OUTLINE OF AN INTRODUCTION TO MATHEMATICAL LOGIC II

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7. Adequacy of the Calculus of Deduction. It is the purpose of the calculus described in the preceding section (i) to yield as theorems all tautologies which can be formulated in terms of the given variables, and (ii) to yield no other formulae as theorems. We shall establish presently that these two conditions are indeed satisfied. However, there are questions of a more general character which can be solved both with regard to the present calculus and with regard to many other calculi, including some with a more comprehensive vocabulary. One of these questions, which is of fundamental importance, is -

(iii) Is the calculus under consideration consistent (non-contradictory)? Can we perhaps obtain by its use not only tautologies and not only sentences which are true for some truth values of the variables and false for others, but even formulae which are identically false? If so, let X be such a formula and let Y be any other formula. Then it will be seen that $X \supset Y$ is a tautology. Hence, if (i) is satisfied, we can derive $X \supset Y$ as a theorem, and hence, we can derive Y, by 6.3. It follows that we can derive all formulae. Thus, for all calculi which include the present calculus of deduction, it is reasonable to define that such a calculus is consistent if not all formulae of the calculus are derivable in it as theorems. In the present case it is clear that if we establish (ii) then we have thereby proved also (iii).

Another important question is -

(iv) Is the calculus complete in the sense of being saturated or maximal? We shall say that the calculus is incomplete if it is possible to add a formula X as an axiom such that the resulting calculus is consistent, although X is not a theorem of the original calculus. It will be shown that the present calculus is indeed complete in this sense.

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We shall now sketch the proof of the fact that (i) above is satisfied. For this purpose it is required to show first that the following are theorems of the calculus.

7.1. $(p \vee (q \vee r)) \supset ((p \vee q) \vee r)$

$((p \vee q) \vee r) \supset (p \vee (q \vee r))$

7.2. $((p \vee q) \wedge (p \vee r)) \supset (p \vee (q \wedge r))$

$(p \vee (q \wedge r)) \supset ((p \vee q) \wedge (p \vee r))$

7.3. $(\sim p \vee \sim q) \supset \sim (p \wedge q)$

$\sim (p \wedge q) \supset (\sim p \vee \sim q)$

7.4. $(\sim p \wedge \sim q) \supset \sim (p \vee q)$

$\sim (p \vee q) \supset (\sim p \wedge \sim q)$

7.5. $p \supset \sim \sim p$

$\sim \sim p \supset p$

In this list, the connective of conjunction is to be regarded as an abbreviation; $p \wedge q$ stands for $\sim (\sim p \vee \sim q)$.

We shall derive the two formulae of 7.5. The derivation of the remaining formulae is left to the reader's (it is hoped, considerable) ingenuity.

In order to derive the first formula of 7.5 we first substitute $\frac{\sim p}{p}$ in (11) of section 6. This yields

(12) $\sim \sim p \vee \sim p$

Next substitute $\frac{\sim \sim p}{p}$, then $\frac{\sim p}{q}$ in (3) of section 6. This yields

(13) $\sim (\sim \sim p \vee \sim p) \vee (\sim p \vee \sim \sim p)$

Applying the rule of modus ponens 6.3 to (12), (13), we then obtain

(14) $\sim p \vee \sim \sim p$

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which is the first formula of 7.5.

Substituting $\frac{\sim p}{p}$ in (14) we obtain

(15) $\sim\sim p \vee \sim\sim\sim p$

Also, making the substitutions $\frac{\sim p}{p}$, $\frac{\sim\sim\sim p}{q}$, and $\frac{p}{r}$ in (4) we obtain

(16) $(\sim p \supset \sim\sim\sim p) \supset ((p \vee \sim p) \supset (p \vee \sim\sim\sim p))$

The implicans of (16) is, except for the notation, the same as (15) and so, again by 6.3.

