# SOME REMARKS ON LOCAL RINGS

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Previously C. Chevalley [1] proved the followings:

1. Let $x_1, \ldots, x_n$ be algebraically independent elements over a field $\mathfrak{k}$ which has infinitely many elements. Then:

a) If $y$ is an element of $\mathfrak{k}[x_1, \ldots, x_n]$ and if $y$ is not in $\mathfrak{k}$, then there exist elements $y_2, \ldots, y_n$ of $\mathfrak{k}[x_1, \ldots, x_n]$ such that $\mathfrak{k}[x_1, \ldots, x_n]$ is integral over $\mathfrak{k}[y, y_2, \ldots, y_n]$.

b) If $\mathfrak{p}$ is a prime ideal of $\mathfrak{k}[x_1, \ldots, x_n]$, then there exist elements $y_1, \ldots, y_n$ of $\mathfrak{k}[x_1, \ldots, x_n]$ such that i) $\mathfrak{k}[x_1, \ldots, x_n]$ is integral over $\mathfrak{k}[y_1, \ldots, y_n]$ and ii) $\mathfrak{p} \cap \mathfrak{k}[y_1, \ldots, y_n] = (y_{m+1}, \ldots, y_n)\mathfrak{k}[y_1, \ldots, y_n]$ (with some $m \leqq n$).

2. Any geometric local ring contains no nilpotent element; more generally, if $\mathfrak{o}$ is a local ring which admits a nucleous and if $\mathfrak{o}$ contains no nilpotent element then the completion of $\mathfrak{o}$ contains no nilpotent element.

Further, O. Zariski [5] proved the following:

3. Let $P$ be a point of an irreducible algebraic variety $V$ and let $\mathfrak{o}$ be the local ring of $P$ on $V$. If $V$ is locally normal at $P$, that is, if $\mathfrak{o}$ is integrally closed, then the completion of $\mathfrak{o}$ is also an integrally closed integrity domain.

On the other hand, P. Samuel [3] stated the following, but his proof contained a falsy argument:[1)]

4. Let $\mathfrak{o}$ be a local ring and let $\mathfrak{o}^*$ be its completion. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of $\mathfrak{o}$ then $(\mathfrak{a} \cap \mathfrak{b})\mathfrak{o}^* = \mathfrak{a}\mathfrak{o}^* \cap \mathfrak{b}\mathfrak{o}^*$.

In the present note, we first give a corrected proof of 4 (for semi-local rings) (§1). In §2 we prove a refinement of 1 dealing with finite ground field too (Theorems 2 and 3). §3 gives a generalization of 2; we define a generalized notion of geometric local rings and that of nuclei and we prove 2 in our generalized sense. In §4, 3 is proved also for geometric local rings in Chevalley's sense.

Received March 13, 1953.

1) Samuel [3] made use of the following lemma: Let $\mathfrak{o}$ be a local ring with maximal ideal $\mathfrak{m}$. If $\mathfrak{b}$ is an ideal of $\mathfrak{o}$ and if $a$ is an element of $\mathfrak{o}$, then $(\mathfrak{b}, \mathfrak{m}^n) : a\mathfrak{o} \subseteqq ((\mathfrak{b} : a\mathfrak{o}), \mathfrak{m}^{s(n)})$ with $s(n)$ which increases infinitely with $n$.

He applied this lemma for a sequence $\{a_n\}$ ($\lim a_n = a$). These $s(n)$ are distinct for distinct $a_m$'s. $s(n)$ must be denoted by, say, $s(n, m)$ for $a_m$. Then the lemma asserts only that $s(n, m)$ increases infinitely with $n$ for a fixed $m$, and we see easily that $s(n, n)$ may not increase. But his proof needs that $s(n, n)$ increases infinitely with $n$.



https://doi.org/10.1017/S0027763000016974 Published online by Cambridge University Press

§ 1. LEMMA 1. Let $\mathfrak{a}$ be an ideal of a semi-local ring $\mathfrak{o}$ and let $b$ an element of $\mathfrak{o}$. If $\mathfrak{o}^*$ denotes the completion of $\mathfrak{o}$, then $\mathfrak{a}\mathfrak{o}^* : b\mathfrak{o}^* = (\mathfrak{a} : b\mathfrak{o})\mathfrak{o}^*$. (Zariski [4])

