Delta\phi + W)\phi - aV\phi - \lambda\phi = 0\ .$$

Multiply by ϕ and integrate to obtain

$$\|\nabla\phi\|_2^2 - a\int V|\phi|^2\,\mathrm{d}x + \int W|\phi|^2\,\mathrm{d}x - \lambda = 0. \tag{2.6}$$

By the critical Sobolev inequality (lemma 1.3) and and Holder inequality

$$\int V|\phi|^2\,\mathrm{d}x \leqslant \|V\|_{\frac{2n}{n-\alpha}}\||\phi|^2\|_{\frac{2n}{n+\alpha}} \leqslant S_{n,\alpha}|||V|||_{\alpha/2}^2\||\phi|^2\|_{\frac{2n}{n+\alpha}} \tag{2.7}$$

By the Gagliardo-Nirenberg interpolation inequality [14]

$$\|\phi\|_p \leqslant C(\theta)\|\nabla\phi\|_2^\theta\|\phi\|_2^{1-\theta}$$

where $C(\theta)$ is independent of ϕ, $p = 2n/(n-2\theta)$ whenever $\theta \in [0,\ 1]$. Since $\|\phi\|_2 = 1$ we get

$$\||\phi|^2\|_{p/2} = \|\phi\|_p^2 \leqslant C^2(\theta)\|\nabla\phi\|_2^{2\theta}. \tag{2.8}$$

Let now $p/2 = 2n/n+\alpha$, corresponding to $\theta = (n-\alpha)/4$, we obtain from (2.7)

$$\int V|\phi|^2\,\mathrm{d}x \leqslant S_{n,\alpha}C^2(\theta)|||V|||_{\alpha/2}^2\|\nabla\phi\|_2^{2\theta}$$

where $\theta < 1$ if $3 \leqslant n < 4+\alpha$. Substitute it in (2.6) we obtain the upper estimate on $\|\nabla\phi\|_2$ and $\int W|\phi|^2\,\mathrm{d}x$. Finally setting $\theta = 1$ corresponding to $p = 2n/(n-2)$ we obtain from (2.8) the estimate on $\|\phi\|_{2n/(n-2)}$. □

Lemma 2.7. *If* $2 < n < 4+\alpha$, $0 < \alpha \leqslant 2$ *then* $C_0^\infty(\mathbb{R}^n) \ni V \mapsto \lambda_j^{(V)}$ *is continuous on bounded sets in* $\dot{\mathbb{H}}_{\alpha/2}(\mathbb{R}^n)$ *with respect to Lebesgue norms* $\mathbb{L}^q(\mathbb{R}^n)$, *where* $n/2 \leqslant q \leqslant \infty$. *In particular* $V \mapsto \lambda_j^{(V)}$ *can be extended as a continuous function on* $\dot{\mathbb{H}}_{\alpha/2}$.

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Proof. By lemma 2.3, there exists $\vec{\phi}^{(V)} \in \mathbb{H}_1^k$ such that

$$G_{\beta,a}(V) = \inf_{\vec{\phi}\in\oplus^k\mathbb{H}^1} \sum_{j=1}^{k} \beta_j \left\langle (L^W - aV)\phi_j, \phi_j \right\rangle .$$

Thus, for $\tilde{V}_1$, $\tilde{V}_2$ bounded in $\dot{\mathbb{H}}^{\alpha/2}$,

$$\begin{aligned} G_{\beta,a}(\tilde{V}_1) - G_{\beta,a}(\tilde{V}_2) &\leqslant \sum_{j=1}^{k} \beta_j \left\langle (L^W - a\tilde{V}_1)\phi_j^{(\tilde{V}_2)}, \phi_j^{(\tilde{V}_2)} \right\rangle \\ &\quad - \sum_{j=1}^{k} \beta_j \left\langle (L^W - a\tilde{V}_2)\phi_j^{(\tilde{V}_2)}, \phi_j^{(\tilde{V}_2)} \right\rangle \\ &= a\sum_{j=1}^{k} \beta_j \left\langle (\tilde{V}_2 - \tilde{V}_1)\phi_j^{(\tilde{V}_2)}, \phi_j^{(\tilde{V}^2)} \right\rangle \\ &\leqslant a\sum_{j=1}^{k} \beta_j \|\tilde{V}_2 - \tilde{V}_1\|_q \||\phi_j^{(\tilde{V}_2)}|^2\|_{\frac{q}{q-1}} . \end{aligned}$$

