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RESEARCH ARTICLE

Minimal Euler characteristics for even-dimensional manifolds with finite fundamental group

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Abstract
We consider the Euler characteristics $\chi(M)$ of closed, orientable, topological $2n$-manifolds with $(n-1)$-connected universal cover and a given fundamental group G of type F_n. We define $q_{2n}(G)$, a generalised version of the Hausmann-Weinberger invariant [19] for 4–manifolds, as the minimal value of $(-1)^n\chi(M)$. For all $n \geq 2$, we establish a strengthened and extended version of their estimates, in terms of explicit cohomological invariants of G. As an application, we obtain new restrictions for nonabelian finite groups arising as fundamental groups of rational homology 4–spheres.

Contents

1. Introduction

In this paper, we address the following problem: if M denotes a closed, orientable even-dimensional manifold with a given fundamental group G, then what restriction does this impose on the Euler characteristic of M? In the particular case when $\chi(M) = 2$, we have the related problem of determining which finite groups can be the fundamental group of a closed, topological, $2n$-manifold M with the rational homology of the $2n$-sphere (see previous work on the 4-dimensional case by Hambleton-Kreck [13] and Teichner [43]). We introduce the following invariant for discrete groups, extending a definition due to Hausmann and Weinberger [19] for 4–manifolds:

Definition 1.1. Given a finitely presented group G, define $q_{2n}(G)$ as the minimum value of $(-1)^n\chi(M)$ for a closed, orientable $2n$-manifold M with $(n-1)$-connected universal cover, such that $\pi_1(M) = G$.

We will first assume that G is a finite group. Recall that Swan [42, p. 193] defined an invariant $\mu_k(G)$, for each $k \geq 1$, by the condition that $(-1)^k|G|\mu_k(G)$ is the minimal value over all partial Euler characteristics of a free resolution of $\mathbb{Z}$ truncated after degree k. We call this a k-step resolution. However, since projective $\mathbb{Z}G$-modules are locally free [40, Section 8], k-step projective resolutions can be used instead to define $\mu'_k(G) \leq \mu_k(G)$ (see [42, Remark, p. 195]).

Let $e_n(G)$ denote the least integer greater than or equal to all the numbers

$$\dim H^n(G,\mathbb{F}) - 2\Big(\dim H^{n-1}(G,\mathbb{F}) - \dim H^{n-2}(G,\mathbb{F}) + \cdots + (-1)^{n-1}\dim H^0(G,\mathbb{F})\Big),$$

where the coefficients range over $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \mathbb{F}_p$ for all primes p. Our main result is the following:

Theorem A. *If G is a finite group and $n \geq 2$, then*

$$\max\{e_n(G), \mu'_n(G) - \mu'_{n-1}(G)\} \leq q_{2n}(G) \leq 2\mu_n(G).$$

Remark 1.2. By [42, Theorem 5.1], $\mu'_k(G) = \mu_k(G)$ unless G has periodic cohomology of (necessarily even) period dividing $k+1$, and G admits no periodic free resolution of period $k+1$. In this case, $k \geq 3$ is odd, and we will say that the pair (G,k) is *exceptional* (see Remark 2.8). For example, $\mu'_3(G) < \mu_3(G)$ for some of the 4-periodic groups $G = Q(8p,q)$ in Milnor's list (see the calculations in [29, 31]). If (G,n) is an exceptional pair, we provide information about $q_{2n}(G)$ in Theorem B and Remark 3.13 below.

The invariants $e_n(G)$ and the $\mu_k(G)$, for $1 \leq k \leq n$, can also be defined for infinite discrete groups of *type* F_n, meaning that there is a model for $K(G,1)$ with finite n-skeleton. In this case, we obtain similar estimates with a slightly weaker lower bound. Recall that a finitely presented group G is said to be *good* if topological surgery with fundamental group G holds in dimension four (see Freedman-Quinn [9, p. 99]).

Theorem A′. *If G is an infinite discrete group of type F_n with $n \geq 2$, then*

$$\max\{e_n(G), \mu_n(G) - \mu''_{n-1}(G)\} \leq q_{2n}(G).$$

If $n \geq 3$, or $n = 2$ and G is good, then $q_{2n}(G) \leq 2\mu_n(G)$.

The invariants $\mu''_k(G) = \mu_k(G)$, for $k \geq 3$, and we define $\mu''_2(G) = 1 - \mathrm{Def}(G)$, and $\mu''_1(G) = d(G) - 1$, where $\mathrm{Def}(G)$ is the *deficiency* of G, defined as the maximum difference $d - r$ of numbers of generators minus relations over all finite presentations of G (see [8]) and $d(G)$ denotes the minimal number of generators for G. These modifications to the previous invariants arise from the additional condition that the resolutions be *geometrically realisable* (see Section 4). For $k = 2$, determining the relation between $\mu_2(G)$ and $1 - \mathrm{Def}(G)$ is part of Wall's (unsolved) D2 problem [45, Section 2], which for infinite groups is related to the Eilenberg-Ganea conjecture [7].

Our results sharpen and generalise the estimate proved by Hausmann-Weinberger [19, Théorème 1]:

$$e_2(G) \leq q_4(G) \leq 2(1 - \mathrm{Def}(G)),$$

since $\mu_2(G) \leq (1 - \mathrm{Def}(G))$ by [42, Proposition 1]. The results of Kirk-Livingston [23] for $q_4(\mathbb{Z}^n)$ show that these bounds can be improved for specific groups.

The proof of the lower bound in Theorem A for $q_{2n}(G)$ is given in Section 2. In Section 3, we establish the upper bound $q_{2n}(G) \leq 2\mu_n(G)$ by generalising the well-known "thickening" construction for groups G which admit a *balanced presentation* with equal numbers of generators and relations (i.e., $\mathrm{Def}(G) = 0$). For $n = 2$, this involves showing that *finite* D2-*complex* es with *good* fundamental groups (e.g., groups of finite order) admit suitable thickenings via methods from topological surgery (see Theorem 3.8).

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Example 1.3. For $E_k = (\mathbb{Z}/p\mathbb{Z})^k$, an elementary abelian p-group, $e_n(E_k) = \mu_n(E_k) - \mu_{n-1}(E_k)$, and this number can be explicitly computed using the Kunneth formula (see Example 3.14). This can be used to show that $q_{2n}(E_k)$ grows like a polynomial of degree n in k, for example

$$\frac{k^4 - 2k^3 + 11k^2 - 34k + 48}{24} \leq q_8(E_k) \leq \frac{k^4 + 2k^3 + 11k^2 - 14k + 24}{12}.$$

For n even, $q_{2n}(G) \geq 2$, as the minimal possible Euler characteristic that can occur in our setting is $\chi(M) = 2$, which holds when M has the rational homology of a $2n$-sphere, and is implied by Theorem A if $\mu_n(G) = 1$. The condition $\mu_2(G) = 1$ also holds for groups of deficiency zero, and there are many groups with this property (see [51]). In contrast, our computations for the groups $E_k = (\mathbb{Z}/p\mathbb{Z})^k$ show that $q_{4n}(E_k) > 2$ for all $n > 1$ and $k \geq 3$. Hence, higher dimensional rational homology spheres with elementary abelian fundamental group of rank larger than 2 cannot occur.

For periodic groups, we can compute $q_{2n}(G)$ in certain cases, which, in particular, provides an alternate argument for [13, Corollary 4.4] and generalises that result to higher dimensions:

Theorem B. *Let G be a finite periodic group of (even) period q. Then $q_{2n}(G) = 2$ if q divides $n + 2$, and $q_{2n}(G) = 0$ if $2q$ divides $n + 1$.*

Remark 1.4. Note that in our setting, $\chi(M) > 0$ if and only if n is even (see Corollary 2.13). Thus, for n odd, the minimal possible value of $q_{2n}(G) = -\chi(M)$ is zero. Apart from the results of Theorem B for periodic groups with twice their period dividing $n + 1$, any finite group G which acts freely and homologically trivially on some product $S^n \times S^n$ will have $q_{2n}(G) = 0$. There are many such examples, including any products $G = G_1 \times G_2$ of periodic groups, many rank two finite p-groups, including the extra-special p-groups of order p^3, and all the finite odd order subgroups of the exceptional Lie group G_2 (see [11, 17, 18]).

We are especially interested in the case of rational homology 4-spheres (called $\mathbb{Q}S^4$ manifolds) with finite fundamental group. In Section 5, we consider the following "inverse" problem, for which the lower bound implies significant restrictions on G.

Question. Which finite groups can be the fundamental group of a closed, topological, 4-manifold M with the rational homology of the 4-sphere?

For example, it was observed in [13, p. 100] that if G is finite abelian, then $d(G) \leq 3$ (see Corollary 5.1). This bound follows directly by estimating the Hausmann-Weinberger invariant $q_4(G)$. Moreover, Teichner [43, Section 4.13] showed that this bound is best possible for abelian groups by explicit construction of examples.

Our methods shed light on more complicated finite groups by making use of cohomology with twisted coefficients to obtain better lower bounds for $q_4(G)$:

Theorem C. *Let $U_k = E_k \times_T C$, where p is an odd prime, $E_k = (\mathbb{Z}/pZ)^k$, and C cyclic of order prime to p acts on each $\mathbb{Z}/p\mathbb{Z}$ factor in E_k via $x \mapsto x^q$, where q is a unit in $\mathbb{Z}/p\mathbb{Z}$.*

(i) *If $x^{q^2} \neq x$ for all $1 \neq x \in E_k$, then for all $k > 4$, U_k does not arise as the fundamental group of any rational homology 4–sphere.*
(ii) *If $q = p-1$, then for all $k > 1$, U_k does not arise as the fundamental group of any rational homology 4–sphere.*

This paper is organised as follows: in Section 2, we analyse free group actions on $(n-1)$-connected $2n$-manifolds using cohomological methods; in Section 3, we discuss minimal complexes and thickenings; in Section 4, we prove Theorem A′; in Section 5, we focus on rational homology 4–spheres; and in Section 6, we collect some remarks, examples, and questions related to the invariants introduced here. Appendix A contains the proof of Theorem 3.8.

2. Free actions on $(n-1)$-connected $2n$-manifolds

In this section, we will apply the cohomological approach outlined in [1, Section 2]. The proofs of Propositions 2.1, 2.3, and 2.5 are straightforward modifications of the results there and details are omitted. We assume that Y is a closed, orientable, $(n-1)$-connected $2n$-manifold with the free orientation-preserving action of a finite group G; its homology has a corresponding $\mathbb{Z}G$–module structure. Both $H_{2n}(Y,\mathbb{Z})$ and $H_0(Y,\mathbb{Z})$ are copies of the trivial module $\mathbb{Z}$, whereas $H_n(Y,\mathbb{Z})$ is a free abelian group with a $\mathbb{Z}G$–module structure which, by Poincaré duality, must be self–dual as a $\mathbb{Z}G$–module, that is, $H_n(Y,\mathbb{Z}) \cong H_n(Y,\mathbb{Z})^*$. We assume, here, that Y admits a finite G–CW complex structure, with cellular chain complex denoted by $C_*(Y)$ (if the action is smooth, this is always true, and holds up to G-homotopy equivalence in the topological case).

