Two Proofs of the ${}_6\Psi_6$ Summation Theorem

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1. *Introduction.* The ${}_6\Psi_6$ summation theorem was first proved by Bailey[1], who deduced it indirectly from a transformation of a well-poised ${}_8\Phi_7$ series into two ${}_4\Phi_3$ series. No direct proof of the theorem has been published, and, since it has interesting applications in the proofs of various identities which occur in combinatory analysis, for example the A series of Rogers[2] and some elegant identities due to Ramanujan[3], we give two new proofs of the theorem in this paper.

The first proof (due to Slater) introduces a basic analogue of the Barnes type integral. The second (due to Lakin) is the basic analogue of an operational method used elsewhere[4], and provides an application of Carlson's theorem.

2. *First Proof.* The notation is that introduced by Bailey, with the addition that

$$\prod^{N}\begin{pmatrix} a\,; \\ b\,; \end{pmatrix} = \prod_{n=0}^{N} \frac{(1-aq^n)}{(1-bq^n)},$$

and Π is written for $\overset{\infty}{\Pi}$. Thus,

$$\Pi\begin{pmatrix} a\,; \\ b\,; \end{pmatrix} = \prod_{n=0}^{\infty} \frac{(1-aq^n)}{(1-bq^n)}.$$

Consider the integral

$$I_N = \frac{1}{2\pi i}\int P_N(s)\,ds$$

where

$$P_N(s) = \prod^{N}\begin{pmatrix} q^{1-d+s},\ q^{1-d-s},\ q^{1-e+s},\ q^{1-e-s},\ q^{1-f+s},\ q^{1-f-s}\,; \\ q^{a+s},\ q^{a-s},\ q^{b+s},\ q^{b-s},\ q^{c+s},\ q^{c-s}\,; \end{pmatrix} q^s,$$

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[1] Bailey [**1**], §4.
[2] See Slater [**5**], for full references.
[3] Bailey [**3**].
[4] Burchnall [**4**].

and $q=e^{-t}$, $t>0$, taken round the contour

$$A(-2N-i\pi/t) \quad B(-2N+i\pi/t) \quad C(2N+i\pi/t) \quad D(2N-i\pi/t),$$

and assume that none of the members of the sequences

$$q^{a\pm n}, \quad q^{b\pm n}, \quad q^{c\pm n}$$

coincide or fall on the contour. By the periodicity of the integrand,

$$\int_{BC}+\int_{DA}=0.$$

Also

$$\int_{CD}=\frac{1}{2\pi i}\int_0^{\pi/t}[P_N(2N-ir)-P_N(2N+ir)]\,dr$$

and

$$\int_{AB}=\frac{1}{2\pi i}\int_0^{\pi/t}[P_N(-2N-ir)-P_N(-2N+ir)]\,dr.$$

Both these integrals tend to zero as $N\to\infty$, provided

$$\mathrm{Rl}\,(5-a-b-c-d-e)>0.$$

Thus we can equate to zero the sum of the residues at the poles of $P_N(s)$ in the s-plane. Now $1/\Pi(q^{a+s};)$ has poles within $ABCD$ at

$$s=-a-n+2\pi ik/t$$

for some integer k. Hence $P_N(s)$ has increasing sequences of poles at $s=a+n$, $b+n$, $c+n$, and decreasing sequences of poles at

$$s=-a-n, \quad -b-n, \quad -c-n, \text{ for } n=0,\ 1,\ 2,\ \ldots.$$

Combining the residues at $s=a+n$ and $s=-a-n$, and using the symmetry in the integrand, we have

$$\sum_{n=0}^{\infty}\Pi\begin{pmatrix} q^{1+a-d+n},\ q^{1-a-d-n},\ q^{1+a-e+n},\ q^{1-a-e-n},\ q^{1+a-f+n},\ q^{1-a-f-n}; \\ q^{2a+n},\ q,\ q^{b+a+n},\ q^{b-a-n},\ q^{c+a+n},\ q^{c-a-n}; \end{pmatrix}$$

$$\times\frac{(q^{a+n}-q^{-a-n})}{(q^{-n})_n}+\text{idem}\,(a;b,c)=0,$$

where "idem $(a;b)$" means that the preceding expression is to be repeated with b and a interchanged.