By lemma 2.6, $\||\phi_j^{(V)}|^2\|_p$ is bounded in terms of the norm of $\||V\||_{\alpha/2}$ for $1 \leqslant p \leqslant n/(n-2)$. It follows that $G_{\beta,a}(\tilde{V}_1) - G_{\beta,a}(\tilde{V}_2)$ is bounded in terms of $\|\tilde{V}_2 - \tilde{V}_1\|_q$ for $n/2 \leqslant q \leqslant \infty$, so $G_{\beta,a}$ is continuous in these norms. Since $n/2 < 2n/n - \alpha$ by assumption and $\dot{\mathbb{H}}_{\alpha/2}(\mathbb{R}^n)$ is embedded in $\mathbb{L}^{2n/(n-\alpha)}(\mathbb{R}^n)$, we obtain the continuous extension of $G_{\beta,a}$ on $\dot{\mathbb{H}}_{\alpha/2}(\mathbb{R}^n)$.

Finally, to continuity of each eigenvalue λ_j is obtained by subtraction $G_{(\beta_1,\ldots\beta_j),a}(V) - G_{(\beta_1,\ldots\beta_{j-1}),\,a}(V) \equiv \beta_j\lambda_j^{(V)}$ by lemma 2.3. □

2.1. Lower limit of the dual functional

Recall the Lieb-Thirring inequality for Schrodinger operator :

THEOREM 2.8 [5]. *For the Schrödinger operator $-\Delta + V$ on $\mathbb{R}^n$ with a real valued potential V the numbers $\mu_1(V) \leqslant \mu_2(V) \leqslant \cdots \leqslant 0$ denote the (not necessarily finite) sequence of its negative eigenvalues. Then, for $n \geqslant 3$ and $\gamma \geqslant 0$*

$$\sum_{j;\mu_j(V)<0} |\mu_j(V)|^\gamma \leqslant L_{\gamma,n} \int V_-^{n/2+\gamma}\, \mathrm{d}x \tag{2.9}$$

where $V_- = \max\{0, -V\}$ and $L_{\gamma,n}$ is independent of V.

PROPOSITION 2.9. *The functional $V \mapsto a/2\,\langle V, V\rangle_{\alpha/2} + G_{\beta,a}(V)$ is bounded from below on $\dot{\mathbb{H}}_{\alpha/2}$ for any $a > 0$ if $3 \leqslant n < 2 + \alpha$. if $n = 3$, $\alpha = 1$ or $n = 4$, $\alpha = 2$ there exists $a = a_c^{(n)}(\vec{\beta}) > 0$ independent of W for which the functional is bounded from below if $a < a_c^{(n)}(\vec{\beta})$ and unbounded if $a > a_c^{(n)}(\vec{\beta})$.*

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Moreover, in the cases $n = 3$ and $n = 4$, $a < a_c^{(n)}(\vec{\beta})$ the functional is coersive on $\dot{\mathbb{H}}_{\alpha/2}(\mathbb{R}^n)$, namely

$$\lim_{|||V|||_{\alpha/2}\to\infty} \frac{1}{2}\langle V, V\rangle_{\alpha/2} + G_{\beta,a}(V) = \infty. \tag{2.10}$$