We denote by $\Omega^r(\mathbb{Z})$ the $\mathbb{Z}G$-module uniquely defined in the stable category (where $\mathbb{Z}G$-modules are identified up to stabilisation by projectives) as the r–fold dimension–shift of the trivial module $\mathbb{Z}$. We refer to [2] and [3] for background on group cohomology.

Proposition 2.1. *Let Y be an $(n-1)$-connected $2n$-manifold with a free action of a finite group G which preserves orientation. Then there is a short exact sequence in the stable category of $\mathbb{Z}G$–modules of the form*

$$0 \to \Omega^{n+1}(\mathbb{Z}) \to H_n(Y;\mathbb{Z}) \to \Omega^{-n-1}(\mathbb{Z}) \to 0. \tag{2.1}$$

Corollary 2.2. *The short exact sequence (2.1) yields a long exact sequence in Tate cohomology*

$$\cdots \to \widehat{H}^{i+n}(G,\mathbb{Z}) \xrightarrow{\cup\sigma} \widehat{H}^{i-n-1}(G,\mathbb{Z}) \to \widehat{H}^{i}(G,H_n(Y;\mathbb{Z})) \to \widehat{H}^{i+n+1}(G,\mathbb{Z}) \to \dots$$

determined by the class $\sigma \in \widehat{H}^{-2n-1}(G,\mathbb{Z})$ which is the image of the generator $1 \in \widehat{H}^0(G,\mathbb{Z}) \cong \mathbb{Z}/|G|$.

We can analyse this sequence just as was done in [1, Section 2].

Proposition 2.3. *The cohomology class $\sigma \in \widehat{H}^{-2n-1}(G,\mathbb{Z}) \cong H_{2n}(G,\mathbb{Z})$ can be identified with the image of the fundamental class $c_*[Y/G]$ under the homomorphism*

$$c_*\colon H_{2n}(Y/G,\mathbb{Z}) \to H_{2n}(BG,\mathbb{Z})$$

induced by the classifying map $c\colon Y/G \to BG$. Under this identification, the class σ determines the extension (2.1).

Remark 2.4. This property of the extension class was proved for $n = 2$ in [13, Corollary 2.4], and the proof in the general case is similar.

Similarly, the map $\Omega^{n+1}(\mathbb{Z}) \to H_n(Y,\mathbb{Z})$ defines an extension class

$$\varepsilon_Y \in H^{n+1}(G, H_n(Y,\mathbb{Z}))$$

which appears in the long exact sequence above as the image of the generator under the map $\widehat{H}^0(G,\mathbb{Z}) \to \widehat{H}^{n+1}(G,\mathbb{Z})$. Algebraically, this responds to mapping the canonical defining extension for $\Omega^{n+1}(\mathbb{Z})$ (identified with the extension class for the module of cycles in $C_n(Y)$) to the extension obtained by reducing by the module of boundaries B_n:

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$$\begin{tikzcd}
 & B_n \arrow[r, equal] \arrow[d] & B_n \arrow[d] & & & & & \\
0 \arrow[r] & Z_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d, equal] & \dots \arrow[r] & C_0 \arrow[r] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H_n(Y,\mathbb{Z}) \arrow[r] & C_n/B_n \arrow[r] & C_{n-1} \arrow[r] & \dots \arrow[r] & C_0 \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}.$$

These two extension classes are related as follows:

Proposition 2.5. *Let G denote a finite group acting freely on an $(n-1)$-connected, orientable $2n$-manifold Y preserving orientation, then $\varepsilon_Y \neq 0$ and $|G| = exp(\sigma) \cdot exp(\varepsilon_Y)$. The class ε_Y has exponent $|G|$ if and only if $\sigma = 0$, in which case, we have a stable equivalence*

$$H_n(Y,\mathbb{Z}) \cong \Omega^{n+1}(\mathbb{Z}) \oplus \Omega^{-n-1}(\mathbb{Z}).$$

Example 2.6. Observing that the cohomology of a group with periodic cohomology is always zero in odd dimensions, we see that if G has periodic cohomology, then there is a stable equivalence $H_n(Y,\mathbb{Z}) \cong \Omega^{n+1}(\mathbb{Z}) \oplus \Omega^{-n-1}(\mathbb{Z})$.

We note the standard identity $\chi(Y) = 2 + (-1)^n \dim H_n(Y,\mathbb{Q})$, and the formula $|G|\chi(Y/G) = \chi(Y)$ from the covering $Y \to Y/G$. Since the transfer map induces an isomorphism $H_i(Y/G;\mathbb{Q}) \cong H_i(Y;\mathbb{Q})^G$, we have $\chi(Y/G) = 2 + (-1)^n \dim H_n(Y,\mathbb{Q})^G$. In particular

$$\dim H_n(Y,\mathbb{Q})^G = (-1)^n(\chi(Y/G) - 2).$$

From the stable sequence

$$0 \to \Omega^{-n}(\mathbb{Z}) \to \Omega^{n+1}(\mathbb{Z}) \to H_n(Y,\mathbb{Z}) \to 0,$$

we infer the existence of projective modules Q_r and Q_s which fit into an exact sequence

$$0 \to \Omega^{-n}(\mathbb{Z}) \oplus Q_s \to \Omega^{n+1}(\mathbb{Z}) \oplus Q_r \to H_n(Y;\mathbb{Z}) \to 0, \tag{2.2}$$

where $Q_i \otimes \mathbb{Q} \cong [\mathbb{Q}G]^i$ for $i = r, s$. Here, we write $\Omega^{j+1}(\mathbb{Z})$ $(j \geq 0)$ for the j-th kernel in a minimal projective resolution of $\mathbb{Z}$, meaning a resolution:

$$0 \to \Omega^{j+1}(\mathbb{Z}) \to P_j \to P_{j-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0 \tag{2.3}$$

realising $\mu'_j(G)$ (see [42, p. 193]), from which we see that

$$\mathrm{rank}_{\mathbb{Z}}\, \Omega^j(\mathbb{Z}) + (-1)^{j-1} = |G|(\mu'_{j-1}(G)) \quad \text{and} \quad \mathrm{rank}_{\mathbb{Z}}\, \Omega^j(\mathbb{Z})^G + (-1)^{j-1} = \mu'_{j-1}(G),$$

where $(-1)^k|G|\mu'_k(G)$ is precisely the minimal value over all partial Euler characteristics of a *projective* resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ (see [42, Remark, p. 195]). The corresponding invariants $\mu_k(G)$ for minimal *free* resolutions of $\mathbb{Z}$ were defined by Swan (see [42, p. 193]).

By dualising, we see that a minimal representative for $\Omega^{-j}(\mathbb{Z})$ is given by $\Omega^j(\mathbb{Z})^*$, the dual module. Thus, for our purposes, we have

$$\mathrm{rank}_{\mathbb{Z}}\, \Omega^{n+1}(\mathbb{Z})^G = \mu'_n(G) + (-1)^{n+1}, \quad \mathrm{rank}_{\mathbb{Z}}\, \Omega^{-n}(\mathbb{Z})^G = \mu'_{n-1}(G) + (-1)^n.$$

Applying invariants after tensoring over $\mathbb{Q}$ to the exact sequence (2.2) yields the formula

$$\mu'_n(G) + (-1)^{n+1} + r = \mu'_{n-1}(G) + (-1)^n + s + (-1)^n[\chi(Y/G) - 2],$$

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whence we obtain

$$s - r = \mu'_n(G) - \mu'_{n-1}(G) + (-1)^{n+1}\chi(Y/G).$$

Theorem 2.7. *If Y is a closed, $(n-1)$-connected $2n$-manifold with a free orientation-preserving action of G, a finite group, then for any subgroup $H \subset G$*

$$\mu'_n(H) - \mu'_{n-1}(H) \leq (-1)^n[G:H]\chi(Y/G).$$

Proof. We will prove this for $H = G$ by contradiction. Assume that $s - r > 0$ and form the diagram

$$\begin{tikzcd}
 & Q_s \arrow[r, equal] \arrow[d] & Q_s \arrow[d] & & \\
0 \arrow[r] & \Omega^{-n}(\mathbb{Z}) \oplus Q_s \arrow[r] \arrow[d] & \Omega^{n+1}(\mathbb{Z}) \oplus Q_r \arrow[r] \arrow[d] & H_n(Y,\mathbb{Z}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^{-n}(\mathbb{Z}) \arrow[r] & L \arrow[r] & H_n(Y,\mathbb{Z}) \arrow[r] & 0,
\end{tikzcd}$$

where L is the quotient of $\Omega^{n+1}(\mathbb{Z}) \oplus Q_r$ in the middle vertical exact sequence. Note that this middle vertical exact sequence splits (since L is torsion-free). Hence,

$$\Omega^{n+1}(\mathbb{Z}) \oplus Q_r \cong L \oplus Q_s.$$

By Swan [41, Lemma 2.1], there is a projective resolution

$$0 \to L \to P_n \oplus Q_r \to P_{n-1} \oplus Q_s \to P_{n-2} \to \cdots \to P_0 \to \mathbb{Z} \to 0.$$

Since $s > r$, this contradicts the minimality of the resolution (2.3) realising $\mu'_n(G)$. Hence, we have shown that $s - r \leq 0$. The full result follows using covering spaces. □

Remark 2.8. As mentioned in the Introduction, Swan proved that $\mu'_k(G) = \mu_k(G)$ unless G has periodic cohomology of period dividing $k+1$, and G admits no periodic free resolution of period $k+1$. In these *exceptional* cases, $\mu_k(G) = 1$ and $\mu'_k(G) = 0$. In contrast, $\mu_k(G) = 0$ if G has a periodic free resolution of period $k+1$ and $G \neq 1$. We also note that if the pair (G,k) is exceptional, then $k \geq 3$ is odd and G is noncyclic. In particular, $\mu'_k(G) = \mu_k(G)$ if G is a finite p-group (see [42, Corollary 5.2]).

If the pair (G,n) is not exceptional, the numbers $\mu_n(G)$ can be computed using group cohomology. By a result of Swan [42, Proposition 6.1], the invariant $\mu_n(G)$ is the least integer greater than or equal to all the numbers

$$(\dim M)^{-1}\Big(\dim H^n(G,M) - \dim H^{n-1}(G,M) + \cdots + (-1)^n \dim H^0(G,M)\Big)$$

as M ranges over all simple $\mathbb{F}_pG$–modules for all primes p dividing $|G|$. As extending the field doesn't change dimensions, we can take $\mathbb{K}_p$, an algebraically closed field of characteristic p, and restrict attention to absolutely irreducible $\mathbb{K}_pG$–modules. Next we introduce

Definition 2.9. For any discrete group G of type F_n, let $e_n(G)$ denote the least integer greater than or equal to all the numbers

$$\dim H^n(G,\mathbb{F}) - 2\Big(\dim H^{n-1}(G,\mathbb{F}) - \dim H^{n-2}(G,\mathbb{F}) + \cdots + (-1)^{n-1}\dim H^0(G,\mathbb{F})\Big),$$

where the coefficients range over $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \mathbb{F}_p$ for all primes p.