*Proof.* Since it is evident that $\mathfrak{a}\mathfrak{o}^* : b\mathfrak{o}^*$ contains $(\mathfrak{a} : b\mathfrak{o})\mathfrak{o}^*$, we have only to prove $\mathfrak{a}\mathfrak{o}^* : b\mathfrak{o}^* \subseteqq (\mathfrak{a} : b\mathfrak{o})\mathfrak{o}^*$. Let $u$ be an element of $\mathfrak{a}\mathfrak{o}^* : b\mathfrak{o}^*$. We take $u_i \in \mathfrak{o}$ such that $u_i \equiv u \pmod{\mathfrak{m}^i\mathfrak{o}^*}$ $(i = 1, 2, \ldots)$, where $\mathfrak{m}$ denotes the intersection of all maximal ideals of $\mathfrak{o}$. Then $u_i b \in (ub, b\mathfrak{m}^i\mathfrak{o}^*)$ and therefore $u_i b \in (\mathfrak{a}\mathfrak{o}^*, b\mathfrak{m}^i\mathfrak{o}^*) = (\mathfrak{a}, b\mathfrak{m}^i)\mathfrak{o}^*$. Therefore $u_i b \in (\mathfrak{a}, b\mathfrak{m}^i)$, which shows $u_i \in ((\mathfrak{a} : b\mathfrak{o}), \mathfrak{m}^i)$.

LEMMA 2. Let $\mathfrak{a}$ be an ideal of a commutative ring $\mathfrak{o}$ and let $b$ be an element of $\mathfrak{o}$. Then $\mathfrak{a} \cap b\mathfrak{o} = b(\mathfrak{a} : b\mathfrak{o})$.

Proof is easy.

Now we prove

THEOREM 1. *Let $\mathfrak{o}^*$ be the completion of a semi-local ring $\mathfrak{o}$ and let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be ideals of $\mathfrak{o}$. Then $(\mathfrak{a}_1 \cap \ldots \cap \mathfrak{a}_n)\mathfrak{o}^* = \mathfrak{a}_1\mathfrak{o}^* \cap \ldots \cap \mathfrak{a}_n\mathfrak{o}^*$.*

*Proof.* It is sufficient to treat the case $n = 2$: $(\mathfrak{a}_1 \cap \mathfrak{a}_2)\mathfrak{o}^* = \mathfrak{a}_1\mathfrak{o}^* \cap \mathfrak{a}_2\mathfrak{o}^*$.

1) When $\mathfrak{a}_2 = (b, \mathfrak{a}_1 \cap \mathfrak{a}_2)$: We may assume that $\mathfrak{a}_1 \cap \mathfrak{a}_2 = (0)$. Then $\mathfrak{a}_2$ is principal: $\mathfrak{a}_2 = b\mathfrak{o}$. By Lemma 2 we have $\mathfrak{a}_1 \cap \mathfrak{a}_2 = b(\mathfrak{a}_1 : b\mathfrak{o})$, $\mathfrak{a}_1\mathfrak{o}^* \cap \mathfrak{a}_2\mathfrak{o}^* = b(\mathfrak{a}_1\mathfrak{o}^* : b\mathfrak{o}^*)$. Now by Lemma 1 we see that $(\mathfrak{a}_1 \cap \mathfrak{a}_2)\mathfrak{o}^* = \mathfrak{a}_1\mathfrak{o}^* \cap \mathfrak{a}_2\mathfrak{o}^*$.

2) The above case being settled, we consider the general case: Take $b_1, \ldots, b_r$ from $\mathfrak{a}_2$ so that $\mathfrak{a}_2 = (b_1, \ldots, b_r, \mathfrak{a}_1 \cap \mathfrak{a}_2)$. We prove our assertion by induction on $r$. Set $\mathfrak{b} = (b_r, \mathfrak{a}_1)$. Then $\mathfrak{a}_1 \cap \mathfrak{a}_2 = \mathfrak{a}_1 \cap \mathfrak{b} \cap \mathfrak{a}_2$, $\mathfrak{b} \cap \mathfrak{a}_2 = (b_r, \mathfrak{a}_1 \cap \mathfrak{a}_2) = (b_r, \mathfrak{a}_1 \cap \mathfrak{b} \cap \mathfrak{a}_2)$ and $\mathfrak{a}_2 = (b_1, \ldots, b_{r-1}, \mathfrak{b} \cap \mathfrak{a}_2)$. Therefore $(\mathfrak{a}_1 \cap \mathfrak{a}_2)\mathfrak{o}^* =$

$(\mathfrak{a}_1 \cap (\mathfrak{b} \cap \mathfrak{a}_2))\mathfrak{o}^* = \mathfrak{a}_1\mathfrak{o}^* \cap (\mathfrak{b} \cap \mathfrak{a}_2)\mathfrak{o}^*$ by 1).