Proof. Recall that $\lambda_j(V)$ are the eigenvalues of $L^W - aV = -\Delta + W - aV$. Since $W \geqslant 0$ it follows that $\lambda_j(V) \geqslant \mu_j(aV)$. Hence $G_{\beta,a}(V) := \sum_{j=1}^k \beta_j\lambda_j(V) \geqslant -\sum_{j;\mu_j(aV)<0} \beta_j|\mu_j(aV)|$. By Holder inequality, for $\gamma \geqslant 1$, $\gamma^{'} = \gamma/(\gamma-1)$ and (2.9)

$$G_{\beta,a}(V) \geqslant -\left(\sum_{j=1}^k |\beta_j|^{\gamma^{'}}\right)^{1/\gamma^{'}} \left(\sum_{j;\mu_j(aV)<0} |\mu_j(aV)|^{\gamma}\right)^{1/\gamma} \geqslant$$

$$-a^{1+n/2\gamma} L_{\gamma,n}^{1/\gamma}\|\vec{\beta}\|_{\gamma'} \left(\int V_-^{n/2+\gamma}\,\mathrm{d}x\right)^{1/\gamma}.$$

Set now $\gamma = \frac{2n}{n-\alpha} - n/2 \equiv \frac{(4+\alpha)n-n^2}{2(n-\alpha)}$. Then, if $2 < n < 2 + \sqrt{1+3\alpha}$ we get $\gamma_{n,\alpha}^{'} \geqslant 1$ and

$$G_{\beta,a}(V) \geqslant -a^{\frac{4}{4+\alpha-n}} L_{\gamma,n}^{1/\gamma}\|\beta\|_{\gamma_{n,\alpha}^{'}} \left(\int_{\mathbb{R}^n} V_+^{\frac{2n}{n-\alpha}}\right)^{\frac{2(n-\alpha)}{(4+\alpha)n-n^2}}.$$

Using the critical Sobolev inequality

$$G_{\beta,a}(V) \geqslant -a^{\frac{4}{4+\alpha-n}} L_{\gamma,n}^{1/\gamma}\|\beta\|_{\gamma_{n,\alpha}^{'}} S_{n,\alpha}^{\frac{4}{(4+\alpha)-n}} \langle V, V\rangle_{\alpha/2}^{\frac{2}{(4+\alpha)-n}}$$

hence

$$\frac{a}{2}\langle V, V\rangle_{\alpha/2} + G_{\beta,a}(V) \geqslant$$

$$a\langle V, V\rangle_{\alpha/2}^{\frac{2}{4+\alpha-n}} \left(\frac{1}{2}\langle V, V\rangle_{\alpha/2}^{1-\frac{2}{4+\alpha-n}} - a^{\frac{n-\alpha}{4+\alpha-n}} L_{\gamma,n}^{1/\gamma}\|\beta\|_{\gamma_{n,\alpha}^{'}} S_{n,\alpha}^{\frac{4}{(4+\alpha)-n}}\right) \tag{2.11}$$

It follows that $\mathcal{H}_{\beta,a}^{W,\alpha}$ is coersive for any $a > 0$ if $3 \leqslant n < 2 + \alpha$. If $n = 2 + \alpha$ then the functional is coersive if $a < \frac{S_{n,\alpha}^{\frac{4}{n-(4+\alpha)}}}{2L_{\gamma,n}^{1/\gamma}}|\beta|_{\gamma_{n,\alpha}^{'}}^{-1}$. Note that $\gamma_{n,\alpha}^{'} = \infty$ for $n = 3$, $\alpha = 1$ and $\gamma_{n,\alpha}^{'} = 2$ for $n = 4$, $\alpha = 2$. Hence coersivity holds if

- $(\alpha, n) = (1, 3)$: $a < \frac{S_{3,1}^{-2}}{2L_{\gamma,3}^{1/\gamma}}|\beta|_{\infty}^{-1}$
- $(\alpha, n) = (2, 4)$: $a < \frac{S_{4,12}^{-2}}{2L_{\gamma,4}^{1/\gamma}}|\beta|_{2}^{-1}$

We now prove the existence of a critical strength $a_c(\beta)$ in both cases. For a given, non-negative $V \in \dot{\mathbb{H}}^{\alpha/2}$, let $k(V)$ be the number of negative eigenvalues of $-\Delta - aV$, enumerated by order $\lambda_1^0(V) < \lambda_2^0(V) \leqslant \dots \lambda_{k(V)}^0(V) < 0$. Denote $G_{\beta,a}^0(V) =$