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Remark 2.10. When $G = P$ is a finite p–group, the trivial module $\mathbb{F}_p$ is the only simple module, and we can verify that $\mu_n(P) - \mu_{n-1}(P) = e_n(P)$.

We have the following elementary inequality:

Lemma 2.11. *Suppose that X is a closed, orientable $2n$-manifold with fundamental group G of type F_n whose universal cover is $(n-1)$-connected. Then, for any subgroup $H \subset G$ of finite index*

$$e_n(H) \le [G : H](-1)^n \chi(X).$$

Proof. Let $\mathbb{F}$ denote any field of coefficients. The connectivity of the universal cover implies that

$$H^i(G, \mathbb{F}) \cong H^i(X, \mathbb{F}) \ \text{ for } \ 0 \le i \le n-1$$

and

$$\dim H^n(G, \mathbb{F}) \le \dim H^n(X, \mathbb{F}).$$

By Poincaré duality, we have

$$H^k(X, \mathbb{F}) \cong H^{2n-k}(G, \mathbb{F}) \ \text{ for } \ n+1 \le k \le 2n.$$

Combining these facts and using covering space theory, we obtain the inequality. □

Applying the mod p coefficient sequence yields an attractive corollary

Corollary 2.12. *If X is a closed, orientable $2n$-manifold with finite fundamental group G whose universal cover is $(n-1)$-connected, then for all primes p dividing $|G|$ and subgroups $H \subset G$*

$$\dim H^{n+1}(H, \mathbb{Z}) \otimes \mathbb{F}_p - \dim H^n(H, \mathbb{Z}) \otimes \mathbb{F}_p \le (-1)^n([G : H]\chi(X) - 2).$$

Proof. Since $[G : H]\chi(X)$ equals the Euler characteristic of the $[G : H]$-fold covering of X, it is enough to do the case $H = G$. Let $h^i(G) = \dim H^i(G; \mathbb{F}_p)$. From the relations noted above, and Lemma 2.11, we have the formula

$$(-1)^n(\chi(X) - 2) \ge h^n(G) - 2\sum_{i=1}^{n-1}(-1)^{i+1}h^{n-i}(G).$$

But by the mod p coefficient sequence, we have

$$h^i(G) = \dim H^{i+1}(G, \mathbb{Z}) \otimes \mathbb{F}_p + \dim H^i(G, \mathbb{Z}) \otimes \mathbb{F}_p, \quad \text{for } \ 1 \le i \le n.$$

The result follows by combining these two relations. □

Applying this to any subgroup $C \subset G$ of prime order, we obtain

Corollary 2.13. *If X is a closed, orientable $2n$-manifold with $(n-1)$-connected universal cover and nontrivial finite fundamental group G, then $\chi(X) > 0$ if and only if n is even.*

Proof. Let $C \subset G$ be a cyclic subgroup of order p, a prime. Then $H^{2k}(C; \mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$ (if $k > 0$), and $H^{2k+1}(C; \mathbb{Z}) = 0$. For n even, applying the inequality above with $H = C$ yields $\chi(X) > 0$. When n is odd, note that $b_n(X) \neq 0$ implies $b_n(X) \ge 2$, since the intersection form of X is nonsingular and skew-symmetric. Hence, $\chi(X) \le 0$. □

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3. Minimal $K(G, n)$-complexes and thickenings

We now turn our attention to the **existence** of orientable $2n$-manifolds having fundamental group of type F_n and $(n-1)$-connected universal cover. We recall the following well-known construction (see Kreck and Schafer [24, Section 2]):

Proposition 3.1. *Let G be a discrete group of type F_n for $n \geq 2$. Then there exists a closed, orientable $2n$-manifold Z, such that $\pi_1(Z) = G$ with $(n-1)$-connected universal cover.*

Proof. Let K denote a finite CW complex of dimension n with $\pi_1(K) = G$ whose universal covering is $(n-1)$-connected. For example, take a finite, cellular model for the classifying space BG, and consider its n-skeleton K. Then we can construct a smooth $2n$-manifold $Z = M(K)$ by doubling a $2n$-dimensional handlebody thickening of K. Thus, the universal cover $\tilde{Z}$ of $M(K)$ is an $(n-1)$-connected, closed, orientable $2n$-manifold with a free action of G, such that

$$\pi_n(M(K)) \cong H_n(M(K); \Lambda) \cong H^n(K; \Lambda) \oplus H_n(K; \Lambda),$$

where $\Lambda := \mathbb{Z}G$ denotes the integral group ring. Moreover, the Euler characteristic $\chi(M(K)) = 2\chi(K)$. A variation of this construction is to let Z denote the boundary of a regular neigbourhood, for some embedding $K \subset \mathbb{R}^{2n+1}$ of the finite n-complex in Euclidean space. □

Definition 3.2. Let G be a discrete group of type F_n. A finite CW complex K of dimension $n \geq 2$, with fundamental group $\pi_1(K) = G$ and $\pi_i(K) = 0$ for $1 \leq i \leq n-1$, is called a $K(G, n)$-*complex*.

The chain complex $C_*(\tilde{K})$ of the universal covering of a $K(G, n)$-complex affords a free n-step resolution of the trivial $\mathbb{Z}G$–module $\mathbb{Z}$. Conversely, we wish to realise a given finitely generated n-step free resolution

$$\mathscr{F} : F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

as the equivariant chain complex of a suitable $K(G, n)$-complex. Note that by Swan [42, Theorem 1.2], we have $\mu_n(G) \leq (-1)^n \chi(\mathscr{F})$ and that the lower bound is attained by some resolution.

Proposition 3.3. *Let G be a discrete group of type F_n, and let $\mathscr{F}$ be an n-step resolution of $\mathbb{Z}$ by finitely generated free ZG-modules. If $n \geq 3$, then there exists a finite $K(G, n)$-complex K and a G-equivariant chain homotopy equivalence $C_*(\widetilde{K}) \simeq \mathscr{F}$.*

Proof. Let $n \geq 3$, we can apply [16, Lemma 8.12] to show that $\mathscr{F}$ is chain homotopy-equivalent to a finitely generated free complex $\mathscr{F}'$ which agrees with the 2-skeleton of a model for $K(G, 1)$. Then the construction of [41, Lemma 3.1] (credited to Milnor) provides the required complex K by successively attaching i-cells equivariantly using the boundary maps from the chain complex $\mathscr{F}'$. □

Remark 3.4. For finite groups, Swan [42, Corollary 5.1] shows that under certain additional assumptions, one can geometrically realise the actual sequence $f_0, f_1, f_2, \ldots$ of ranks for the i-chains of $\mathscr{F}$. We also record the facts due to Swan that $\mu_n(G) \geq 1$ for n even, and $\mu_n(G) \geq 0$ for n odd if $G \neq 1$ is finite (see [42, Section 1]).

Corollary 3.5. *If $n \geq 3$, then for any discrete group of type F_n, we have $q_{2n}(G) \leq 2\mu_n(G)$. In particular, if n is even and G is a finite group with $\mu_n(G) = 1$, then G is the fundamental group of a rational homology $2n$-sphere.*

Proof. We apply Proposition 3.3 to a minimal n-step resolution $\mathscr{F}$ with $\chi(\mathscr{F}) = \mu_n(G)$, and obtain a finite $K(G, n)$-complex K. The manifold $Z = M(K)$ constructed in Proposition 3.1 provides the upper bound $q_{2n}(G) \leq \chi(Z) = 2\mu_n(G)$. □

We now consider the case $n = 2$, where the argument above fails at the first step. To establish our upper bound for $q_4(G)$, we need a more general construction and some results of C. T. C. Wall [45, 46].

Definition 3.6. A finite complex X *satisfies Wall's* D2-*conditions* if $H_i(\widetilde{X}) = 0$, for $i > 2$, and $H^3(X; \mathcal{B}) = 0$, for all coefficient bundles $\mathcal{B}$. Here, $\widetilde{X}$ denotes the universal covering of X. If these conditions hold, we will say that X is a D2-*complex*. If every D2-complex with fundamental group G is homotopy-equivalent to a finite 2-complex, then we say that G *has the* D2-property.

In [45, p. 64], Wall proved that a finite complex X satisfying the D2-conditions is homotopy-equivalent to a finite 3-complex. We will therefore assume that all our D2-complexes have $\dim X \leq 3$. It is not known at present whether all discrete groups have the D2-property. Note that $\mu_2(G) \leq (1 - \mathrm{Def}(G))$ by [42, Proposition 1], and equality holds if G has the D2-property.

Proposition 3.7 [12, Corollary 2.4]. *Any finitely generated free resolution*

$$\mathcal{F} : F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

over $\mathbb{Z}G$ *is chain homotopy-equivalent to* $C_*(X)$*, where* X *is a finite* D2-*complex.*

If we apply this to a minimal resolution with $\chi(\mathcal{F}) = \mu_2(G) = \mu_2'(G)$, then if G is finite, the module $H_2(\widetilde{X}; \mathbb{Z})$ is a minimal $\mathbb{Z}$-rank representative of the stable module $\Omega^3(\mathbb{Z})$. The following result may also be of independent interest (it applies to any finitely presented group G which is *good* in the sense of Freedman [9, p. 99], in particular, to poly-(finite or cyclic) groups).

Theorem 3.8. *For any finite* D2-*complex* X *with good fundamental group, there exists a closed, topological* 4-*manifold* $M(X)$ *with* $\pi_1(M(X)) = \pi_1(X)$ *and* $\chi(M(X)) = 2\chi(X)$.

For continuity, we defer the proof of this result to Appendix A.

Corollary 3.9. *For* G *a finitely presented good group,* $q_4(G) \leq 2\mu_2(G)$*. In particular,* $\mu_2(G) = 1$ *and* G *finite implies that* G *is the fundamental group of a rational homology* 4-*sphere.*

Proof. We apply Proposition 3.7 to realise a minimal 2-step resolution by a finite D2-complex, and then Theorem 3.8 provides a suitable $\mathbb{Q}S^4$ manifold. □

The proof of Theorem A. Concatenating our previous results, we have obtained the estimates

$$\max\{e_n(G), \mu_n'(G) - \mu_{n-1}'(G)\} \leq q_{2n}(G) \leq 2\mu_n(G)$$

for any finite group G. For the lower bound, we apply Theorem 2.7 and Lemma 2.11. For the upper bound, we apply Corollary 3.5 if $n > 2$, and Corollary 3.9 for $n = 2$.

We now prepare for the proof of Theorem B. The next result, due to Swan and Wall, shows that arbitrary periodic groups appear as fundamental groups of rational homology spheres.

Lemma 3.10. *If* G *is a finite group with periodic cohomology of period dividing* $2k+2$*, then* $\mu_{2k}(G) = 1$ *for* $k \geq 1$.