$(\mathfrak{b} \cap \mathfrak{a}_2)\mathfrak{o}^* = \mathfrak{b}\mathfrak{o}^* \cap \mathfrak{a}_2\mathfrak{o}^*$ by our induction assumption.

Thus $(\mathfrak{a}_1 \cap \mathfrak{a}_2)\mathfrak{o}^* = \mathfrak{a}_1\mathfrak{o}^* \cap \mathfrak{b}\mathfrak{o}^* \cap \mathfrak{a}_2\mathfrak{o}^* = \mathfrak{a}_1\mathfrak{o}^* \cap \mathfrak{a}_2\mathfrak{o}^*$.

COROLLARY. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be ideals of a semi-local ring $\mathfrak{o}$ such that $\mathfrak{a}_1 \cap \ldots \cap \mathfrak{a}_n = (0)$. Then $\mathfrak{o}$ is a closed subspace of the direct sum of semi-local rings $\mathfrak{o}/\mathfrak{a}_1, \ldots, \mathfrak{o}/\mathfrak{a}_n$.

For this, cf. Nagata [2], Theorems 2 and 3.

§ 2. THEOREM 2. *Let $x_1, \ldots, x_n$ be algebraically independent elements over a field $\mathfrak{k}$. If an element $y$ of $\mathfrak{k}[x_1, \ldots, x_n]$, which is not in $\mathfrak{k}$, is given, we can choose elements $y_2, \ldots, y_n$ of $\mathfrak{k}[x_1, \ldots, x_n]$ so that $\mathfrak{k}[x_1, \ldots, x_n]$ is integral over $\mathfrak{k}[y, y_2, \ldots, y_n]$.*

*Proof.* Let $M_i = x_1^{\alpha_{i,1}} \cdots x_n^{\alpha_{i,n}}$ $(i = 1, \ldots, N)$ be monomials which occur in the polynomial $y : y = \sum_{i=1}^{N} a_i M_i$ $(a_i \in \mathfrak{k},\ a_i \neq 0)$. Then we can find non-negative integers $m_2, \ldots, m_n$ so that there exists one $i$, say 1, such that

$$\alpha_{1,1} + m_2\alpha_{1,2} + \ldots + m_n\alpha_{1,n} > \alpha_{i,1} + m_2\alpha_{i,2} + \ldots + m_n\alpha_{i,n} \qquad (2 \leqq i \leqq N).$$

Set $y_i = x_i + x_1^{m_i}$ $(2 \leqq i \leqq N)$. Then evidently $\mathfrak{k}[x_1, \ldots, x_n] = \mathfrak{k}[x_1, y_2, \ldots, y_n]$,

https://doi.org/10.1017/S0027763000016974 Published online by Cambridge University Press

and $x_1$ is integral over $\mathfrak{k}[y, y_2, \ldots, y_n]$ as follows readily from our construction on $m_i$. Therefore $\mathfrak{k}[x_1, \ldots, x_n]$ is integral over $\mathfrak{k}[y, y_2, \ldots, y_n]$.

THEOREM 3. *If* $\mathfrak{a}$ *is an ideal of the polynomial ring* $\mathfrak{k}[x_1, \ldots, x_n]$ *(in Theorem* 2*), then there exist elements* $y_1, \ldots, y_n$ *of* $\mathfrak{k}[x_1, \ldots, x_n]$ *such that* 1) $\mathfrak{k}[x_1, \ldots, x_n]$ *is integral over* $\mathfrak{k}[y_1, \ldots, y_n]$ *and* 2) $\mathfrak{a} \cap \mathfrak{k}[y_1, \ldots, y_n] = (y_1, \ldots, y_r)\mathfrak{k}[y_1, \ldots, y_n]$ *(with some* $r \leqq n$*)*.

