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SYMMETRIC GREEN'S FUNCTION FOR A CLASS OF CIV BOUNDARY VALUE PROBLEMS

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ABSTRACT. Generalized boundary value problems are considered for hyperbolic equations of the form $u_{tt} - u_{ss} + \lambda p(s, t)u = 0$. By constructing symmetric Green's functions appropriate to such problems the existence of eigenvalues is established.

1. **Introduction.** The eigenvalue problem

$$\begin{aligned} &\frac{d^2u}{dt^2} + \lambda p(t)u = 0 \\ &u(0) = 0 = u(1) \end{aligned} \tag{1.1}$$

has a simple physical interpretation in terms of the motion of a particle, subject to a linear restoring force, which is required to pass through its equilibrium position at times $t = 0$ and $t = 1$. Attempts to generalize these ideas to vibrating strings has led [2] to the consideration of hyperbolic equations of the form

$$u_{tt} - u_{ss} + \lambda p(s, t)u = 0 \tag{1.2}$$

subject to appropriate boundary conditions.

As in [2], we consider (1.2) in a characteristic triangle of the form

$$R(1, 1) = \{ (s, t): t \leqq s \leqq 2 - t;\ 0 \leqq t \leqq 1 \}$$

in which $p(s, t)$ is assumed to be continuous and positive. Our goal is to establish spectral properties for various eigenvalue problems associated with (1.2) by constructing corresponding fundamental solutions for

$$L[u] \equiv u_{tt} - u_{ss} \quad \text{in } R \times R.$$

For example, in analogy with the boundary conditions of (1.1), we can now impose

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$$(1.3) \qquad u(s, 0) = 0 = u(1, 1); \ 0 \leqq s \leqq 2,$$

so that the vertex (1, 1) of R serves as a generalized conjugate point for the initial condition $u(s, 0) = 0$. The techniques of the present paper lead to the construction of Green's functions $G(s, t; \sigma, \tau)$ which satisfy prescribed boundary conditions such as (1.3) and are symmetric in (s, t) and (σ, τ). This form of symmetry is closely related to auxiliary conditions of the form

$$(1.4) \qquad u(t, t) = u(1 + t, 1 - t); \ 0 \leqq t \leqq 1$$

which have also been used by Kalmenov [1].

Equation (1.2) subject to

$$(1.5) \qquad u(t, t) = f(t); \ u(1 + t, 1 - t) = g(t); \ 0 \leqq t \leqq 1$$

constitutes a classical characteristic initial value (CIV) problem which has a solution in R for all real values of λ. However, the replacement of (1.5) by (1.3) and (1.4) leads to a "CIV boundary value problem" whose spectral properties are remarkably similar to those of (1.1).

One approach to the construction of Green's functions is to find an integral equation equivalent to

$$L[u] \equiv u_{tt} - u_{ss} = f(s, t); \ (s, t) \in R(1, 1)$$

$$u(s, 0) = 0 = u(1, 1); \ u_t(s, 0) = kg(s),$$

where $g(s)$ is continuous and positive for $0 \leqq s \leqq 2$ and k is a constant yet to be determined. Using D'Alembert's formula

$$u(s, t) = \frac{k}{2} \int_{s-t}^{s+t} g(\sigma) d\sigma - \frac{1}{2} \int \int_{R(s,t)} f(\sigma, \tau) d\sigma d\tau$$

with

$$R(s, t) = \{ (\sigma, \tau) : s - (t - \tau) \leqq \sigma \leqq s + (t - \tau); \ 0 \leqq \tau \leqq t \}$$

one obtains [2] a Green's function

$$\begin{aligned} G(s, t; \sigma, \tau) &= \frac{g_0(s, t)}{2g_0(1, 1)} \quad \text{for} \quad (\sigma, \tau) \text{ in } R(1, 1) - R(s, t) \\ &= \frac{g_0(s, t)}{2g_0(1, 1)} - \frac{1}{2} \quad \text{for} \quad (\sigma, \tau) \text{ in } R(s, t), \end{aligned}$$

where

$$g_0(s, t) = \frac{1}{2} \int_{s-t}^{s+t} g(\sigma) d\sigma.$$

As observed in [2], this Green's function satisfies (1.3) but is *not* symmetric in (s, t) and (σ, t). This asymmetry prevents one from drawing the desired analogies between the eigenvalue problems (1.1) and (1.2)-(1.4).