Proof. We will discuss the case $k = 1$ for groups of period 4. Swan [41] constructed a finitely dominated Poincaré 3-complex Y with $\pi_1(Y) = G$, and Wall [47, Corollary 2.3.2] shows that Y is obtained from a D2-complex by attaching a single 3-cell. The chain complex $C_*(\widetilde{Y})$ provides a projective resolution

$$\mathcal{F}' : P \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

with $\chi(\mathcal{F}') = 1$, where P is projective, F_1 and F_0 are free and $I(G)^* = \ker d_2(\mathcal{F}')$. This shows that $\mu_2'(G) = 1$ and so $\mu_2(G) = \mu_2'(G) = 1$ by Swan's results.

One can give a direct argument for this last step. By adding a projective Q so that $P \oplus Q = F$ is free, we obtain a free resolution

$$\mathcal{F} : F \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

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with $I(G)^* \oplus Q = \ker d_2(\mathscr{F})$. By the 'Roiter replacement lemma' (see [36, Proposition 5], or [21, Theorem 3.6]), $I(G)^* \oplus Q = J \oplus F'$, where F' is free and J is locally isomorphic to $I(G)^*$, so $\mathrm{rank}_{\mathbb{Z}}(J) = \mathrm{rank}_{\mathbb{Z}} I(G)^*$. We now divide out the image of F' in F (a direct summand) to obtain a free resolution

$$\mathscr{F}'' : F'' \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

with $J = \ker d_2(\mathscr{F}'')$ and $\chi(\mathscr{F}'') = 1$. Hence, $\mu_2(G) = 1$.

A similar argument shows that $\mu_{2k}(G) = 1$, for all $k > 1$, if G has periodic cohomology with period dividing $2k + 2$. Details will be left to the reader. □

Remark 3.11. The calculation in Lemma 3.10 together with Theorem 3.8 provides an alternate proof of [13, Corollary 4.4]. However, the essential ingredients are the same in both arguments.

The proof of Theorem B. By assumption, the group G is periodic of even period q. In the first case, if q divides $n + 2$, then n is even and $\mu_n(G) = 1$ by Lemma 3.10. By Theorem A, we have the inequalities

$$2 \le q_{2n}(G) \le 2\mu_2(G) = 2,$$

and hence, $q_{2n}(G) = 2$.

In the second case, n is odd and the minimal Euler characteristic $q_{2n}(G) \ge 0$ by Corollary 2.13. We will show that the lower bound is realised when G is a periodic group of even period q, provided that $2q$ divides $n + 1$.

This follows from the solution of the space form problem: Madsen, Thomas and Wall [28, Theorem 1], [49, Corollary 12.6] proved that there exists a finite Poincaré duality complex X (called a *finite Swan complex*) of dimension $(2k - 1)$, with $\pi_1(X) = G$ and universal covering $\widetilde{X} \simeq S^{2k-1}$, whenever $k \equiv 0 \pmod{e(G)}$, where $e(G)$ is the Artin exponent of G [25, p. 94]. Moreover, a detailed analysis of the group cohomology of periodic groups shows that $2e(G)$ is equal to q or $2q$, depending on the structure of its 2-hyperelementary subgroups (see Wall [49, p. 542], where the notation $2d(\pi)$ is used for the period of a periodic group π).

For any finite Swan complex X, there exists a degree one normal map $(f, b)\colon N \to X$, where N^n is a closed, topological n-manifold (see [44, Corollary 3.3]). We then have a degree one normal map of pairs

$$(f \times \mathrm{id}, b \times \mathrm{id})\colon (N \times D^{n+1}, N \times S^n) \to (X \times D^{n+1}, X \times S^n).$$

By Wall's 'π-π Theorem' [50, Theorem 3.3], this normal map is normally cobordant to a homotopy equivalence of pairs. It follows that $X \times S^n$ is homotopy-equivalent to a closed, topological $2n$-manifold. Since $X \times S^n$ has Euler characteristic zero, these examples show that $q_{2n}(G) = 0$ as required.

Remark 3.12 (Smooth examples). If G satisfies the $2p$*-conditions* (meaning that every subgroup of order $2p$ is cyclic, for p prime), Madsen, Thomas and Wall [28, Theorem 5] proved that there exists a closed, oriented, smooth $(2k - 1)$-manifold N^{2k-1} with $\pi_1(N) = G$ and universal covering $\widetilde{N} = S^{2k-1}$, whenever $k \equiv 0 \pmod{e(G)}$. Under this extra assumption, the products $N^n \times S^n$, for $n = 2qr - 1$, provide *smooth manifolds* realising the minimum value $q_{2n}(G) = 0$.

Remark 3.13 (The exceptional case). In the arguments above, we have not used the full strength of the Madsen-Thomas-Wall results, which produce smooth space forms in the minimal dimension $q - 1$ whenever $q = 2e(G)$ (see the discussion on [28, p. 142]). This observation does give additional examples of periodic groups with $q_{2n}(G) = 0$, for example, when $n + 1 \equiv 2 \pmod 4$, but deciding whether $2e(G)$ equals q or $2q$ for a given G involves difficult number theory.

If the pair $(G.n)$ is exceptional, then surgery theory can be used to study $q_{2n}(G)$ as follows (see [27, Sections 2–3] for background on the space form problem):

(i) For any periodic group with period $n + 1$, there exists a *finitely dominated* Swan complex X with $\pi_1(X) = G$ and universal covering $\widetilde{X} \simeq S^n$ (see [41, Proposition 3.1]).

(ii) For any finitely dominated Swan complex X, there exists a degree one normal map $(f, b)\colon N \to X$, where N^n is a closed, oriented, topological n-manifold (see [44, Corollary 3.3] and [48, Proposition 2]).

(iii) The product $X \times S^n$ is homotopy-equivalent to a finite Poincaré complex (by the product formula for Wall's finiteness obstruction [10, Theorem 0.1]).

(iv) We have a degree one normal map

$$(f \times \mathrm{id}, b \times \mathrm{id})\colon N \times S^n \to X \times S^n,$$

with surgery obstruction $\lambda(f, b) \in L^h_{2n}(\mathbb{Z}G)$ determined by the Wall finiteness obstruction $\sigma(X) \in \widetilde{K}_0(\mathbb{Z}G)$ (see [34, p. 244]).

(v) If $\lambda(f, b) = 0$ (this is the hard step), then this normal map would be normally cobordant to a homotopy equivalence. In other words, $X \times S^n$ would be homotopy-equivalent to a closed, topological $2n$-manifold with Euler characteristic zero.

We conclude this section with a sample computation of the estimates for elementary abelian p-groups.

Example 3.14. If $E_k = (\mathbb{Z}/p\mathbb{Z})^k$ then we can use the Kunneth formula to compute these invariants. The term $\mu_n(E_k)$ has a polynomial of degree n as its leading term. For $n = 2, 3, 4$, we have

$$\frac{k^2 - 3k + 4}{2} \le q_4(E_k) \le k^2 - k + 2$$

$$\frac{k^3 - 3k^2 + 8k - 12}{6} \le q_6(E_k) \le \frac{k^3 + 5k - 6}{3}$$

$$\frac{k^4 - 2k^3 + 11k^2 - 34k + 48}{24} \le q_8(E_k) \le \frac{k^4 + 2k^3 + 11k^2 - 14k + 24}{12}$$

For instance, for $k = 2$, this only gives the rough estimate $1 \le q_8(E_2) \le 6$, but we know that $q_8(E_2) = 2$ by performing surgery[1] on $L^7(\mathbb{Z}/p\mathbb{Z}) \times S^1$. However, for $k = 3$, the lower bound gives $q_8(E_3) \ge 3$, and hence, E_3 is not the fundamental group of a rational homology 8-sphere.

4. The proof of Theorem A′

In this section, we establish a lower bound for $q_{2n}(G)$, for G an infinite discrete group of type F_n. With the results of Lemma 2.11, Corollary 3.5 and Corollary 3.9, this will complete the proof of Theorem A′.

The invariants $\mu''_k(G)$ used in the statement of Theorem A′ can also be defined as follows.

Definition 4.1. For $k \ge 2$, let $\mu''_k(G) = (-1)^k \cdot \min\{\chi(\mathscr{F})\}$, where $\mathscr{F}$ varies over all k-step resolutions

$$\mathscr{F} : F_k \to F_{k-1} \to \cdots \to F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

of $\mathbb{Z}$ by finitely generated free $\mathbb{Z}G$-modules, which arise geometrically as the chain complex of the universal covering for a finite CW-complex of dimension k with fundamental group G.

The sign $(-1)^k$ is introduced to agree with Swan's conventions. Note the inequalities

$$\mu'_k(G) \le \mu_k(G) \le \mu''_k(G)$$

relating these invariants to those defined by Swan. We define $\mu''_1(G) = d(G) - 1$, where $d(G)$ denotes the minimal number of generators for G.

[1]Here, $L^7(\mathbb{Z}/p\mathbb{Z})$ denotes the 7-dimensional lens space with fundamental group $G = \mathbb{Z}/p\mathbb{Z}$, and the surgery is performed on the S^1 factor by removing $D^6 \times S^1$ and gluing in $S^5 \times D^2$.

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Remark 4.2. By Proposition 3.3, we have $\mu_k(G) = \mu''_k(G)$ if $n \geq 3$. Note that $\mu''_2(G) = 1 - \mathrm{Def}(G)$. If $\mu_2(G) < \mu''_2(G)$ for some finitely presented group G, then there would be a counter-example to Wall's D2 problem (but no such examples are known at present). In addition, we do not know if the strict inequality $\mu_1(G) < d(G) - 1$ can occur.

We now establish the lower bound for infinite groups.

Theorem 4.3. *Let G be a discrete group of type F_n, for $n \geq 2$. If Y is a closed, $(n-1)$-connected $2n$-manifold with a free orientation-preserving action of G, a finite group, then for any subgroup $H \subset G$ of finite index*

$$\mu_n(H) - \mu''_{n-1}(H) \leq (-1)^n [G : H] \chi(Y/G).$$

Proof. It suffices to prove this inequality for $H = G$, and then apply covering space theory. Let M denote a closed, orientable, $2n$-dimensional manifold with $n \geq 2$ and fundamental group G of type F_n, such that $\pi_i(M) = 0$ for $1 < i < n$. Let $K \simeq M$ be a finite CW-complex homotopy-equivalent to M, and let $C := C(K; \Lambda) = C(\widetilde{K})$ denote the chain complex of its universal covering. It is a finite chain complex, with each C_i a finitely generated free $\mathbb{Z}G$-module. We note that the homology of M is computed from the chain complex $C \otimes_{\mathbb{Z}G} \mathbb{Z}$, and therefore, $\chi(M) = \sum_{i=0}^{2n} (-1)^i c_i$, where $c_i := \mathrm{rank}_{\mathbb{Z}G}\, C_i$.

We may assume that the $(n-1)$-skeleton $K^{(n-1)} \subset K$ has $(-1)^{n-1}\chi(\widetilde{K}^{(n-1)}) = \mu''_{n-1}(G)$, by applying Wall's construction of a *normal form* to replace K by a homotopy-equivalent complex if necessary (see [47, p. 238]).

