Hence in this case $\Sigma f(n)$ is divergent.

(2°) Suppose $\underset{x \to \infty}{\mathrm{L}} xf(x=0$, and that the approach to the limit is steady.

We can choose r an integer so that $b_1 < a^r < b_2$ where b_1 and b_2 are integers $\geqq 2$.

$\Sigma a^n f(a^n)$.

$$
\begin{array}{lll}
 & =\{af(a)+\ldots+a^r f(a^r)\} & > rb_2 f(b_2) \\
rb_1 f(b_1) > & +\{a^{r+1}f(a^{r+1})+\ +a^{2r}f(a^{2r})\} & > rb_2^2 f(b_2^2) \\
rb_1^2 f(b_1^2) > & +\{a^{2r+1}f(a^{2r+1})+\ldots+a^{3r}f(a^{3r}\} & > rb_2^3 f(b_2^3) \\
 & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
rb_1^{m-1}f(b_1^{m-1}) > & +\{a^{\overline{m-1}\,r+1}f(a^{\overline{m-1}\,r+1})+\ldots+a^{mr}f(a^{mr})\} & > rb_2^m f(b_2^m) \\
rb_1^m f(b_1^m) > & +\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &
\end{array}
$$

Hence we have

$$\{af(a)+\ldots+a^r f(a^r)\}+r\Sigma b_1^m f(b_1^m) > \Sigma a^n f(a^n) > r\Sigma b_2^m f(b_2^m).$$

Hence if $\Sigma a^n f(a^n)$ converges so does $\Sigma b_2^m f(b_2^m)$ and $\therefore$ $\Sigma f(m)$
and if $\Sigma a^n f(a^n)$ diverges so does $\Sigma b_1^m f(b_1^m)$ and $\therefore$ $\Sigma f(m)$.

Hence $\Sigma f(n)$ converges or diverges with $\Sigma a^2 f(a^n)$ where $a > 1$.

Q.E.D.

ROBERT VICKERS

To show that the equation

$$\begin{vmatrix} x-a & f & e \\ f & x-b & d \\ e & d & x-c \end{vmatrix} = 0$$

has three real roots.—Let x_1 be any one root. Then ξ, η, ζ can be chosen uniquely to satisfy simultaneously the three equations

$$(x_1-a)\xi+f\eta+e\zeta=0 \ldots\ldots\ldots\ldots \text{(i)}$$
$$f\xi+(x_1-b)\eta+d\zeta=0 \ldots\ldots\ldots\ldots \text{(ii)}$$
$$e\xi+d\eta+(x_1-c)\zeta=0 \ldots\ldots\ldots\ldots \text{(iii)}$$

If x_1 is complex, so also will be ξ, η, ζ. Let their conjugate complexes be $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$ so that

$$\xi=\xi_1+\iota\xi_2,\ \bar{\xi}=\xi_1-\iota\xi_2;\ \text{etc.}$$

Multiplying (i), (ii), (iii) respectively by $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$ and adding we have

$$\Sigma_{a,b,c}(x_1-a)\xi\bar{\xi}+\Sigma_{d,e,f}f(\eta\bar{\xi}+\xi\bar{\eta})=0 \ldots\ldots\ldots \text{(iv)}$$

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Now $\xi\bar{\xi} = \xi_1^2 + \xi_2^2$ and is real.

$$\eta\bar{\xi} + \xi\bar{\eta} = (\eta_1 + i\eta_2)(\xi_1 - \iota\xi_2) + (\xi_1 + \iota\xi_2)(\eta_1 - \iota\eta_2)$$
$$= 2(\eta_1\xi_1 + \eta_2\xi_2), \text{ and is also real.}$$

It therefore follows from (iv), that the root x_1' is real. Similarly the other roots are real.

The proof for the corresponding equation of the nth degree is deducible at once, on the same lines.

This method of proof shows very clearly the manner in which the fact that the roots are all real depends upon the cross coefficients being equal, for $f_1\eta\bar{\xi} + f_2\xi\bar{\eta}$ is only real if $f_1 = f_2$.

H. Spencer Jones