(17) $(p \vee \sim p) \supset (p \vee \sim\sim\sim p)$

is a theorem. Now $p \vee \sim p$ is a theorem, by (3) and (11) (compare the passage from (12) to (14). Hence $p \vee \sim\sim\sim p$ is a theorem and hence, by the same argument,

(18) $\sim\sim\sim p \vee p$

is a theorem. But this is the second formula of 7.5.

It will be seen that the pairs of formulae 7.1 - 7.5 are all of the form $X \supset Y$, $Y \supset X$. To obtain further pairs of formulae of this type, we only have to substitute arbitrary formulae X, Y, Z, for p, q, r. Other pairs can be obtained by means of the following theorems which are again stated here without proof.

7.6. $(p \supset q) \supset (\sim q \supset \sim p)$

7.7. $(p \supset q) \supset ((p \vee r) \supset (q \vee r))$

7.8. $(p \supset q) \supset ((r \wedge p) \supset (r \wedge q))$

$(p \supset q) \supset ((p \wedge r) \supset (q \wedge r))$

For example, if $X \supset Y$, $Y \supset X$ are theorems then we see from axiom (4) and 7.7 that $(Z \vee X) \supset (Z \vee Y)$, $(Z \vee Y) \supset (Z \vee X)$, and again $(X \vee Z) \supset (Y \vee Z)$, $(Y \vee Z) \supset (X \vee Z)$ also are theorems. Similarly, if $X \supset Y$ and $Y \supset X$ are theorems, then by 7.6 $\sim Y \supset \sim X$ and $\sim X \supset \sim Y$ also are theorems.

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More generally, we have the following rule of replacement.

7.9. Let $W(p)$ be a formula which contains the propositional variable p, possibly more than once. We write $W(X)$, $W(Y)$ for the formula obtained from $W(p)$ by replacing p everywhere by X and Y, respectively. Then if $X \supset Y$ and $Y \supset X$ are theorems of the calculus, so are the formulae $W(X) \supset W(Y)$ and $W(Y) \supset W(X)$.

Proof. Suppose first that $W(p)$ contains p only once. Then $W(p)$ is obtained from p by applying to p the operations of negation, $\sim$, and of disjunction with some other formula (which does not contain p)successively, in a specified manner. Also, $W(X)$ and $W(Y)$ are obtained by applying the same operations to X and Y respectively. As we have just seen, theorems (4), 7.6, and 7.7 ensure that at each step we obtain formulae X', Y', such that $X' \supset Y'$ and $Y' \supset X'$ are theorems. This establishes 7.9 in the particular case that p occurs only once in $W(p)$. If p occurs more than once, then we replace p in each occurrence by a different variable p_i which does not occur elsewhere in W, X, or Y. The resulting formula may be written $W'(p_1, p_2, \ldots p_k)$. By what has already been proved, we have

$$W'(X, p_2 \ldots , p_k) \supset W'(Y, p_2, \ldots p_k)$$

and

$$W'(Y, p_2, \ldots , p_k) \supset W'(X, p_2, \ldots , p_k)$$

and moreover

$$W'(X, X, \ldots , p_k) \supset W'(Y, X, \ldots p_k)$$

$$W'(Y, X, \ldots , p_k) \supset W'(X, X, \ldots , p_k)$$

$$W'(Y, Y, \ldots , p_k) \supset W'(Y, X, \ldots , p_k)$$

$$W'(Y, X, \ldots , p_k) \supset W'(Y, Y, \ldots , p_k)$$

Hence, by (5) and 6.3

$$W'(X, X, \ldots p_k) \supset W'(Y, Y, \ldots , p_k)$$

and

$$W'(Y, Y, \ldots p_k) \supset W'(X, X, \ldots , p_k)$$

Continuing in this manner, we obtain

$$W(X) \supset W(Y) \quad , \quad W(Y) \supset W(X).$$

Finally, we require the following rules.