The long exact sequences of the triples $(K, K^{(i)}, K^{(i-1)})$, for cohomology with $\mathbb{Z}G$-coefficients gives:

$$0 \to H^i(K, K^{(i-1)}) \to H^i(K^{(i)}, K^{(i-1)}) \to H^{i+1}(K, K^{(i)}) \to H^{i+1}(K, K^{(i-1)}) \to 0.$$

If we let $Z^i := \ker \delta^i$ and $B^i := \mathrm{im}\, \delta^{i-1}$ (for later use) in the cochain complex (C^*, δ^*), where $C^i = \mathrm{Hom}_\Lambda(C_i, \Lambda)$, then the sequence above becomes

$$0 \to Z^i \to C^*_i \to Z^{i+1} \to H^{i+1}(C) \to 0.$$

Since $H^i(C) = H_{2n-i}(C) = 0$, for $n+1 \leq i \leq 2n-1$, and $H^{2n}(C) = \mathbb{Z}$, we can splice the short exact sequences

$$0 \to Z^i \to C^*_i \to Z^{i+1} \to 0$$

for $n \leq i \leq 2n-1$, and obtain a long exact sequence

$$0 \to Z^n \to C^*_n \to C^*_{n+1} \to \cdots \to C^*_{2n-1} \to C^*_{2n} \to \mathbb{Z} \to 0.$$

Since this a resolution of $\mathbb{Z}$ by finitely generated $\mathbb{Z}G$-modules, with $\mathrm{rank}_{\mathbb{Z}G}(C^*_i) := c_i$, we have

$$(-1)^n \chi_{upper}(C) := (-1)^n \sum_{i=n}^{2n} (-1)^i c_i \geq \mu_n(G).$$

On the other hand, by the normal form construction, we have

$$(-1)^n \chi_{lower}(C) := (-1)^n \sum_{i=0}^{n-1} (-1)^i c_i = (-1)^n (-1)^{n-1} \mu''_{n-1}(G) = -\mu''_{n-1}(G).$$

Therefore, $q_{2n}(G) \geq (-1)^n \chi(M) \geq \mu_n(G) - \mu''_{n-1}(G)$, as required. □

Remark 4.4. Note that $\mu'_1(G) \leq \mu_1(G) \leq d(G) - 1$ by Swan [42, Proposition 1], so this is slightly different than the estimate in Theorem A for finite groups if $n = 2$.

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5. Rational homology 4–spheres

We now specialise our results to the case when M is a rational homology 4–sphere with finite fundamental group G. We would like to find restrictions on G by computing $\mu_2(G)-\mu_1(G)$. Note that $\mu_1(G) = \mu_1'(G)$ and $\mu_2(G) = \mu_2'(G)$, and that for any solvable finite group, $\mu_1(G) = d(G) - 1$ [6, Proposition 1]. Let A denote a finite abelian group minimally generated by d elements, then using Theorem 2.7, we have the estimate

$$\mu_2(A) - \mu_1(A) = \frac{d^2 - 3d + 4}{2} \leq \chi(M) = 2$$

and so we recover the estimate proved in [43, 3.4]:

Corollary 5.1. *If G is a finite abelian group minimally generated by $k > 3$ elements, then it cannot be realised as the fundamental group of a closed 4–manifold which is a rational homology sphere.*

Our next objective will be to consider examples where twisted coefficients can be used to establish conditions for nonabelian groups. Recall the result due to Swan [42, Theorem 1.2 and Proposition 6.1]: for any finite group G, $\mu_n(G)$ is the smallest integer which is an upper bound on

$$(\dim M)^{-1}\Big(h^n(G,M) - h^{n-1}(G,M) + \cdots + (-1)^n h^0(G,M)\Big),$$

where K is a field, M is a KG-module and $h^n(G;M) := \dim_K H^n(G;M)$. Moreover, we can assume that K is algebraically closed and has characteristic p dividing $|G|$, and it suffices to verify the upper bound on absolutely irreducible modules.

We now focus on an interesting class of nonabelian groups for which the absolutely irreducible modules are easy to determine. The following proposition follows from elementary representation theory (see [52, Corollary 6.2.2]).

Proposition 5.2. *Let K be a field of characteristic p, G a finite group with maximal normal p–group denoted by $O_p(G)$. Then the simple KG–modules are precisely the simple $K[G/O_p(G)]$–modules, made into KG–modules via the quotient homomorphism $G \to G/O_p(G)$.*

Corollary 5.3. *Let $U_k = E_k \times_T C$ denote a semidirect product, where $E_k \cong (\mathbb{Z}/p\mathbb{Z})^k$ with $k > 1$ and C is cyclic of order relatively prime to p. Then for any algebraically closed field $\mathbb{K}_p$ of characteristic p, the absolutely irreducible $\mathbb{K}_p U_k$–modules are one-dimensional characters $\alpha\colon C \to \mathbb{K}_p^\times$ on which E_k acts trivially.*

The cyclic group C acts on the vector spaces $H^i(E_k;\mathbb{K}_p)$ via one-dimensional characters $\alpha\colon C \to \mathbb{K}_p^\times$. Using the multiplicative structure in cohomology and the Bockstein, this is determined by $N_k = H^1(E_k;\mathbb{K}_p) \cong \mathrm{Hom}(E_k,\mathbb{K}_p)$ as an $\mathbb{K}_p[C]$–module.

Recall that by [2, Corollary II.4.3, Theorem II.4.4], the mod p cohomology ring of E_k is given by

$$H^*(E_k,\mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[x_1,\ldots,x_k] & \text{for } p = 2 \\ \Lambda(x_1,\ldots,x_k) \otimes \mathbb{F}_p[y_1,\ldots,y_k] & \text{for } p \text{ odd} \end{cases},$$

where $x_1,\ldots,x_k \in H^1(E_k,\mathbb{F}_p)$, $y_1,\ldots,y_k \in H^2(E_k,\mathbb{F}_p)$ and $\Lambda(x_1,\ldots,x_k)$ denotes the exterior algebra on these one-dimensional generators. Moreover, if we let $B\colon H^1(E_k,\mathbb{F}_p) \to H^2(E_k,\mathbb{F}_p)$ denote the Bockstein, then we can assume that for p odd $B(x_i) = y_i$, whereas for $p = 2$, $B(x_i) = x_i^2$ for all $i = 1,\ldots,k$. By extending coefficients, we obtain the same structure for $H^*(E_k,\mathbb{K}_p)$.

The map B is compatible with respect to the C action and defines an isomorphism onto its image, thus giving rise to an exact sequence

$$0 \to N_k \to H^2(E_k;\mathbb{K}_p) \to \Lambda^2(N_k) \to 0$$

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as $\mathbb{K}_p C$–modules. If $N_k \cong \bigoplus_{1\leq i\leq k} L(\alpha_i)$ then $\Lambda^2(N_k) \cong \bigoplus_{1\leq i<j\leq k} L(\alpha_i\alpha_j)$. Note that if we tensor the sequence with any other character and take C–invariants, it will still be exact, as $(|C|, p) = 1$.

Using the fact that for any $\mathbb{K}_p U_k$–module M, $H^t(U_k; M) \cong H^t(E_k; M_{|E})^C$ for every $t \geq 0$, for any character $L(\beta)$, we obtain the formula

$$h^2(U_k; L(\beta)) - h^1(U_k; L(\beta)) + h^0(U_k; L(\beta)) = \dim[\Lambda^2(N_k) \otimes \beta]^C + \dim L(\beta)^C.$$

At primes q dividing $|C|$, we work over the field $\mathbb{K}_q$, and note that $H^i(U_k, L) = H^i(C, L^{E_k})$. Hence, an absolutely irreducible L with some $h^i(U_k, L) \neq 0$ must also have a trivial action of E_k. Arguing, as before, L is the inflation of a character $C/O_q(C) \to \mathbb{K}_q^\times$. Thus, we have $H^i(U_k, L) = [H^i(O_q(C), \mathbb{K}_q) \otimes L]^{C/O_q(C)}$. As $O_q(C)$ is cyclic, all these terms are isomorphic and of nonzero rank (equal to one) if and only if the action of $C/O_q(C)$ is trivial, and we obtain that

$$h^2(U_k; L) - h^1(U_k; L) + h^0(U_k; L) \leq 1.$$

We apply our analysis to obtain a calculation for $\mu_2(U_k)$ and $e_2(U_k)$:

Proposition 5.4. *For $U_k = E_k \times_T C$ as above, with $N_k = H^1(E_k; \mathbb{K}_p)$,*

$$\mu_2(U_k) = \max\{\dim[\Lambda^2(N_k) \otimes L(\beta)]^C + \dim L(\beta)^C\}$$

as $L(\beta)$ ranges over all characters $\beta\colon C \to \mathbb{K}_p^\times$, and

$$e_2(U_k) = \dim \Lambda^2(N_k)^C - \dim N_k^C + 2.$$

We apply this to the special case when p is an odd prime and the action on $N_k = H^1(E_k, \mathbb{K}_p)$ is *isotypic*, that is it is the direct sum of copies of a fixed character $L(\alpha)$.

Corollary 5.5. *Let $U_k = E_k \times_T C$, where p is odd and the action of C on the vector space E_k gives rise to the sum of k copies of a fixed character $L(\alpha)$ over the splitting field $\mathbb{K}_p$, with $k > 1$.*

(i) *If $\alpha^2 \neq 1$, $e_2(U_k) = 2$, $\mu_2(U_k) = \frac{k(k-1)}{2}$ and $\mu_2(U_k) - \mu_1(U_k) = \frac{k(k-3)}{2}$.*
(ii) *If $\alpha^2 = 1$, $e_2(U_k) = \frac{k(k-1)}{2} + 2$, $\mu_2(U_k) = \frac{k(k-1)}{2} + 1$ and $\mu_2(U_k) - \mu_1(U_k) = \frac{k(k-3)}{2} + 1$.*

Proof. We apply Proposition 5.4 to compute $\mu_2(U_k)$ and $e_2(U_k)$. Choose $\beta = \alpha^{-2}$, then $\Lambda^2(H^1(E_k; \mathbb{K}_p)) \otimes L(\beta)$ is a trivial $\mathbb{K}_p[C]$–module of dimension equal to $\frac{k(k-1)}{2}$. In the special case $\beta = 1$, we obtain the extra term. The calculation for $e_2(U_k)$ follows from its expression in terms of invariants. As U_k is solvable, we have $\mu_1(U_k) = k$, and the proof is complete. □

Corollary 5.6. *Let $U_k = E_k \times_T C$, where p is odd, $E_k = (\mathbb{Z}/p\mathbb{Z})^k$ and C cyclic of order prime to p acts on each $\mathbb{Z}/p\mathbb{Z}$ factor in E_k via $x \mapsto x^q$, where q is a unit in $\mathbb{Z}/p\mathbb{Z}$.*

(i) *If $x^{q^2} \neq x$ for all $1 \neq x \in E_k$, then*

$$\max\{2, \frac{k(k-3)}{2}\} \leq q_4(U_k) \leq k(k-1).$$

(ii) *If $q = p - 1$, then*

$$\frac{k(k-1)}{2} + 2 \leq q_4(U_k) \leq k(k-1) + 2.$$

Remark 5.7. Note that for $\alpha^2 \neq 1$, $e_2(U_k) = 2 < \mu_2(U_k) - \mu_1(U_k)$ for $k \geq 5$. Hence, 5.6 improves on the lower bound given in [19]. On the other hand, if $\alpha^2 = 1$, then $\mu_2(U_k) - \mu_1(U_k) < e_2(U_k)$ for all $k > 1$. This shows that the two invariants play a role in establishing lower bounds for $q_4(G)$.