*Proof.* Let $y_1$ be a non-zero element of $\mathfrak{a}$. Then by virtue of Theorem 2 we can find $y_{2,1}, \ldots, y_{n,1}$ of $\mathfrak{k}[x_1, \ldots, x_n]$ such that $\mathfrak{k}[x_1, \ldots, x_n]$ is integral over $\mathfrak{k}[y_1, y_{2,1}, \ldots, y_{n,1}]$. Now we assume that there exist $y_1, \ldots, y_s, y_{s+1,s}, \ldots, y_{n,s}$ so that i) $\mathfrak{k}[x_1, \ldots, x_n]$ is integral over $\mathfrak{k}[y_1, \ldots, y_s, y_{s+1,s}, \ldots, y_{n,s}]$ and ii) $y_1, \ldots, y_s \in \mathfrak{a}$. When $\mathfrak{a} \cap \mathfrak{k}[y_1, \ldots, y_s, y_{s+1,s}, \ldots, y_{n,s}] = (y_1, \ldots, y_s)$, we may set $y_{s+j} = y_{s+j,s}$ $(j \geqq 1)$. In the contrary case, we can find a non-zero element $y_{s+1}$ of $\mathfrak{a} \cap \mathfrak{k}[y_{s+1,s}, \ldots, y_{n,s}]$. Then by Theorem 2 we can choose elements $y_{s+2,s+1}, \ldots, y_{n,s+1}$ of $\mathfrak{k}[y_{s+1,s}, \ldots, y_{n,s}]$ so that $\mathfrak{k}[y_{s+1,s}, \ldots, y_{n,s}]$ is integral over $\mathfrak{k}[y_{s+1}, y_{s+2,s+1}, \ldots, y_{n,s+1}]$. Then evidently $\mathfrak{k}[x_1, \ldots, x_n]$ is integral over $\mathfrak{k}[y_1, \ldots, y_{s+1}, y_{s+2,s+1}, y_{n,s+1}]$ and $y_1, \ldots, y_{s+1} \in \mathfrak{a}$.

COROLLARY 1.[2] Let $\mathfrak{o}$ be a ring which is generated by a finite number of elements over a field $\mathfrak{k}$. Then there exist elements $x_1, \ldots, x_r$ of $\mathfrak{o}$ such that i) $x_1, \ldots, x_r$ are algebraically independent over $\mathfrak{k}$ and ii) $\mathfrak{o}$ is integral over $\mathfrak{k}[x_1, \ldots, x_r]$.

COROLLARY 2. Let $\mathfrak{o}$ be the same as in Corollary 1 and let $\mathfrak{p} \supset \mathfrak{q}$ be prime ideals of $\mathfrak{o}$. Then $\dim \mathfrak{q} - \dim \mathfrak{p} = \operatorname{rank} \mathfrak{p} - \operatorname{rank} \mathfrak{q}$. Therefore all maximal descending chains of prime ideals which begin from $\mathfrak{p}$ and end to $\mathfrak{q}$ have the same length.

§ **3.** We define the notions of geometric local rings, nuclei of local rings, rings of type $\mathfrak{r}(n:\mathfrak{k})$ and rings of type $\bar{\mathfrak{r}}(n, m:\mathfrak{k})$ by a similar way as in Chevalley [1] but we drop the conditions on basic field $\mathfrak{k}$ that $\mathfrak{k}$ has infinitely many elements and that $[\mathfrak{k}:\mathfrak{k}^p] < \infty$. Then by virtue of our Theorem 3 we see easily in a same way as in Chevalley [1] that *every geometric local ring admits a nucleous.*

First we observe the following

LEMMA 3. Let $\mathfrak{o}$ be a local integrity domain which admits a nucleous $\mathfrak{r}$. If a field $L$ is a finite algebraic extension of the quotient field of $\mathfrak{o}$, then the totality $\bar{\mathfrak{o}}$ of $\mathfrak{o}$-integers in $L$ is a finite $\mathfrak{o}$-module. Therefore if $\mathfrak{m}$ is a maximal ideal of $\bar{\mathfrak{o}}$, then $\mathfrak{r}$ is also a nucleous of $\bar{\mathfrak{o}}_{\mathfrak{m}}$.

Proof is easy.

The following two lemmas are due to Zariski [4].