7.10. For any formula X, $\sim X \vee X$ is a theorem, in particular $\sim q \vee q$ is a theorem for any variable q.

7.11. For any theorem X, the formula $X \vee Y$ also is a theorem.

7.12. If $X \vee Y$ is a theorem then $Y \vee X$ also is a theorem.

7.13. If $X \vee (Y \vee Z)$ is a theorem then $(X \vee Y) \vee Z$ also is a theorem; if $(X \vee Y) \vee Z$ is a theorem, then $X \vee (Y \vee Z)$ is a theorem.

7.14. If X and Y are theorems then $X \wedge Y$ also is a theorem.

7.10 follows directly from (11), 7.11 follows from (2) and 6.3, 7.12 follows from (3) and 6.3, 7.13 follows from 7.1 and 6.3. 7.14 requires the prior derivation of the theorem $p \supset (q \supset (p \wedge q))$ which will be omitted.

Now let X be a formula which is a disjunction of propositional variables and (or) of the negation of each variable taken in any order and in any manner of association. Suppose that X represents a tautology, then X contains at least one variable, p say, together with its negation, $\sim p$. We denote by $q_1, \ldots, q_n$ the remaining variables of X, or their negations, if the latter appear in X, and we use a separate q_i for each occurrence of a variable (possibly again p or $\sim p$). Then the formula

$$X^* = (\ldots (((\sim p \vee p) \vee q_1^*) \vee q_2^*) \vee \ \ldots \ \vee q_n^*)$$

differs from X (if at all) only in the order of the variables and in the manner of their association in disjunction.

Now X^* is a theorem by rules 7.10 and 7.11. But X can be obtained from X^* by applying the associative and commutative laws to the disjunctions. Hence X also is a theorem, by 7.12 and 7.13.

Next, let X be a formula which is in conjunctive normal form (see section 4), and which represents a tautology. Then the conjuncts of X are disjunctions of the type just considered and each of them is a tautology and hence a theorem. If follows that X also is a theorem, by 7.14.

Now let X be any formula which represents a tautology. It was shown in section 4 that by means of the equivalences 3.2, 3.4, 3.5, 3.7, 3.8, 3.9, we can transform X into a formula X^* in conjunctive normal form such that X^* also represents a tautology. It follows that X^* is a theorem of our calculus. In order to deduce from this fact that X also is a theorem, we only have to apply the procedure of section 7 "in reverse". Now the possibility of carrying out this procedure follows directly from 7.1 - 7.5 together with the rule of replacement 7.9. For example, it follows from 7.3 in conjunction with 7.9 that in any theorem we may replace $\sim X \vee \sim Y$ by $\sim(X \wedge Y)$. In this way we may reduce X^* to X step by step, showing that X also is a theorem. This settles (i) above.

Next, we wish to show that all theorems of the calculus of deduction represent tautologies. For this purpose, we check first that the axioms (1) - (4) in section 6 are tautologies. Next we reflect that the substitutions of an arbitrary formula in a tautology yields a tautology. Finally, we observe that the rule of modus ponens when applied to tautologies X, $X \supset Y$ yields a tautology Y. This shows that the derivation of new theorems by means of 6.2, 6.3, can only lead to tautologies. Thus, we have settled (ii) (see the beginning of this section), and thereby, (iii).

We shall now show that the calculus is complete in the sense defined under (iv).

Let X be a formula which is not a theorem of the deductive calculus considered so far, and hence, is not a tautology. We have to show that the addition of X to the axioms (1) - (4) renders the system contradictory.

Let X^* be a formula in conjunctive normal form, such that $X \supset X^*$ and $X^* \supset X$ are tautologies and hence theorems. We know how to determine X^* by the rules of section (3) (compare an earlier argument in this section). Then X^* becomes a theorem of the augmented calculus, although it is not a tautology. Moreover, we may assume that X^* is of the form $X_1 \wedge X_2$ where X_1 is a disjunction of variables and (or) of the negation of such variables, such that no variable occurs in X_1 together with its negation (since X^* is not a tautology.) Then X_1 is a theorem, in view of the tautology $p \wedge q \supset p$, i.e. $(\sim p \vee \sim q) \vee p$.