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We now apply our estimate to homology 4–spheres.

Theorem 5.8. *Let $U_k = E_k \times_T C$, where p is an odd prime, $E_k = (\mathbb{Z}/p\mathbb{Z})^k$ and C cyclic of order prime to p acts on each $\mathbb{Z}/p\mathbb{Z}$ factor in E_k via $x \mapsto x^q$, where q is a unit in $\mathbb{Z}/p\mathbb{Z}$.*

(i) *If $x^{q^2} \neq x$ for all $1 \neq x \in E_k$, then for all $k > 4$, U_k does not arise as the fundamental group of any rational homology 4–sphere.*
(ii) *If $q = p-1$, then for all $k > 1$, U_k does not arise as the fundamental group of any rational homology 4–sphere.*

Proof. For the groups $U_k = E_k \times_T C$, where $x^{q^2} \neq x$ for $x \in E_k$, we consider the inequality

$$\max\{2, \frac{k(k-3)}{2}\} \leq q_4(U_k) \leq k(k-1).$$

If a rational homology 4–sphere X with fundamental group U_k exists, then $q_4(U_k) = 2$ and we'd have $\frac{k(k-3)}{2} \leq 2$, which implies that $k \leq 4$. Note that the upper bound implies the existence of a rational homology 4–sphere with fundamental group U_2. In the case when $q = p-1$, our estimate 5.6 shows that $2 < q_4(U_k)$ for all $k > 1$. □

Example 5.9. Let $\mathbb{F}_q$ denote a field with $q = 2^k$ elements. Then the cyclic group of units $C = \mathbb{Z}/(q-1)\mathbb{Z}$ acts transitively on the nonzero elements of the underlying mod 2 vector space $E_k = (\mathbb{Z}/2\mathbb{Z})^k$. If we write $\mathbb{F}_q = \mathbb{F}_2[u]/(p(u))$, where $p(u)$ is an irreducible polynomial of degree k over $\mathbb{F}_2$, then the action can be described as multiplication by u. Expressing it in terms of the basis $\{1, u, \dots, u^{k-1}\}$, we obtain a faithful representation $C \to GL(k, \mathbb{F}_2)$ with characteristic polynomial $p(t)$. This gives rise to a semidirect product $J_k = E_k \times_T C$, where the action of C on $N_k = H^1(E_k, \mathbb{K}_2)$ decomposes into nontrivial, distinct characters determined by the roots of $p(t)$. If α is a root of this polynomial, so are all the powers $\{\alpha^{2^i}\}_{i=0,\dots,k-1}$, and these appear as a complete set of eigenvalues for the action on the k–dimensional vector space. In other words, we have $N_k \cong \bigoplus_{0\leq i\leq k-1} L(\alpha^{2^i})$. We propose to compute the invariants $\mu_2(J_k)$ and $e_2(J_k)$.

Proposition 5.10. *For the groups J_k described above, we have*

(i) $\mu_2(J_2) = 2$, *whereas* $\mu_2(J_k) = 1$ *for all* $k > 2$.
(ii) $e_2(J_2) = 3$, *whereas* $e_2(J_k) = 2$ *for all* $k > 2$.

Proof. As $N_k \cong \bigoplus_{0\leq i\leq k-1} L(\alpha^{2^i})$, we have that $\Lambda^2(N_k) \cong \bigoplus_{0\leq i<j\leq k-1} L(\alpha^{2^i+2^j})$. For $k > 2$, this is a sum of distinct, nontrivial characters. This follows from the fact that for $k > 2$,

$$2^k - 1 = 2^{k-1} + 2^{k-2} + \cdots + 2 + 1 > 2^i + 2^j,$$

and each $\alpha^{2^i+2^j}$ is a distinct, nontrivial $2^k - 1$ root of unity. Hence, if $k > 2$, the module $\Lambda^2(N_k)$ has **no trivial summands** and **no repeated summands**. Now, if we take any character $L(\beta)$, we see that at most one summand in $\Lambda^2(N_k) \otimes L(\beta)$ can be trivial. And if this occurs, then $\beta \neq 1$. Hence, applying the formula in Proposition 5.4, we conclude that $\mu_2(J_k) = 1$. Similarly, we see that $e_2(J_k) = 2$ for all $k > 2$. Also $\Lambda^2(N_2) \cong L(1)$, whence we see that $\mu_2(J_2) = 2$ and $e_2(J_2) = 3$. □

From these examples, we conclude that there exist rational homology 4–spheres with fundamental group equal to J_k for $k > 2$, and so groups of arbitrarily high rank can occur as such groups, in contrast to the situation for abelian groups appearing in Corollary 5.1. For $J_2 \cong A_4$, the alternating group on four letters, we have $e_2(J_2) = 3$, $\mu_2(J_2) = 2$, and so $3 \leq q_4(A_4) \leq 4$. The cohomological computations also imply that $\mu_4(A_4) = 1$, whence, there does exist a rational homology 8–sphere with fundamental group A_4.

Proposition 5.11. *For the alternating groups $G = A_4$ or $G = A_5$, we have $q_4(G) = 4$.*

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Proof. By the estimates above, for $G = A_4$, we only need to rule out $q_4(G) = 3$, so suppose that there exists M^4 with $\pi_1(M) = A_4$ and $\chi(M) = 3$. Applying the universal coefficient theorem, we can use the computation at $p = 2$ (see [2, Theorem 1.3, Chapter III]) to show that $H_4(A_4;\mathbb{Z}) = 0$. Hence, $\widehat{H}^{-5}(A_4,\mathbb{Z}) = 0$ and applying Proposition 2.5, we infer that $\pi_2(M)$ is stably isomorphic to $J \oplus J^*$, where J denotes a minimal representative of $\Omega^3(\mathbb{Z})$. Since $\chi(M) = 3$, we have $H^0(G;\pi_2(M)) = \mathbb{Z}$. From the exact sequence for Tate cohomology [3, Chapter IV.4], we have a surjection $\mathbb{Z} = H^0(G;\pi_2(M)) \twoheadrightarrow \widehat{H}^0(G;\pi_2(M))$. However,

$$\widehat{H}^0(G;\pi_2(M)) = \widehat{H}^0(G;J \oplus J^*) = \widehat{H}^{-3}(G;\mathbb{Z}) \oplus \widehat{H}^3(G;\mathbb{Z}) \cong H_2(G;\mathbb{Z}) \oplus H_2(G;\mathbb{Z})$$

and since $H_2(G;\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, this is impossible. For $G = A_5$, we apply the fact that for every nonperiodic finite subgroup G of $SO(3)$, $\mu_2(G) = 2$ (see Remark 6.3). The rest of the argument is analogous to that for A_4, since the restriction map $H^*(A_5;\mathbb{Z}/2\mathbb{Z}) \to H^*(A_4;\mathbb{Z}/2\mathbb{Z})$ is an isomorphism. This is true because both groups share the same 2–Sylow subgroup $(\mathbb{Z}/2\mathbb{Z})^2$, with normaliser A_4 (see [2, Theorem 6.8, Chapter II]). This implies that $e_2(A_5) = 3$, $H_4(A_5;\mathbb{Z}) = 0$ and $H_2(A_5;\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ (note that the other two p–Sylow subgroups are cyclic, so don't contribute to even degree homology). Therefore, we can rule out $q_4(A_5) = 3$ whence $q_4(A_5) = 4$. □

6. Some further remarks and questions in dimension four

In this section, we will briefly discuss some questions about rational homology 4-spheres whose fundamental groups are finite.

Section 6A. Existence via surgery. The main open problem is to characterise the finite groups G for which $q_4(G) = 2$. To make progress, we need more constructions of rational homology 4-spheres.

Examples of $\mathbb{Q}S^4$-manifolds can be constructed by starting with a rational homology 3-sphere X, forming the product $X \times S^1$, and then doing surgery on an embedded $S^1 \times D^3 \subset X \times S^1$ representing a generator of $\pi_1(S^1) = \mathbb{Z}$. This construction is equivalent to the 'thickened double' construction $Z = M(K)$ for a finite 2-complex of Proposition 3.1 (compare [13, Section 4]).

Example 6.1. The groups $G = \mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}/p\mathbb{Z}$ are $\mathbb{Q}S^4$-groups, since we can do surgery on an embedded circle $L^3(p,1) \times S^1$ representing p-times a generator of $\pi_1(S^1) = \mathbb{Z}$. These examples are *not* of the thickened double form $M(K)$ because the minimal rank of $\pi_2(K)$ representing $\Omega^3(\mathbb{Z})$ is greater than $|G| - 1$, and hence, the extension describing $\pi_2(M)$ is nontrivial (by Proposition 2.3).

Since the quotient of a free finite group action on a rational homology 3-sphere is again a rational homology 3-sphere, one could use the examples $X = Y/G$ studied by [1], where Y is a $\mathbb{Q}S^3$ and G is a finite group acting freely on Y. However, to obtain a $\mathbb{Q}S^4$ with finite fundamental group by this construction, Y must, itself, have finite fundamental group.

Remark 6.2. The finite fundamental groups of closed, oriented 3-manifolds have periodic cohomology of period 4, but not all 4-periodic groups arise this way. A complete list of 4-periodic groups is given in Milnor [32, Section 3], and those which can act freely and orthogonally on S^3 were listed by Hopf [20]. Perelman [26] showed that the remaining groups in Milnor's list do not arise as the fundamental group of any closed, oriented 3-manifold, and that the closed 3-manifolds with finite fundamental group are exactly the 3-dimensional spherical space forms.

Remark 6.3. For every nonperiodic finite subgroup G of $SO(3)$, we have $\mu_2(G) = 2$, and hence, $q_4(G) \leq 4$ (see [15, Proposition 2.4]). Note that each such subgroup has a 2-fold central extension $G^* \subset SU(2)$ which acts freely on S^3, and let $X = S^3/G^*$ denote the quotient 3-manifold. On $N := X\times S^1$, we can do surgery on disjoint circles representing (i) a generator of the central subgroup of G^* and (ii) a generator of $\mathbb{Z}$, to reduce the fundamental group from $\pi_1(N) = G^* \times \mathbb{Z}$ to G. We thus obtain a 4-manifold M with $\chi(M) = 4$ and $\pi_1(M) = G$, realising the upper bound for $q_4(G)$. Our estimates give $2 \leq q_4(G) \leq 4$ for the cases not yet determined, namely, where G is dihedral of order $4n$ or G is the symmetric group S_4.

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Remark 6.4. Teichner [43, 3.7] indicated that topological surgery could produce examples with finite fundamental group from certain 4-manifolds with infinite fundamental group. This technique should be investigated further.