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$\sum_1^{k\wedge k(V)} \beta_j \lambda_j^0(V)$. Let $\bar\phi_j^0$ be the corresponding eigenfunctions of $-\Delta - aV$. From the variational characterization of $G_{\beta,a}$ introduced in lemma 2.3 we may obtain

$$G^0_{\beta,a}(V) \leqslant G_{\beta,a}(V) \leqslant G^0_{\beta,a}(V) + \sum_{j=1}^{k\wedge k(V)} \beta_j \int W|\phi_j^0|^2 + O(1) \tag{2.12}$$

where $O(1)$ stands for some constant independent of V.[5]

Substitute now now $V/\sqrt{a}$ for V. Then $\frac{a}{2}\langle V/\sqrt{a}, V/\sqrt{a}\rangle_{\alpha/2} + G^0_{\beta,a}(V/\sqrt{a}) = \frac{1}{2}\langle V, V\rangle_{\alpha/2} + G^0_{\beta,a}(V/\sqrt{a})$. By definition $G^0_{\beta,a}(V/\sqrt{a}) = \sum_1^{k\wedge k(V/\sqrt{a})} \beta_j \lambda_j^0(V/\sqrt{a})$, while $\lambda_j^0(V/\sqrt{a})$ is a negative eigenvalue of $-\Delta + W + \sqrt{a}V$. Thus, if $V<0$ somewhere then $\lim_{a\to\infty} G^0_{\beta,a}(V/\sqrt{a}) = -\infty$. In particular it follows that

$$\text{if } \ a>0 \ \text{ large enough then } \exists V \in \dot{\mathbb{H}}_{\alpha/2} \text{ for which } \frac{a}{2}\langle V, V\rangle_{\alpha/2} + G^0_{\beta,a}(V) < 0 \tag{2.13}$$

Now apply the transformation $V \mapsto V_\delta(x) := \delta^2 V(\delta x)$, where $\delta > 0$. We obtain that $\lambda_j^0(V_\delta) = \delta^2 \lambda_j^0(V)$ (in particular, $k(V_\delta) = k(V)$), while $\phi_j^{0,\delta} := \delta^{n/2}\phi_j^0(\delta x)$ is the corresponding normalized eigenfunction. Hence $G^0_{\beta,a}(V_\delta) = \delta^2 G^0_{\beta,a}(V)$ so, by (2.12), $G_{\beta,a}(V_\delta) \leqslant \delta^2 G^0_{\beta,a}(V) + \sum_{j=1}^{k\wedge k(V)} \beta_j \int W|\phi^0_{j,\delta}|^2 + O(1)$.

Next, we obtain for both $n=3$, $\alpha=1$ and $n=4$, $\alpha=2$ cases that the quadratic form scale the same: $\langle V_\delta, V_\delta\rangle_{\alpha/2} = \delta^2 \langle V, V\rangle_{\alpha/2}$ so

$$\begin{aligned} \frac{a}{2}\langle V_\delta, V_\delta\rangle_{\alpha/2} + G_{\beta,a}(V_\delta) &\leqslant \delta^2\left[\frac{a}{2}\langle V, V\rangle_{\alpha/2} + G^0_{\beta,a}(V)\right] \\ &\quad + \sum_{j=1}^{k\wedge k(V)} \beta_j \int W|\phi^0_{j,\delta}|^2 + O(1). \end{aligned} \tag{2.14}$$

By (1.21) we also get $\lim_{\delta\to\infty} \int W|\phi^0_{j,\delta}|^2 = W(0) = 0$, so, using (2.13) we obtain the existence of V for which $\frac{a}{2}\langle V_\delta, V_\delta\rangle_{\alpha/2} + G_{\beta,a}(V_\delta) \to -\infty$ as $\delta\to\infty$, if $a>0$ is large enough.