If we put a for q^a, and so on, this gives, in the more conventional notation,

$$\frac{1}{a}\Pi\begin{pmatrix} aq/d,\ q/ad,\ aq/e,\ q/ae,\ aq/f,\ q/af; \\ a^2q,\ ab,\ b/a,\ ac,\ c/a; \end{pmatrix}$$

$$\times\, {}_8\Phi_7\left[\begin{matrix} a^2,\ aq,\ -aq,\ ab,\ ac,\ ad,\ ae,\ af; \\ a,\ -a,\ aq/b,\ aq/c,\ aq/d,\ aq/e,\ aq/f; \end{matrix}\ \frac{q^2}{abcdef}\right]$$

$$+\text{idem}\,(a;\ b,\ c)=0, \qquad (1)$$

where the restriction that q is real can now be removed. This is the basic analogue of a result due to Whipple[1]. In (1) put $c=q/a$. The first and third series combine to give

$$\frac{1}{a}\Pi\left[\begin{matrix} aq/d,\ aq/e,\ aq/f,\ q/ad,\ q/ae,\ q/af; \\ a^2q,\ ab,\ q,\ b/a,\ q/a^2; \end{matrix}\right]$$

$$\times\, {}_6\Psi_6\left[\begin{matrix} aq,\ -aq,\ ab,\ ad,\ ae,\ af; \\ a,\ -a,\ aq/b,\ aq/d,\ aq/e,\ aq/f; \end{matrix}\ \frac{q}{bdef}\right]$$

and the second series reduces to

$${}_6\Phi_5\left[\begin{matrix} b^2,\ bq,\ -bq,\ bd,\ be,\ bf; \\ b,\ -b,\ bq/d,\ bq/e,\ bq/f; \end{matrix}\ \frac{q}{bdef}\right]=\Pi\begin{pmatrix} b^2q,\ q/de,\ q/ef,\ q/df; \\ bq/d,\ bq/e,\ bq/f,\ q/bdef; \end{pmatrix}.$$

Hence, after a little reduction, we have the required result,

$${}_6\Psi_6\left[\begin{matrix} aq,\ -aq,\ ab,\ ad,\ ae,\ af; \\ a,\ -a,\ aq/b,\ aq/d,\ aq/e,\ aq/f; \end{matrix}\ \frac{q}{bdef}\right]$$

$$=\Pi\left[\begin{matrix} a^2q,\ q/bd,\ q/be,\ q/bf,\ q/de,\ q/df,\ q/ef,\ q,\ q/a^2; \\ q/ab,\ q/ad,\ q/ae,\ q/af,\ aq/b,\ aq/d,\ aq/e,\ aq/f,\ q/bdef; \end{matrix}\right]. \qquad (2)$$

3. *Second Proof.* Let

$$\Psi={}_4\Psi_4\left[\begin{matrix} ab,\ ac,\ ad,\ ae; \\ aq/b,\ aq/c,\ aq/d,\ aq/e; \end{matrix}\ x\right]$$

and let Q be the operator $q^{x\,d/dx}$ with the property

$$Qf(x)=f(qx),$$

[1] Bailey [1], § (4 . 6).

where $f(x)$ is a polynomial or power series in x. Then

$$\left(1-\frac{a}{b}Q\right)(1-abQ)\Psi=\left(1-\frac{a}{b}\right)(1-ab)$$

$$\times{}_4\Psi_4\left[\begin{matrix}abq, & ac, & ad, & ae; \\ a/b, & aq/c, & aq/d, & aq/e;\end{matrix}\ x\right], \quad (3)$$

the effect of the operator being to multiply b by q whenever it occurs in the series, which remains well-poised. Further,

$$(1-a^2Q^2)\Psi=(1-a^2){}_6\Psi_6\left[\begin{matrix}aq, & -aq, & ab, & ac, & ad, & ae; \\ a, & -a, & aq/b, & aq/c, & aq/d, & aq/e;\end{matrix}\ x\right], \quad (4)$$

which introduces the first and second parameters of special form.

The q-difference equation satisfied by Ψ is

$$\left[\left(1-\frac{a}{b}Q\right)\left(1-\frac{a}{c}Q\right)\left(1-\frac{a}{d}Q\right)\left(1-\frac{a}{e}Q\right)\right.$$

$$\left.-x(1-abQ)(1-acQ)(1-adQ)(1-aeQ)\right]\Psi=0. \quad (5)$$

The operator in (5) may be written

$$[(1-\sigma_{-1}aQ+\sigma_{-2}a^2Q^2-\sigma_{-3}a^3Q^3+\sigma_{-4}a^4Q^4)$$

$$-x(1-\sigma_1aQ+\sigma_2a^2Q^2-\sigma_3a^3Q^3+\sigma_4a^4Q^4)],$$

where σ_r is the r-th elementary symmetric function of the four parameters b, c, d and e. Put $x=1/bcde=\sigma_{-4}$; then since $\sigma_1\sigma_{-4}=\sigma_{-3}$, etc., the operator may be written

$$[(1-\sigma_{-4})(1-a^4Q^4)-(\sigma_{-1}-\sigma_{-3})aQ(1-a^2Q^2)]$$

or

$$\left[B(1-abQ)\left(1-\frac{a}{b}Q\right)+C(1-acQ)\left(1-\frac{a}{c}Q\right)\right](1-a^2Q^2), \quad (6)$$

where B and C are undetermined constants. These may be evaluated by putting $Q=1/ac$, $1/ab$ in turn in (5) and (6), whence we find

$$B=\frac{(1-1/cd)(1-1/ce)}{1-b/c}, \quad \text{and } C=\frac{(1-1/bd)(1-1/be)}{1-c/b}.$$