2. **A Fundamental Solution.** In order to determine a symmetric Green's function for (1.2)-(1.4), it is convenient to begin with a change of variables

$$s = x + y;\ t = -x + y \tag{2.1}$$

and to adopt the notation $F(x, y) = f(x + y, -x + y)$. Then (1.2) becomes

$$\begin{gathered} -U_{xy} + \lambda P(x, y)U = 0 \\ U(0, 1) = U(x, x) = 0 \quad \text{for} \quad 0 \leqq x \leqq 1. \end{gathered} \tag{2.2}$$

while R is mapped into the unit square $Q = \{(x, y): 0 \leqq x \leqq 1, 0 \leqq y \leqq 1\}$.

In accordance with [1], we note that

$$-U_{xy} = \left(i\frac{\partial}{\partial y}\right)\left(i\frac{\partial}{\partial x}\right)U$$

and that the operator $L_x w \equiv i(\partial w/\partial x)$ is formally selfadjoint when subject to boundary conditions of the form

$$w(0) + w(1) = 0. \tag{2.3}$$

These observations motivate us to find the Green's function for d/dx and (2.3) by solving the boundary value problem

$$\frac{dv}{dx} = f(x);\ v(0) + v(1) = 0 \tag{2.4}$$

in the form

$$v(x) = \int_0^x f(\xi)d\xi - \frac{1}{2}\int_0^1 f(\xi)d\xi. \tag{2.5}$$

From (2.5) it is evident that the Green's function for (2.4) is given by

$$\begin{aligned} G_1(x;\xi) &= \frac{1}{2} \quad \text{for} \quad 0 \leqq \xi < x \\ &= -\frac{1}{2} \quad \text{for} \quad x < \xi \leqq 1. \end{aligned}$$

Analogously, the Green's function for

$$\frac{dv}{dx} = f(y);\ v(0) + v(1) = 0 \tag{2.6}$$

is given by

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$$G_2(y;\eta) = \frac{1}{2} \quad \text{for} \quad 0 \leqq \eta < y$$

$$= -\frac{1}{2} \quad \text{for} \quad y < \eta \leqq 1.$$

Thus in the square

$$Q = \{(\xi, \eta): 0 \leqq \xi \leqq 1, 0 \leqq \eta \leqq 1\}$$

it is plausible that a fundamental solution for $U_{xy} = f(x, y)$ can be obtained by direct multiplication:

$$(2.7) \qquad \begin{aligned} S(x, y; \xi, \eta) &= \frac{1}{4} \quad \text{for} \quad (x - \xi)(y - \eta) > 0 \\ &= -\frac{1}{4} \quad \text{for} \quad (x - \eta)(y - \eta) < 0. \end{aligned}$$

A direct computation shows that (2.7) is in fact a fundamental solution, in the sense that

$$U(x, y) = \int\int_Q S(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta$$

does satisfy $U_{xy} = f(x, y)$ for $f(x, y)$ continuous in Q.

Using (2.1) to transform these results back into the (s, t) coordinate system, we obtain a fundamental solution $S(s, t; \sigma, \tau)$ for $L[u] \equiv u_{tt} - u_{ss}$ which is defined for $(s, t; \sigma, \tau) \in R \times R$ as follows:

$$(2.8) \qquad \begin{aligned} S(s, t; \sigma, \tau) &= \frac{1}{4} \quad \text{for} \quad (\sigma, \tau) \text{ in } Q_L \cup Q_R \\ &= -\frac{1}{4} \quad \text{for} \quad (\sigma, \tau) \text{ in } Q_T \cup T_B. \end{aligned}$$

To see that the fundamental solution given by (2.8) is symmetric in $(s, t; \sigma, \tau)$ we need only choose a point (σ, τ) in (say) Q_L; an interchange of the roles of (σ, τ) and (s, t) would then replace (s, t) in the new Q_R. Since $S(s, t; \sigma, \tau)$ has the same values in Q_L and $Q_R(Q_T$ and $T_B)$, the symmetry of S follows. However S does not satisfy the boundary conditions (1.3) and therefore does not provide the desired Green's function.