Now let

$$a_c(\vec\beta) = \inf\left\{a>0; \quad \inf_{V\in\dot{\mathbb{H}}^{\alpha/2}} \frac{a}{2}\langle V, V\rangle_{\alpha/2} + G^0_{\beta,a}(V) < 0\right\}.$$

It follows that $\infty > a_c(\beta) > 0$ and is independent of W for any $\vec\beta$ in the cases $n=3$, $\alpha=1$ and $n=4$, $\alpha=2$. □

2.2. Existence of minimizers of the local problem

When attempting to prove the existence of minimizers to the functional $\mathcal{H}^{W,\alpha}_{\beta,a}$ (2.3) we face the problem of compactness of the space $\dot{\mathbb{H}}_{\alpha/2}$. So, we start by considering the subspace of $\dot{\mathbb{H}}_{\alpha/2}(B^n_R) \subset \dot{\mathbb{H}}_{\alpha/2}\mathbb{R}^n)$, obtained by the closure of C_0^∞

[5]We may estimate this constant by $\sum_{k\wedge k(V)+1}^k \beta_j \lambda_j^w$ where λ_j^w is the $j-$ eigenvalue of $H_0 = -\Delta + W$.

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of functions *supported on the ball* $B_R^n := \{x \in \mathbb{R}^n;\ |x| < R\}$ under the induced $|||\cdot|||_{\alpha/2}$ norm (§§ 1.4).

Note that, by this definition, $V \in \dot{\mathbb{H}}_{\alpha/2}(B_R^n)$ is defined *over the whole* of $\mathbb{R}^n$, and is identically zero on $\mathbb{R}^n - B_R^n$.

By

LEMMA 2.10. *Given $\vec{\beta}$ satisfying (1.19), $R > 0$, $3 \leqslant n < 2+\alpha$ or either $n = 3$, $\alpha = 1$ or $n = 4$, $\alpha = 2$. Then there exists a minimizer $\bar{V}_R$ of $\mathcal{H}_{\beta,a}^{W,\alpha}$ restricted to $\dot{\mathbb{H}}_{\alpha/2}(B_R^n)$. Moreover,*

$$I_\alpha^{B_R^n}(\bar{V}_R) = \left(\sum_{j=1}^{k} \beta_j |\bar{\phi}_j^R|^2\right) \tag{2.15}$$

where $\bar{\phi}_j^R$ are the normalized eigenstates of $L^W - a\bar{V}_R$ in $\mathbb{R}^n$.

Proof. Let $V_n \subset \dot{\mathbb{H}}_{\alpha/2}(B_R^n)$ be a minimizing sequence of $\mathcal{H}_{\beta,a}^{W,\alpha}$. Since $\mathcal{H}_{\beta,a}^{W,\alpha}$ is bounded from below by proposition 2.9 we get

$$\lim_{n\to\infty} \mathcal{H}_{\beta,a}^{W,\alpha}(V_n) = \inf_{V \in \dot{\mathbb{H}}_{\alpha/2}(B_R^n)} \mathcal{H}_{\beta,a}^{W,\alpha}(V).$$

Since $\dot{\mathbb{H}}_{\alpha/2}(B_R^n)$ is weakly compact and the functional is coersive (2.10) there exists a weak limit $\bar{V}_R \in \dot{\mathbb{H}}_{\alpha/2}(B_R^n)$ of this sequence. Moreover, by Sobolev compact embedding, V_n converges strongly to $\bar{V}_R$ in $\mathbb{L}^q(B_B^n)$ for any $1 \leqslant q < 2n/(n-\alpha)$. Since $n < 2+\alpha$ then $n/2 < 2n/(n-\alpha)$ and by lemma 2.7

$$\lim_{n\to\infty} G_{\beta,a}(V_n) = G_{\beta,a}(\bar{V}_R). \tag{2.16}$$

Since $V \mapsto |||V|||_{\alpha/2}^2$ is l.s.c , it follows that

$$\lim_{n\to\infty} \langle V_n, V_n\rangle_{\alpha/2}\, \mathrm{d}x \geqslant \left\langle \bar{V}_R, \bar{V}_R\right\rangle_{\alpha/2} .$$

This and (2.16) imply that $\bar{V}_R$ is, indeed, a minimizer of $\mathcal{H}_{\beta,a}^{W,\alpha}$ on $\dot{\mathbb{H}}_{\alpha/2}(B_R^n)$.