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Using (3) and (4) to perform on Ψ the operation indicated by (6), we have

$$\frac{(1-1/cd)(1-1/ce)(1-ab)(1-a/b)}{1-b/c}\,{}_6\Psi_6\left[\begin{matrix}aq, & -aq, & abq, & ac, & ad, & ae; & \frac{1}{bcde}\\ a, & -a, & a/b, & aq/c, & aq/d, & aq/e; & \end{matrix}\right]$$

$$+\frac{(1-1/bd)(1-1/be)(1-ac)(1-a/c)}{1-c/b}$$

$$\times\,{}_6\Psi_6\left[\begin{matrix}aq, & -aq, & ab, & acq, & ad, & ae; & \frac{1}{bcde}\\ a, & -a, & aq/b, & a/c, & aq/d, & aq/e; & \end{matrix}\right]=0.$$

If we arrange this and write b/q for b, then

$$\Psi(b,\ c)\equiv{}_6\Psi_6\left[\begin{matrix}aq, & -aq, & ab, & ac, & ad, & ae; & \frac{q}{bcde}\\ a, & -a, & aq/b, & aq/c, & aq/d, & aq/e; & \end{matrix}\right]$$

$$=\frac{(1-q/bd)(1-q/be)(1-1/ac)(1-a/c)}{(1-1/cd)(1-1/ce)(1-q/ab)(1-aq/b)}\,\Psi(b/q,\ cq),$$

or, on applying the transformation N times,

$$\Psi(b,\ c)=\prod^{N-1}\left(\begin{matrix}q/bd, & q/be, & 1/ac, & a/c;\\ 1/cd, & 1/ce, & q/ab, & aq/b;\end{matrix}\right)\Psi(bq^{-N},\ cq^{N}). \qquad (7)$$

This equation is a two-term difference relation satisfied by the series. Such a relation must exist in order that a hypergeometric series should be summable. To show that (7) is still true for non-integral values of N we apply Carlson's theorem to the function

$$f(z)=\Pi(q^{1+z}/bd,\ q^{1+z}/be,\ q/ac,\ aq/c,\ q^{1-z}/cd,\ q^{1-z}/ce,\ q/ab,\ aq/b;)\,\Psi(b,\ c)$$

$$-\,\Pi\,(q/bd,\ q/be,\ q^{1-z}/ac,\ aq^{1-z}/c,\ q/cd,\ q/ce,\ q^{1+z}/ab,\ aq^{1+z}/b;)$$

$$\times\,\Psi(bq^{-z},\ cq^{z}), \quad (8)$$

which is, in effect, (7) multiplied by a suitable factor, with N replaced by z.

It is easy to establish by the usual arguments [1] that for $\mathrm{Rl}(z)\geqslant 0$ this function is regular and of the required order for large values of z, subject to certain restrictions on the parameters which may be removed from the final result. By (7), $f(z)=0$ if $z=0,\ 1,\ 2,\ \ldots$, and therefore by Carlson's theorem it is identically zero. In particular it is zero if $q^{z}=b/a$, when $\Psi(bq^{-z},\ cq^{z})$ reduces to a summable ${}_6\Phi_5$ and the required result (2) follows immediately.

[1] Bailey [2], (5 . 3) and (5 . 4).

The operator (6) may be written in another form, thus:

$$\left[A(1-abQ)\left(1-\frac{a}{b}Q\right)+B(1-a\lambda Q)\left(1-\frac{a}{\lambda}Q\right)\right](1-a^2Q^2)$$

$$\equiv\left[\left(1-\frac{a}{b}Q\right)\left(1-\frac{a}{c}Q\right)\left(1-\frac{a}{d}Q\right)\left(1-\frac{a}{e}Q\right)\right.$$

$$\left.-\frac{1}{bcde}(1-abQ)(1-acQ)(1-adQ)(1-aeQ)\right], \qquad (9)$$

where λ is dependent on a, b, c, d and e.

Using the operator in this form, we obtain

$$A(1-ab)\left(1-\frac{a}{b}\right){}_6\Psi_6\left[\begin{matrix} aq, & -aq, & abq, & ac, & ad, & ae; \\ a, & -a, & a/b, & aq/c, & aq/d, & aq/e; \end{matrix}\ \frac{1}{bcde}\right]$$

$$+B(1-a\lambda)(1-a/\lambda)$$

$$\times{}_8\Psi_8\left[\begin{matrix} aq, & -aq, & a\lambda q, & aq/\lambda, & ab, & ac, & ad, & ae; \\ a, & -a, & a/\lambda, & a\lambda, & aq/b, & aq/c, & aq/d, & aq/e; \end{matrix}\ \frac{1}{bcde}\right]=0, \qquad (10)$$

where A and B are constants which can be determined. The ${}_6\Psi_6$ is summable, and so therefore is the ${}_8\Psi_8$. It is interesting to notice the existence of this summable ${}_8\Psi_8$, though there is little point in stating the result in detail.

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