3. **A Symmetric Green's Function.** Given (2.8) one can adapt the "method of images" to obtain the desired Green's function which satisfies (1.3) and (1.4). In particular, we consider

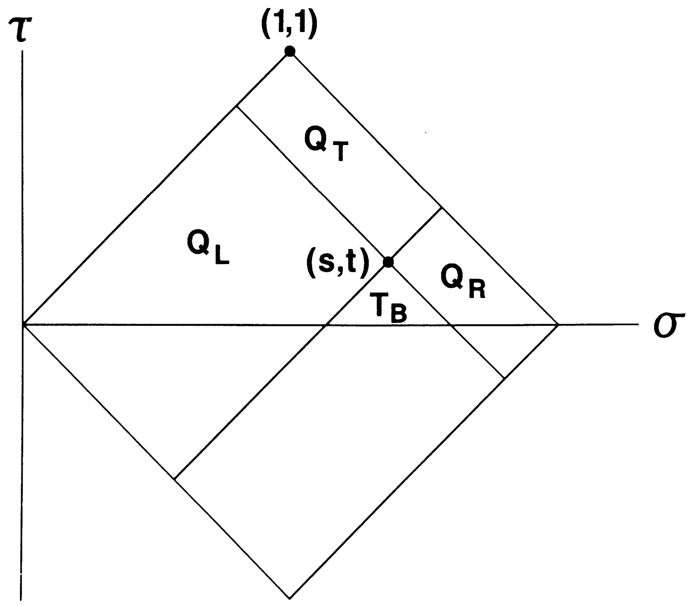

FIGURE 1.

$$G(s, t; \sigma, \tau) = S(s, t; \sigma, \tau) - S(s, -t; \sigma, \tau) \tag{3.1}$$

for $(s, t; \sigma, \tau) \in R \times R$. The computation implicit in (3.1) is most easily carried out in graphical form by considering R and its mirror image.

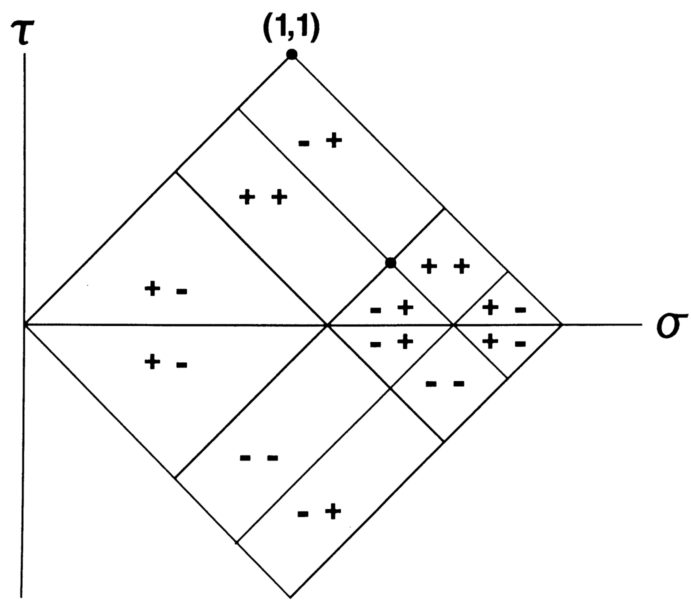

FIGURE 2.

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Here the $+(-)$ denotes the addition (subtraction) of the number $1/4$ in the quadrilateral or triangle indicated. The result of this computation is the following formula for (3.1):

$$
\begin{aligned}
G(s, t; \sigma, \tau) &= \frac{1}{2} \quad \text{for} \quad (\sigma, \tau) \text{ in } Q_L \cup Q_R \\
&= 0 \quad \text{for} \quad (\sigma, \tau) \text{ in } Q_T \cup T_L \cup T_B \cup T_R
\end{aligned} \tag{3.2}
$$

where the regions used to specify $G(s, t; \sigma, \tau)$ in (3.2) are defined below.

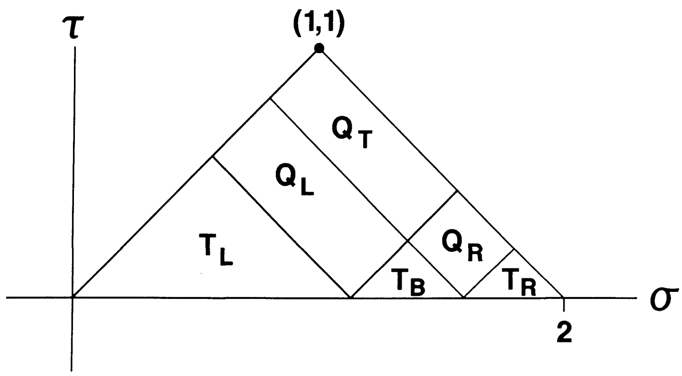

FIGURE 3.