Finally, (2.15) follows from (1.26) while taking $\Omega = B_R^n$. □

2.3. Proof of theorem 1.6(ii, iii)

Let V_m be a minimizing sequence for $\mathcal{H}_{\beta,a}^{W,\alpha}$ in $\dot{\mathbb{H}}_{\alpha/2}(\mathbb{R}^n)$. Since $C_0^\infty(\mathbb{R}^n)$ is dense in $\dot{\mathbb{H}}_{\alpha/2}(\mathbb{R}^n)$ by definition, we can assume that there exists a sequence $R_m \to \infty$ such that V_m is supported in $B_{R_m}^n$.

Let $\bar{V}_m$ be the minimizers of $\mathcal{H}_{\beta,a}^{W,\alpha}$ on $\dot{\mathbb{H}}_{\alpha/2}(B_R^n)$.

Since $\mathcal{H}_{\beta,a}^{W,\alpha}(\bar{V}_m) \leqslant \mathcal{H}_{\beta,a}^{W,\alpha}(V_m)$ then $\bar{V}_m$ is a minimizing sequence of $\mathcal{H}_{\beta,a}^{W,\alpha}$ on $\dot{\mathbb{H}}_{\alpha/2}(\mathbb{R}^n)$ as well. Now, under the conditions of the Theorem we know by proposition 2.9 that $\mathcal{H}_{\beta,a}^{W,\alpha}$ is bounded from below on $\dot{\mathbb{H}}_{\alpha/2}(\mathbb{R}^n)$ and coersive (2.10), so $|||\bar{V}_m|||_{\alpha/2}$ are uniformly bounded. Let $\bar{\phi}_j^m$ be the normalized eigenfunctions of $L^W - a\bar{V}_m$. By lemma 2.6 we obtain that $\|\nabla\bar{\phi}_j^m\|_2$ and $\int_{\mathbb{R}^n} W|\bar{\phi}_j^m|^2$ and

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$\||\bar\phi_j^R|^2\|_{n/(n-2)}$ are uniformly bounded on $\mathbb{R}^n$. In addition, $\|\bar\phi_j^m\|_2 = 1$ by definition. In particular, $\bar\phi_j^m$ are in the space $\mathbb{H}^1$ (c.f Definition 1.1) . Using the first part of Lemma 2.1 we obtain a subsequence (denoted by the index m) along which $\bar\rho_m := \sum_{j=1}^k \beta_j|\bar\phi_j^m|^2$ converges in $\mathbb{L}^p(\mathbb{R}^n)$ for any $p < n/(n-2) \equiv 2^*/2$, while $\bar V_m = I_\alpha^{B_R^n}(\bar\rho_m)$. Since $n/2 \leqslant n/(n-2)$ for $n = 3, 4$, lemma 1.4 implies the convergence of $\bar V_m$ to $\bar V$ in $\mathbb{L}^q(\mathbb{R}^n)$ for any $1 \leqslant q < \infty$. By lower semi continuity we obtain that $\bar V \in \dot{\mathbb{H}}_{\alpha/2}(\mathbb{R}^n)$ and $\left\langle \bar V, \bar V\right\rangle_{\alpha/2} \leqslant \lim_{m\to\infty}\left\langle \bar V_m, \bar V_m\right\rangle_{\alpha/2}$.

In addition, lemma 2.7 implies that $G_{\beta,a}(\bar V_m)$ converges to $G_{\beta,a}(\bar V)$. This implies

$$\mathcal{H}^{W,\alpha}_{\beta,a}(\bar V) \leqslant \inf_{V\in\dot{\mathbb{H}}_{\alpha/2}(\mathbb{R}^n)} \mathcal{H}^{W,\alpha}_{\beta,a}(V)$$

so $\bar V \in \dot{\mathbb{H}}_{\alpha/2}(\mathbb{R}^n)$ is, indeed, a minimizer. The proof of theorem 1.6 follows now from lemma 1.5 and Corollary 2.5.