Section 6B. Groups of deficiency zero. There are many finite groups with deficiency zero: for example, Wamsley [51] showed that a metacyclic group G with $H_2(G;\mathbb{Z}) = 0$ has $\mathrm{Def}(G) = 0$. In particular, the class of finite groups arising as fundamental groups of rational homology 4–spheres includes groups with periodic cohomology of arbitrarily high period. There is an extensive literature on this problem: for example, see [4, 5, 6, 8, 22, 30, 33, 37, 38, 39].

According to Swan, $1 \leq \mu_2(G) \leq 1 - \mathrm{Def}(G)$ (see [42, Proposition 1, Corollary 1.3]), hence, if G is a finite group of deficiency zero, we have $\mu_2(G) = 1$. Thus, for such groups by Proposition 3.1, we can construct an orientable 4–manifold M with $\pi_1(M) = G$ and $\chi(M) = 2$. More generally, this can be done whenever $\mu_2(G) = 1$ by Theorem B (see Corollary 3.9 and the series of groups J_k considered in Example 5.9). Then M is a rational homology 4–sphere, and in these cases, there is a minimal representative J for the stable module $\Omega^3(\mathbb{Z})$ with $\mathrm{rank}_{\mathbb{Z}}(J) = |G| - 1$ (compare [13, Corollary 4.4]). For example, if G is the fundamental group of a closed, oriented 3-manifold, then $J \cong I(G)^*$.

Remark 6.5. We are indebted to Mike Newman and Özgün Ünlü for showing that some of the groups J_k do have deficiency zero (e.g. at least for $3 \leq k \leq 6$). It is a challenging, open problem to decide whether this is true for all $k \geq 3$. Note that any group in this range which does not admit a balanced presentation would give a negative answer to Wall's D2 problem.

Example 6.6. Teichner [43, 3.4, 4.15] proved that if G is a finite $\mathbb{Q}S^4$-group, then $d(H_1(G)) \leq 7$, and used a mapping torus construction to produce a nonabelian $\mathbb{Q}S^4$-group G with $d(H_1(G)) = 4$.

Section 6C. Algebraic questions. For the rational homology 4-spheres M with $\pi_1(M) = G$ constructed in Theorem B, we have $\pi_2(M) = H_2(\widetilde{M};\mathbb{Z}) = J \oplus J^*$, where J is a minimal representative for $\Omega^3(\mathbb{Z})$ over $\mathbb{Z}G$, with $\mathrm{rank}_{\mathbb{Z}}(J) = |G| - 1$.

Moreover, J is locally, and hence, rationally isomorphic to the augmentation ideal $I(G)$, and the equivariant intersection form s_M on $\pi_2(M) = J \oplus J^*$ is metabolic, with totally isotropic submodule $0 \oplus J^*$. Similar results hold for the higher-dimensional examples constructed in Proposition 3.1.

More generally, for any finite group G, the existence of a representative J for the stable module $\Omega^3(\mathbb{Z})$ with $\mathrm{rank}_{\mathbb{Z}}(J) = |G| - 1$ is equivalent to the condition $\mu_2(G) = 1$ (see Proposition 3.7).

Question. Is there a finite group G with $\mu_2(G) = 1$, such that G is neither periodic nor admits a balanced presentation?

For any closed, oriented 4-manifold M with finite fundamental group G, we have seen in 2.3 that $\pi_2(M)$ is *stably* given by an extension of $\Omega^{-3}(\mathbb{Z})$ by $\Omega^3(\mathbb{Z})$ (see also [13, Proposition 2.4]) and that the extension class in $\mathrm{Ext}^1_{\mathbb{Z}G}(\Omega^{-3}(\mathbb{Z}), \Omega^3(\mathbb{Z})) \cong H_4(G;\mathbb{Z})$ is given by the image of the fundamental class of M. For any rational homology 4-sphere M with finite fundamental group G, the condition $\chi(M) = 2$ implies that $\mathrm{rank}_{\mathbb{Z}}(\pi_2(M)) = 2(|G| - 1)$ and $H^0(G;\pi_2(M)) = 0$.

Question. If M is a $\mathbb{Q}S^4$, what is the (unstable) structure of $\pi_2(M)$ as an integral representation? Is the equivariant intersection form s_M always metabolic (in the sense defined in [14, Section 2])?

Finally, we point out that many questions in the representation theory of finite groups can be investigated by induction and restriction to proper subgroups. At present, we do not see how to apply this technique in our setting.

Question. If M is a $\mathbb{Q}S^4$-manifold with finite fundamental group G, then its nontrivial finite coverings have Euler characteristic > 2 (and, hence, are not $\mathbb{Q}S^4$-manifolds). How can we decide if proper subgroups of G are also $\mathbb{Q}S^4$-groups?

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7. Appendix A: The Proof of Theorem 3.8

In this section, we give a direct construction of the minimal 4-manifold needed for Theorem 3.8. The idea is to use a handlebody thickening (see Definition 7.4) of a finite 2-complex K instead of starting with an embedding of K in $\mathbb{R}^5$. The advantage of this thickening is that we can identify the intersection form of its 4-manifold boundary, and then apply a recent refinement of Freedman's work due to Teichner, Powell and Ray (see [35, Corollary 1.4]).

Section 7A. Metabolic forms. To analyse the intersection form of the handlebody thickening, we will need some algebraic preparations.

Definition 7.1. Let $(E, [q])$ denote a quadratic metabolic form on a Λ-module $E = N \oplus N^*$, where N is a left Λ-module and N^* inherits a left Λ-module structure via the standard anti-involution $a \mapsto \bar{a}$ on $\Lambda = \mathbb{Z}G$. Then

$$q((x, \phi), (x', \phi')) = \phi(x') + g(\phi, \phi'),$$

where $x, x' \in N$, $\phi, \phi' \in N^*$ and $g \in \operatorname{Hom}(N^* \otimes N^*, \Lambda)$ is a sesquilinear form. We use the notation $(E, [q]) = \operatorname{Met}(N, g)$ for this metabolic form (see [14, Section 2] for metabolic forms defined on a nonsplit extension of N and N^*).

The associated Hermitian form $h = q + q^*$ is nonsingular, and $N \oplus 0 \subset E$ is a totally isotropic direct summand. More explicitly,

$$h((x, \phi), (x', \phi')) = \phi(x') + \overline{\phi'(x)} + g(\phi, \phi') + \overline{g(\phi', \phi)}.$$

In our geometric setting, the metabolic forms arise on modules $E = H^2(K) \oplus H_2(K)$, where K is a finite 2-complex with fundamental group G (take coefficients in $\Lambda = \mathbb{Z}G$). If G is finite, then $H^2(K; \Lambda) \cong \operatorname{Hom}_\Lambda(H_2(K), \Lambda)$, and the definition above applies. If G is infinite, then we slightly generalise our notion of metabolic form.

Definition 7.2. Let $E = N \oplus \widehat{N}$, and let $\alpha\colon \widehat{N} \to N^*$ be a Λ-module homomorphism. Define a *generalised* metabolic form $(E, [q]) := \operatorname{Met}(N, \widehat{N}, \alpha, g)$ by the formula

$$q((x, \phi), (x', \phi')) = \alpha(\phi)(x') + g(\phi, \phi'),$$

where $x, x' \in N$, $\phi, \phi' \in \widehat{N}$, and $g \in \operatorname{Hom}(\widehat{N} \otimes \widehat{N}, \Lambda)$ is a given sesquilinear form.

Example 7.3. For a finite 2-complex K, we have the evaluation map $\alpha\colon H^2(K) \to \operatorname{Hom}_\Lambda(H_2(K), \Lambda)$, which in general is neither injective nor surjective. In this case, we will shorten the notation of Definition 7.2 to $(E, [q]) = \operatorname{Met}(H_2(K), g)$, where $E = H_2(K) \oplus H^2(K)$ as above.

Here are some preliminary remarks.

- Let $(E, [q])$ be any quadratic form, and suppose that U is a finitely generated submodule on which the restriction λ_0 of $\lambda = q + q*$ to U is nonsingular. Then there is is orthogonal splitting $(E, [q]) \cong U \perp L$.

  Proof. Consider the following sequence

$$0 \to U \to E \xrightarrow{\operatorname{ad}\lambda} E^* \to U^* \to 0,$$

  where the composition $\operatorname{ad}\lambda_0\colon U \to U^*$ is an isomorphism by assumption. Therefore, the inclusion $U \subset E$ is a split injection, and $E = U \perp L$, where $L := U^\perp$. To check this last point, note that a splitting map for the inclusion $i\colon U \to E$ is given by

$$r := (\operatorname{ad}\lambda_0)^{-1} \circ i^* \circ \operatorname{ad}\lambda \circ i.$$

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For $e \in E$, we compute

$$\lambda(e - i(r(e)), i(h)) = \lambda(e, i(h)) - \lambda(i(r(e)), i(h)) = \lambda(e, i(h)) - \lambda_0(r(e), h) = 0$$

after substituting the formula for r. Therefore, $E = U + U^\perp$, and $U \cap U^\perp = 0$ since λ_0 is nonsingular. □

- Let $(E, [q]) = \mathrm{Met}(H, g)$ be a metabolic quadratic form on $E = H \oplus H^*$, where $H = \Lambda^r$ is a finitely generated free Λ-module. Then $(E, [q]) \cong H(\Lambda^r)$ is a hyperbolic form.

  Proof. This is a standard fact (see [50, Lemma 5.3]). □

Section 7B. A handlebody thickening. Let K be a finite 2-complex with $\pi_1(K) = G$. We construct a suitable thickening of K to be used in the proof of Theorem 3.8.

Definition 7.4. We first consider a 4-dimensional parallelisable thickening $A(K)$ of K constructed by attaching suitable 2-handles to a connected sum $\sharp\, \ell(S^1 \times D^3)$. Then $A(K)$ is a compact 4-manifold with boundary, and we let $N(K) = A(K) \times I$. Note that $N(K)$ is a 5-dimensional thickening of K but may not embed in $\mathbb{R}^5$, and that $\partial N(K) = A(K) \cup -A(K)$ is the double of A along the common boundary.

Then $M := \partial N(K)$ has the intersection form $\lambda_M = \mathrm{Met}(H_2(K), g)$, since $H_2(\partial N(K)) = H^2(K) \oplus H_2(K)$ and the direct summand $H^2(K)$ is totally isotropic (compare [24, Section 2]). All the homology groups have coefficients in $\Lambda := \mathbb{Z}G$.

Remark 7.5. Note that the quadratic intersection form $\mathrm{Met}(H_2(K), g)$ is a *generalised*metabolic form (see Example 7.3). It is nonsingular if $\pi_1(K) = G$ is a finite group. If G is infinite, this form has radical $H^2(G; \Lambda)$, and the cokernel of its adjoint is $H^3(G; \Lambda)$ by the exact sequence

$$0 \to H^2(G;\Lambda) \to H^2(M;\Lambda) \xrightarrow{\mathrm{ad}\,\lambda_M} \mathrm{Hom}_\Lambda(H_2(M), \Lambda) \to H^3(G;\Lambda) \to 0$$

arising from the universal coefficient theorem.