[2] When $\mathfrak{o}$ is an integrity domain, our result is the well known normalization theorem, which was proved by Noether when $\mathfrak{k}$ contains infinitely many elements, and was proved in general case by Cohen (see Zariski [6]).

LEMMA 4. Let $\mathfrak{o}$ be an integrally closed local integrity domain and let $\mathfrak{o}^*$ be its completion. Let $\mathfrak{p}$ be a prime ideal of $\mathfrak{o}$ of rank 1. Assume that $\mathfrak{p}\mathfrak{o}^*$ is an intersection of prime ideals $\mathfrak{p}_1^*, \ldots, \mathfrak{p}_h^*$ ($\mathfrak{p}_i^* \not\supseteq \mathfrak{p}_j^*$ if $i \neq j$). Then $\mathfrak{o}^*_{\mathfrak{p}_i^*}$ is a valuation ring, and therefore, each $\mathfrak{p}_i^*$ contains a unique divisor $\mathfrak{p}^*$ of the zero ideal of $\mathfrak{o}^*$ and every formal power $\mathfrak{p}_i^{*(n)}$[3] of $\mathfrak{p}_i^*$ contains $\mathfrak{P}^*$. Further $\mathfrak{p}^{(n)}\mathfrak{o}^* = \bigcap_i \mathfrak{p}_i^{*(n)}$.

*Proof.* Let $w$ be an element of $\mathfrak{p}$ which is not in $\mathfrak{p}^{(2)}$ and let $a^*$ be an element of $\mathfrak{p}_2^* \cap \ldots \cap \mathfrak{p}_h^*$ which is not in $\mathfrak{p}_1^*$ (when $h=1$, we may set $a^*=1$). We take an element $b$ of $w\mathfrak{o} : \mathfrak{p}$ which is not in $\mathfrak{p}$ and set $c^* = a^*b$. Then $c^* \notin \mathfrak{p}_1^*$, $\mathfrak{p}_1^* c^* \subseteq \mathfrak{o}^*\mathfrak{p}b \subseteq w\mathfrak{o}^*$. Since $\mathfrak{o}^*_{\mathfrak{p}_1^*}$ is a local ring with maximal ideal $\mathfrak{p}_1^*\mathfrak{o}^*_{\mathfrak{p}_1^*}$, it is a principal ideal ring with unique maximal ideal $w\mathfrak{o}^*_{\mathfrak{p}_1^*}$. Since there exists a prime divisor of zero ideal of $\mathfrak{o}^*$ which is contained in $\mathfrak{p}_1^*$, $\mathfrak{o}^*_{\mathfrak{p}_1^*}$ is a valuation ring. This being proved, the else is easy.

LEMMA 5. Let $\mathfrak{o}$ and $\mathfrak{o}^*$ be the same as in Lemma 4. Assume that there exists a non-unit $d$ of $\mathfrak{o}$ such that for every prime divisor $\mathfrak{p}_i$ ($1 \leqq i \leqq h$) of $d\mathfrak{o}$, $\mathfrak{p}_i\mathfrak{o}^*$ is an intersection of prime ideals $\mathfrak{p}_{i,1}^*, \ldots, \mathfrak{p}_{i,m(i)}^*$ ($\mathfrak{p}_{i,j}^* \not\supseteq \mathfrak{p}_{i,k}^*$ if $j \neq k$). Then $\mathfrak{o}^*$ contains no nilpotent element.

*Proof.* Let $\mathfrak{P}_1^*, \ldots, \mathfrak{P}_g^*$ be the totality of prime divisors of zero ideal of $\mathfrak{o}^*$ which are contained in at least one $\mathfrak{p}_{i,j}^*$. Then

$$\mathfrak{P}_1^* \cap \ldots \cap \mathfrak{P}_g^* \subseteq \bigcap_{i,j} (\mathfrak{p}_{i,1}^{*(j)} \cap \ldots \mathfrak{p}_{i,m(i)}^{*(j)}) = \bigcap_{i,j} \mathfrak{p}_i^{(j)}\mathfrak{o}^* \subseteq \bigcap_k d^k\mathfrak{o}^* = (0).^{4)}$$

Now we prove

THEOREM 4. *If a class $\mathfrak{S}$ of local rings satisfies the following three conditions, then the completion $\mathfrak{o}^*$ of a member $\mathfrak{o}$ of $\mathfrak{S}$ has no nilpotent element:*