An argument analogous to that given at the end of Section 2 now shows that $G(s, t; \sigma, \tau)$ is symmetric in $(s, t; \sigma, \tau)$. We also note that for fixed (s, t), $G(s, t; \sigma, \tau)$ satisfies the boundary conditions of (1.3) as well as the selfadjointness condition (1.4), which together assure that

$$\int\int_R (uLv - vLu)d\sigma d\tau = 0.$$

Accordingly, the generalized CIV eigenvalue problem given by (1.2)-(1.4) is equivalent to the integral equation

$$u(s, t) = \lambda \int_R \int G(s, t; \sigma, \tau)p(\sigma, \tau)u(\sigma, \tau)d\sigma d\tau \tag{3.3}$$

where the integral on the right defines an operator G in the weighted L_2 space $L_2^p(R)$ consisting of functions $u(\sigma, \tau)$ satisfying

$$\int\int_R |u(\sigma, \tau)|^2 p(\sigma, \tau)d\sigma d\tau < \infty.$$

The symmetry and boundedness of $G(s, t; \sigma, \tau)$ in $R \times R$ assures that

$$\mathcal{G}[u] \equiv \int\int_R G(s, t; \sigma, \tau)p(\sigma, \tau)u(\sigma, \tau)d\sigma d\tau$$

defines a selfadjoint completely continuous operator in $L_2^p(R)$. A standard result [3] now yields the following.

THEOREM 3.1. *The eigenvalue problem* (1.2)-(1.4) *has a real discrete spectrum of eigenvalues* λ_k *with* $|\lambda_k| \to \infty$ *and a corresponding set of orthogonal eigenfunctions which are complete in* $L_2(R)$.

For the special case $p(s, t) \equiv 1$ these eigenvalues and eigenfunctions can be calculated explicitly and are given in [1].

4. **A Focal Point Problem.** Finally, we address the question of whether one can vary the boundary conditions in (1.2) and still find a corresponding Green's function with the properties of (3.2). In particular, we consider the generalized right focal point problem defined by

$$u_t(s, 0) = 0 = u(1, 1); \; 0 \leqq s \leqq 2 \tag{4.1}$$

in place of (1.3).

The corresponding symmetric Green's function can now be deduced from the assumption that it is constant in the regions given in Figure 3. For in order to satisfy $u(1, 1) = 0$ we must then choose $G(s, t; \sigma, \tau) = 0$ for (σ, τ) in T_B, and symmetry then requires that G also vanish in Q_T. Furthermore, to assure that G is a fundamental solution, its constant values in Q_L and Q_R must sum to one, and the condition of symmetry then requires that $G(s, t; \sigma, \tau) = 1/2$ in $Q_L \cup Q_R$. Thus G is the same as in (3.2) in the regions so far considered.

It remains to determine G in T_L and T_R so that $u_t(s, 0) = 0$ for $0 \leqq s \leqq 2$. To that end we consider $G(s_0, 0; \sigma, \tau)$ and $G(s_0, \Delta t; \sigma, \tau)$ in the context of Figure 3.

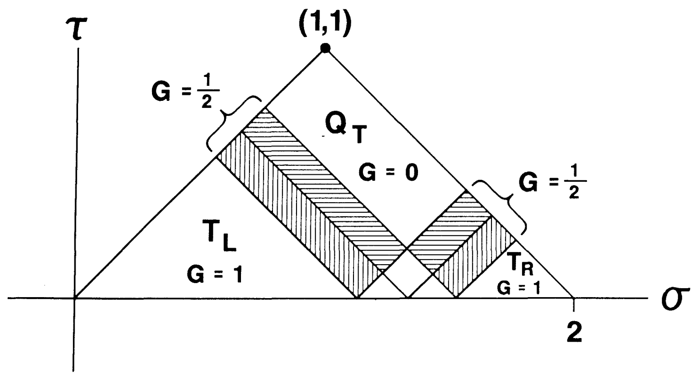

FIGURE 4.

In order to assure that

$$\begin{aligned} &u(s_0, \Delta t) - u(s_0, 0) \\ &= \int\int_R [G(s_0, \Delta t; 0, t) - G(s_0, 0; \sigma, t)] f(\sigma, \tau) d\sigma d\tau \end{aligned}$$

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be $o(\Delta t)$ as $\Delta t \rightarrow 0$, we define G in T_L and T_R so that the contribution from the horizontally shaded region in Figure 4 essentially cancel the contribution from the vertically shaded region. Given the values already assigned in Q_T, Q_L and Q_R, this occurs if and only if $G(s, t; \sigma, \tau) = 1$ for σ, τ in T_L and T_R. Since these last definitions do not affect the fundamental singularity at (s, t), the Green's function (which is readily seen to be symmetric) is fully defined for (4.1).

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