Section 7C. A self-homotopy equivalence. Let $N(K)_r := N(K) \natural\, r(S^2 \times D^3)$ denote this new thickening of $K \vee r(S^2)$. We recall the construction of a useful homotopy self-equivalence of $K \vee r(S^2)$.

Lemma 7.6 [12, Lemma 2.1]. *Let X be a finite* D2*-complex, and let $u\colon K \subset X$ denote the* 2*-skeleton of X. Then, for $r = b_3(X)$, there is a simple self-homotopy equivalence $h\colon K \vee r(S^2) \to K \vee r(S^2)$ inducing a simple homotopy equivalence $f\colon X \vee r(S^2) \simeq K$.*

Proof. We recall some of the notation from [12, Section 2]. There is an identification

$$\pi_2(K \vee r(S^2)) \cong \pi_2(K) \oplus \Lambda^r \cong \pi_2(X) \oplus C_3(X) \oplus F, \tag{7.1}$$

and we fix free Λ-bases $\{e_1, \ldots, e_r\}$ for $C_3(X) \cong \Lambda^r$ and $\{f_1, \ldots, f_r\}$ for $F \cong \Lambda^r$. The same notation $\{e_i\}$ and $\{f_j\}$ will also be used for continuous maps $S^2 \to K \vee r(S^2)$ in the homotopy classes of $\pi_2(K \vee r(S^2))$ defined by these basis elements. Notice that the maps $f_j\colon S^2 \to K \vee r(S^2)$ may be chosen to represent the inclusions of the S^2 wedge factors.

An examination of the proof of [12, Lemma 2.1] shows that the simple homotopy equivalence $f\colon X \vee r(S^2) \simeq K$ is obtained by extending a certain simple homotopy equivalence $h\colon K \vee r(S^2) \to K \vee r(S^2)$ over the (stabilised) inclusion

$$\begin{tikzcd} X \vee r(S^2) \arrow[r, dashed, "h'"] & K' \\ K \vee r(S^2) \arrow[u, hook, "u\vee\mathrm{id}"] \arrow[r, "h"] & K \vee r(S^2) \arrow[u, hook] \end{tikzcd}$$

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by attaching the 3-cells of X in domain by the maps $e_i = [\partial D^3_i]$, $1 \le i \le r$, and 3-cells in the range via the maps $f_i = [\partial D^3_i]$, $1 \le i \le r$, which homotopically cancel the S^2 wedge factors, to obtain a complex $K' \simeq K$.

Then we have $h \circ [\partial D^3_i] = f_i$ by the construction of h (see [12, p. 364]). Hence, we can extend h over X by the identity on the 3-cells attached in domain and range along the maps $\{f_i : S^2 \to K \vee r(S^2)\}$. We obtain a map

$$h' : X \vee r(S^2) \to K' := K \vee r(S^2) \cup \bigcup \{D^3_i : [\partial D^3_i] = f_i, 1 \le i \le r\}$$

extending h. From the construction of the map h (see [12, p. 364]), it follows that h' is a (simple) homotopy equivalence, which induces a simple homotopy equivalence $f : X \vee r(S^2) \simeq K$, after composition with the obvious projection $K' \to K$. □

Section 7D. Topological surgery. We will now apply some results of topological surgery due to Freedman. Recall that $N(K)$ is the 5-dimensional thickening of K constructed above and $N(K)_r = N(K) \natural r(S^2 \times D^3)$ is its stabilisation. We have introduced the notation $M = \partial N(K)$, and let $M_r = \partial N(K)_r = \partial N(K) \# r(S^2 \times S^2)$.

Lemma 7.7. *Suppose that $\pi_1(K)$ is a good group. There is a self-homeomorphism $\beta : \partial N(K)_r \approx \partial N(K)_r$ extending the simple homotopy self-equivalence $h : K \vee r(S^2) \to K \vee r(S^2)$.*

Proof. Since $\pi_1(K)$ is a good group, the topological s-cobordism theorem [9, Theorem 7.1A] implies that the given simple homotopy self-equivalence $h : K \vee r(S^2) \to K \vee r(S^2)$ extends to a self-homeomorphism $\hat{h} : N(K)_r \to N(K)_r$. This follows since we may assume (by general position) that the image $h(K \vee r(S^2)) \subset N(K))_r$ is embedded in the interior of the 5-manifold $N(K))_r$. Since h is a simple homotopy self-equivalence, the complement of a small tubular neighbourhood of $h(K \vee r(S^2))$ will then be an s-cobordism, and hence, a product. Since $N(K))_r$ is a thickening of $K \vee r(S^2)$, we can construct the self-homeomorphism $\hat{h}$ by identifying the tubular neighbourhoods in domain and range, and then using the product structures. Let $\beta := \partial\hat{h}$ denote the restriction of $\hat{h}$ to $\partial N(K)_r$. □

We now combine these ingredients. Recall that $X \vee r(S^2) \simeq K$, so that $H_2(K) \cong H_2(X) \oplus H$, where $H \cong \Lambda^r$. We have the isomorphism

$$H_2(N(K)_r) = H_2(K \vee r(S^2)) \cong H_2(K) \oplus F \cong H_2(X) \oplus H \oplus F, \tag{7.2}$$

where $F \cong \Lambda^r$. We fix free Λ-bases $\{e_1, \ldots, e_r\}$ for $H \cong \Lambda^r$, and $\{f_1, \ldots, f_r\}$ for $F \cong \Lambda^r$. It follows that $M_r := \partial N(K)_r$ has intersection form

$$\lambda_{M_r} = \lambda_M \oplus H(F) = \mathrm{Met}(H_2(K), g) \oplus H(\Lambda^r),$$

where the classes $\{f_1, f_2, \ldots, f_r\}$ and their duals provide a standard hyperbolic base for the second summand $H(F)$. By construction, $h_*(e_i) = f_i$, $h_*(f_i) = e_i$ for $1 \le i \le r$, and $h_*(x) = x$ for all $x \in H_2(X)$. Note that $H^2(K) = H^2(X) \oplus H^*$ is totally isotropic under λ_M, and orthogonal to, the summand $H(F)$.

Lemma 7.8. *There is a closed, topological 4-manifold M_0 and a homeomorphism $M_r = \partial N(K)_r \approx M_0 \sharp 2r(S^2 \times S^2)$, such that $\chi(M_0) = 2\chi(X)$.*

Proof. We have the decomposition:

$$H_2(M_r) = H^2(X) \oplus H_2(X) \oplus H \oplus H^* \oplus F \oplus F^*$$

in the notation introduced in (7.2).

The metabolic intersection form $\lambda_{M_r} = \lambda_M \oplus H(F)$ admits a self-isometry β_* (induced from the map β constructed in Lemma 7.7) extending the map $h_* : H_2(K) \oplus \Lambda^r \to H_2(K) \oplus \Lambda^r$ constructed above.

Since the images of the basis elements $h_*(e_i) = f_i \in F$ have dual classes $f_i^* \in F^*$, it follows that

$$\lambda_{M_r}(\beta_*(f_i^*), e_j) = \lambda_{M_r}(\beta_*(f_i^*), \beta_*(f_j)) = \lambda_{M_r}(f_i^*, f_j) = \delta_{ij}. \tag{7.3}$$

Similarly, we have the formulas

$$\lambda_{M_r}(\beta_*(f_i^*), \beta_*(f_j^*)) = \lambda_{M_r}(f_i^*, f_j^*) = 0, \quad \text{for all } \ 1 \le i, j \le r, \tag{7.4}$$

and

$$\lambda_{M_r}(e_i, e_j) = \lambda_{M_r}(\beta_*(f_i), \beta_*(f_j)) = \lambda_{M_r}(f_i, f_j) = 0, \quad \text{for all } \ 1 \le i, j \le r. \tag{7.5}$$

Let $U = \langle \beta_*(F^*); H; F \oplus F^* \rangle$ denote the submodule of $H_2(M_r)$ generated by $H(F) = F \oplus F^*$, together with the classes $\{\beta_*(f_i^*)\}$, and the classes $\{e_i\}$, for $1 \le i \le r$. Then we claim that $U \cong \beta_*(F^*) \oplus H \oplus H(F)$ is a *free direct summand* of $H_2(M_r)$, with indicated basis elements, on which the restriction of λ_{M_r} is *a nonsingular form*.

We check that $U \cong \beta_*(F^*) \oplus H \oplus F \oplus F^*$ is a free submodule (of rank $4r$) in $H_2(M_r)$ by first showing that

$$\beta_*(F^*) \cap (H \oplus F \oplus F^*) = 0.$$

We then observe that the restriction λ_U of the intersection form to $U \subset H_2(M_r)$ is nonsingular. It follows that (U, λ_U) is an orthogonal direct summand of $(H_2(M_r), \lambda_{M_r})$.

Here are the details: suppose that $u \in \beta_*(F^*) \cap (H \oplus F \oplus F^*)$. We can express

$$u = \sum a_i \beta_*(f_i^*) = \sum b_i e_i + \sum c_i f_i + \sum d_i f_i^*$$

as Λ-linear combinations of the basis elements. Then by the formula (7.3) above, we have $\lambda_{M_r}(\beta_*(f_i^*), e_j) = \delta_{ij}$, and hence, $\lambda_{M_r}(u, e_i) = a_i$. Since H is totally isotropic by (7.5), the summand $H \oplus F \oplus F^*$ is orthogonal to H, and it follows that $\lambda_{M_r}(u, e_i) = 0$. Hence, all the a_i are zero and $u = 0$.

Now let λ_U denote the restriction of λ_{M_r} to U. The submodule $H \oplus F$ is a totally isotropic-based free direct summand of rank $2r$ in U, and the dual basis elements under λ_U form the basis of the complementary direct summand $\beta_*(F^*) \oplus F^*$. Hence, λ_U is nonsingular, and in fact, $\lambda_U \cong \mathrm{Met}(H \oplus F, g)$, where g encodes the intersections of $\beta_*(F^*)$ with F^* (which may be nonzero). In this situation, it follows that $\lambda_U \cong H(\Lambda^{2r})$ is isomorphic to a nonsingular hyperbolic form (see [50, Lemma 5.3]).

Hence, there is a splitting for the intersection form

$$\lambda_{M_r} = \mathrm{Met}(H_2(K), g) \perp H(F) \cong (E, \lambda_0) \perp \lambda_U$$

with respect to the orthogonal complement $(E, \lambda_0) = (\lambda_U)^\perp$. Since M has good fundamental group and λ_{M_r} contains the hyperbolic subform

$$\lambda_U \cong H(\Lambda^{2r}) \cong H(\Lambda^r) \perp H(F),$$

topological surgery [35, Corollary 1.4] shows that $M \approx M_0 \sharp 2r(S^2 \times S^2)$. The resulting closed, topological 4-manifold M_0 has $\chi(M_0) = 2\chi(X)$. □

The construction of the manifold $M(X) := M_0$ completes the proof of Theorem 3.8.

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