1) *If $\mathfrak{o} \in \mathfrak{S}$, then $\mathfrak{o}$ contains no nilpotent element;*

2) *If $\mathfrak{o} \in \mathfrak{S}$ and if $\mathfrak{p}$ is a prime ideal of $\mathfrak{o}$, then $\mathfrak{o}/\mathfrak{p}$ is in $\mathfrak{S}$;*

3) *If $\mathfrak{o} \in \mathfrak{S}$ and if $\mathfrak{o}$ is an integrity domain, then* i) *the integral closure $\bar{\mathfrak{o}}$ of $\mathfrak{o}$ in its quotient field is a finite $\mathfrak{o}$-module and* ii) *for every maximal ideal $\mathfrak{m}$ of $\bar{\mathfrak{o}}$, $\bar{\mathfrak{o}}_{\mathfrak{m}}$ is in $\mathfrak{S}$.*

*Proof.* When $\dim \mathfrak{o} = 0$ our assertion is evident. We prove our assertion by induction on the dimension of $\mathfrak{o}$ ($\mathfrak{o} \in \mathfrak{S}$).

By Theorem 1 and by conditions 1) and 2), we may assume that $\mathfrak{o}$ is an integrity domain. Further by condition 3), we may assume that $\mathfrak{o}$ is integrally closed. Then applying Lemma 5, we see that $\mathfrak{o}^*$ contains no nilpotent element.

COROLLARY 1. Let $\mathfrak{o}$ be a local ring which admits a nucleous. If $\mathfrak{o}$ con-

[3] When $\mathfrak{p}$ is a prime ideal of a ring $\mathfrak{o}$, $\mathfrak{p}^{(n)}$ denotes the formal $n$-th power of $\mathfrak{p}$, i.e., $\mathfrak{p}^{(n)} = \mathfrak{p}^n\mathfrak{o}_{\mathfrak{p}} \cap \mathfrak{o}$.

[4] That $\bigcap_{i,j} \mathfrak{p}_i^{(j)}\mathfrak{o}^* \subseteq \bigcap_k d^k\mathfrak{o}^*$ follows from Theorem 1 and from that $d\mathfrak{o}$ has no imbedded prime divisor (since $\mathfrak{o}$ is integrally closed).

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tains no nilpotent element, then also the completion of $\mathfrak{o}$ contains no nilpotent element.

COROLLARY 2. Any geometric local ring contains no nilpotent element.

§ 4. LEMMA 6. Let $\mathfrak{o}$ be an integrally closed local integrity domain and let $\mathfrak{o}^*$ be its completion. Further let $\bar{\mathfrak{o}}^*$ be the integral closure of $\mathfrak{o}^*$ in its total quotient ring. Assume that there exists an element $d(\neq 0)$ of $\mathfrak{o}$ such that i) $d\bar{\mathfrak{o}}^* \subseteqq \mathfrak{o}^*$ and ii) for every prime divisor $\mathfrak{p}$ of $d\mathfrak{o}$, $\mathfrak{p}\mathfrak{o}^*$ is an intersection of prime ideals. Then $\mathfrak{o}^*$ is an integrally closed integrity domain. (Zariski [5])

Proof is easy by virtue of Lemma 4.

Now we prove

THEOREM 5. *Let $\mathfrak{o}$ be an integrally closed local integrity domain which admits a nucleous $\mathfrak{r}$. Let $R$ and $K$ be the quotient field of $\mathfrak{r}$ and $\mathfrak{o}$ respectively. Assume that there exists a finite algebraic extension field $R'$ of $R$ such that* i) *the totality $\mathfrak{r}'$ of $\mathfrak{r}$-integers in $R'$ is a regular local ring and* ii) *$L = R'K$ is separable over $R'$. Then the completion $\mathfrak{o}^*$ of $\mathfrak{o}$ is an integrally closed integrity domain.*