3. Further remarks

It is interesting to consider the dependence of the solution to the Choquard system on the probability vector $\vec\beta$. In particular, the relation between the critical interaction strength $a(\beta)$ at dimension $n-4$ and the universal critical value $\bar a_c$ corresponding to the scalar case $k = 1$ (see (1.8)).

(a) *Estimate on* $\bar a_c$: In [**8**] the critical value in case $\alpha = n-2$ is implicitly given as the $\mathbb{L}^2$ norm of the solution of equation (1.8). However, these solutions are not known explicitly. Here we introduce an estimate based on Hardy inequality

$$\int_{\mathbb{R}^n} |\nabla f|^2 \geqslant \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|f|^2}{|x|^2}$$

for any $f \in C_0^\infty(\mathbb{R}^n)$. In particular it implies that the operator $-\Delta - V$ is non-negative in $\mathbb{R}^n$ for any $V \leqslant (\frac{n-2}{2})^2|x|^{-2}$.

As discussed in §§ 1.1, the functional E_a^W is bounded from below on the unit ball of $\mathbb{L}^2$ iff $a \leqslant \bar a_c$. This implies, in particular, that if $A > \bar a_c$ there exists $\tilde\phi \in \mathbb{H}^1$ for which

$$E_a^0(\tilde\phi) := \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\tilde\phi|^2 - \frac{A}{4}\int_{\mathbb{R}^n}\left(I_{n-2} * |\tilde\phi|^2\right)|\tilde\phi|^2 < 0\ . \tag{3.1}$$

Moreover, by Riesz's rearrangement theorem we can assume that this $\tilde\phi$ is radially symmetric.

In particular, for any $V \geqslant I_{n-2} * |\tilde\phi|^2$

$$-\Delta - (A/2)V \ngeqslant 0. \tag{3.2}$$

In the special case $n = 4$, $I_2 = (-\Delta)^{-1}$ is the fundamental solution of the Laplacian. Let $\rho := |\tilde\phi|^2$ be this radial function. Then $U := I_2 * \rho$ is a solution

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of $\Delta U + \rho = 0$. Thus

$$r^{-3}\left(r^3 U^{'}\right)^{'} + \rho(r) = 0. \tag{3.3}$$

Let $m(r) = 2\pi^2 \int_0^r s^3\rho(s)\mathrm{d}s$. In particular, $m(\cdot)$ is non-decreasing on $\mathbb{R}_+$, $m(0) = 0$, and $m(r) \leqslant 1$ by assumption. Integrating (3.3) we get

$$r^3 U^{'}(r) = -(2\pi^2)^{-1} m(r) \Longrightarrow U(r) = (2\pi)^2 \int_r^\infty \frac{m(s)}{s^3}\mathrm{d}s \leqslant 2\pi^2 r^{-2}.$$

Thus, taking $V = 2\pi^2 r^{-2}$ in (3.2) we obtain a violation of the Hardy inequality if $\pi^2 A$ is below the Hardy constant. Since the Hardy constant $(\frac{n-2}{2})^2 = 1$ for $n = 4$ we get $A > \pi^{-2}$ for any $A > \bar{a}_c$, that is

$$\bar{a}_c \geqslant \frac{1}{\pi^2}$$

if $n = 4$.

It is not clear, at this point, if the above estimate holds for general dimension, since $I_{n-2} = (-\Delta)^{-1}$ only if $n = 4$. There is, indeed, an estimate of the form

$$|x; I_\alpha * \rho(x)| > t| \leqslant c\left(\frac{c}{t}\|\rho\|_2\right)^{n/(n-\alpha)}$$

(c.f [**16**], eq. (2.12)) which, if ρ is radial, is equivalent to

$$I_\alpha * \rho(r) \leqslant c^\alpha \omega_n^{(n-\alpha)/n} \|\rho\|_1 r^{\alpha-n}$$

where ω_n is the surface area of the unit sphere $\mathbb{S}^{n-1}$. This suggests a similar estimate for $\bar{a}_c$ in for general n and $\alpha = n-2$ using Hardy inequality. However, there is now known estimate (as far as we know) for the constant c.