*Proof.* By Theorem 4, we have only to prove the existence of a non-zero element $d$ of $\mathfrak{o}$ which satisfies the condition i) in Lemma 6. Let $\mathfrak{J}$ and $\mathfrak{i}$ be the totalities of $\mathfrak{r}$-integers in $L$ and $K$ respectively. Further let $\mathfrak{J}^*$ and $\mathfrak{i}^*$ be the completions of $\mathfrak{J}$ and $\mathfrak{i}$ respectively and let $\bar{\mathfrak{J}}^*$ and $\bar{\mathfrak{i}}^*$ be the integral closure of $\mathfrak{J}^*$ and $\mathfrak{i}^*$ in their respective total quotient rings. We take an element $a$ of $\mathfrak{J}$ so that $L = R'(a)$. Let $d'$ be the discriminant of the irreducible polynomial over $R'$ which is satisfied by $a$. Then $d'\bar{\mathfrak{J}}^* \subseteqq \mathfrak{r}'^*[a] \subseteqq \mathfrak{J}^*$, where $\mathfrak{r}'^*$ is the completion of $\mathfrak{r}'$. Therefore $\mathfrak{J}^*$ is integrally closed by virtue of Lemma 6.[5] Now let $1, a_1, \ldots, a_s \in \mathfrak{i}$ be a linearly independent basis of $K$ over $R$ and let $1, b_1, \ldots, b_r \in \mathfrak{J}$ be a linearly independent basis of $L$ over $R$ with $a_i = b_i$ for $i = 1, \ldots, s$. Now $\bar{\mathfrak{i}}^* \subseteqq \mathfrak{J}^*$, because $\mathfrak{J}^*$ is integrally closed. Let $d$ be a non-zero element of $\mathfrak{r}$ such that $d\mathfrak{J} \subseteqq \mathfrak{r}[b_1, \ldots, b_r]$. Then $d\mathfrak{J}^* \subseteqq \mathfrak{r}^*[b_1, \ldots, b_r]$ and therefore $d\bar{\mathfrak{i}}^* \subseteqq \mathfrak{r}^*[b_1, \ldots, b_r]$, where $\mathfrak{r}^*$ is the completion of $\mathfrak{r}$. That $d\bar{\mathfrak{i}}^* \subseteqq \mathfrak{r}^*[b_1, \ldots, b_r]$ shows $d\bar{\mathfrak{i}}^* \subseteqq \mathfrak{r}^*[a_1, \ldots, a_s]$ and therefore $d\bar{\mathfrak{i}}^* \subseteqq \mathfrak{i}^*$, which shows, by virtue of Lemma 6, $\mathfrak{i}^*$ and therefore $\mathfrak{o}^*$ are integrally closed again.

COROLLARY 1. Let $\mathfrak{o}$ be an integrally closed local integrity domain such that either $\mathfrak{o}$ admits a nucleous of type $\mathfrak{r}(n:\mathfrak{k})$ or $\mathfrak{o}$ admits a nucleous and has a basic field $\mathfrak{k}$ such that $[\mathfrak{k}:\mathfrak{k}^p] < \infty$. Then the completion of $\mathfrak{o}$ is an integrally closed integrity domain.

COROLLARY 2. Let $\mathfrak{o}$ be a local integrity domain such that either $\mathfrak{o}$ admits a nucleous of type $\mathfrak{r}(n:\mathfrak{k})$ or $\mathfrak{o}$ admits a nucleous and has a basic field $\mathfrak{k}$ such

[5] It is easy to generalize Lemma 6 for integrally closed semi-local integrity domains.

that $[\mathfrak{k}:\mathfrak{k}^p]<\infty$. Let $\bar{\mathfrak{o}}$ be the integral closure of $\mathfrak{o}$ in its quotient field. Then the completion $\bar{\mathfrak{o}}^*$ of $\bar{\mathfrak{o}}$ is the integral closure of the completion $\mathfrak{o}^*$ of $\mathfrak{o}$ in its total quotient ring.

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*Added in Proof*

1. Our proof of Theorem 5 is not correct unless that $\mathfrak{o}^*$ is an integrity domain is proved. We can correct our proof. Further we can prove that if an integrally closed local integrity domain $\mathfrak{o}$ admits a nucleous (in our generalized sense), then the completion of $\mathfrak{o}$ is an integrally closed integrity domain. This will be proved in a latter paper "Some remarks on local rings II" to appear in Memo. Kyôto.

2. As for the proof of Lemma 3 for the case that the nucleous $\mathfrak{r}$ is of type $\bar{\mathfrak{r}}(n, m : \mathfrak{k})$, see appendix of the above paper.

3. It was communicated to the writer that some of our results was discussed independently by P. Samuel (Algèbre Locale, Mémo. Sci. Math. No. 128 (1953)).