(b) *Relation between $\bar{a}_c$ and a_β*: The inequality $a(\beta) \geqslant \bar{a}_c$ can be easily obtained for the critical case for any $\alpha = n-2$, $n \geqslant 3$, and any $\vec{\beta}$ satisfying (1.10). Indeed, using Definition 1.1 and the polar inequality

$$\left\langle |\phi_j|^2, I_\alpha * |\phi_i|^2 \right\rangle \leqslant \frac{1}{2}\left[\left\langle |\phi_j|^2, I_\alpha * |\phi_j|^2\right\rangle + \left\langle |\phi_i|^2, I_\alpha * |\phi_i|^2 \right\rangle\right]$$

we obtain

$$\mathcal{E}^{(\alpha)}_{\beta,a}(\vec{\phi}) \geqslant \frac{1}{2}\sum_{j=1}^k \beta_j \left[\langle\langle \phi_j, \phi_j\rangle\rangle_W - \frac{a}{2}\left\langle |\phi_j|^2, I_\alpha * |\phi_j|^2\right\rangle\right] = \sum_{j=1}^k \beta_j E_a^W(\phi_j)$$

where E_a^W as defined in (1.7). It follows that $\mathcal{E}^{(\alpha)}_{\beta,a}$ is bounded on $\oplus^k \mathbb{H}^1$ if E_a^W is bounded on $\mathbb{H}^1$. Since E_a^W is bounded from below iff $a \leqslant \bar{a}_c$ ([**8**]), the inequality $a(\beta) \geqslant \bar{a}_c$ follows.

In the case $n = 3$, $\alpha = 1$ and $n = 4$, $\alpha = 2$ we can say more about $a_c(\beta)$. By definition, $a > a_c(\beta)$ iff $\mathcal{H}^{W,\alpha}_{\beta,a}$ is unbounded from below on $\dot{\mathbb{H}}_{\alpha/2}$. Using

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(2.11) we obtain that $a_c(\beta) > O(|\beta|_\infty^{-1})$ for $n = 3$ and $a_c(\beta) > O(|\beta|_2^{-1})$ for $n = 4$.

For an interesting conclusion from the above estimate, let $\vec{\beta}$ be the uniform vector $\vec{\beta} = \mathbf{1}_k := k^{-1}(1, \ldots 1) \in \mathbb{R}^k$. Then $|\vec{\beta}|_2 = k^{-1/2}$ (resp. $|\vec{\beta}|_\infty = k^{-1}$) so

$$n = 4 \Rightarrow a_c(\mathbf{1}_k) \geqslant O(k^{1/2})\text{resp. } n = 3 \Rightarrow \ (a_c(\mathbf{1}_k) \geqslant O(k)$$

for large k.

(c) The following alternative definitions of I_α and $(-\Delta)^{\alpha/2}$ is known [4, 16]:

$$I_\alpha = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha/2-1} \mathrm{e}^{t\Delta}\,\mathrm{d}t; (-\Delta)^{\alpha/2} = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha/2-1} \left(\mathrm{e}^{t\Delta} - I\right) \mathrm{d}t$$

where $\mathrm{e}^{t\Delta}$ is the heat kernel on $\mathbb{R}^n$:

$$\mathrm{e}^{t\Delta} = (4\pi t)^{-n/2} \mathrm{e}^{-|x|^2/4t}.$$

We may, at least formally, substitute the kernel $\mathrm{e}^{\Delta_\Omega t}$ of the killing, Dirichlet problem for the heat flow in a domain $\Omega \subset \mathbb{R}^n$ in the above expression, and obtain (again, at least formally...) an explicit expressions for I_α^Ω and $(-\Delta_\Omega)^{\alpha/2}$, introduced implicitly in (1.26). Such a representation can provide some insight on the trace of I_α^Ω for $\alpha < 2$.

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