Now suppose that p is a variable which does not occur in X, and substitute p for each variable of X_1 which occurs without the negation sign, and $\sim p$ for each variable which occurs with the negation sign. Write p for the double negations $\sim \sim p$ which appear after the substitutions. The resulting formula, X_3, is still a theorem by the rule of replacement, 7.9.

X_3 is a (repeated) disjunction of the single variable p. Hence, $X_3 \supset p$ is a tautology, and therefore a theorem, and p is a theorem. But if so, then any other formula Y is a theorem, since it can be obtained from p by means of the substitution $\frac{Y}{p}$.

This completes the discussion of the points raised at the beginning of this section. There is another question which is of a somewhat less fundamental character, although historically it is the question most frequently discussed in axiomatics - the question of the independence of the axioms. It can be shown that the four axioms (1) - (4) are indeed independent. That is to say, none of them can be derived from the remaining three axioms by means of 6.2 and 6.3.

8. Boolean Algebras. While the deductive calculus which has been explained in the last two sections is a step in the direction of the algebraisation of logic, it cannot be said to be in line with the "classical" axiomatic theories of algebra such as the theory of groups or the theory of rings. We now present a corresponding theory for the logic of propositions.

A Boolean algebra is a non empty set of objects B in which the following two operations are defined.

For every a, b ε B, there exists a uniquely determined element $c = a \cup b$ (read "a cup b") and for every a ε B there exists a uniquely determined $b = a'$ (read "complement of a") such that the following conditions are satisfied.

8.1. For any a, b, ε B, $(a \cup b) \cup c = a \cup (b \cup c)$.

This is the associative law for the cup operation. It follows that the generalized associative law is satisfied for any set of $n \geq 3$ elements, and we may write

$a \cup b \cup c = a \cup (b \cup c) = (a \cup b) \cup c,$

$a \cup b \cup c \cup d = a \cup (b \cup c \cup d) = (a \cup b \cup c) \cup d = (a \cup b) \cup$

$(c \cup d) = a \cup (b \cup c) \cup d$, etc.

8.2. For any $a, b \varepsilon B$, $a \cup b = b \cup a$.

This is the commutative law.

8.3. For any $a, b, c \varepsilon B$, $a \cup b = a$ entails $a \cup b' = c \cup c'$.

8.4. For any $a, b, c \varepsilon B$, $a \cup b' = c \cup c'$ entails $a \cup b = a$.

(Except for dualisation, this is essentially the system of axioms given by P.C. Rosenbloom in "The Elements of Mathematical Logic," Dover Publications, New York, 1950).

The following two properties of Boolean Algebras are immediate consequences of the axioms.

8.5. $a \cup a = a$.

For the proof, substitute a for b and a for c in 8.4. This yields $a \cup a' = a \cup c'$ in the hypothesis and $a \cup a = a$ in the conclusion.

8.6. $a \cup a'$ is the same for all $a \varepsilon B$.

For the proof, substitute a for b in 8.3. This yields $a \cup a = a$ in the hypothesis (which is true, by 8.5) and $a \cup a' = c \cup c'$ in the conclusion.

We denote this uniquely determined element by V, so that $V = a \cup a'$ for all $a \varepsilon B$. The complement of V will be denoted by Λ, so that $\Lambda = V'$.

We write $b \leq a$ if $a \cup b = a$, for any $a, b \varepsilon B$. By 8.3 and 8.4, an equivalent condition is $a \cup b' = V$. If $a \leq b$ and $b \leq a$ then $a \cup b = a = b$, and so $a = b$. Also, $a \leq a$, by 8.5 and $a \leq b$,

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$b \leq c$ entail $a \leq c$. For according to the assumption $a \cup b = b$, $b \cup c = c$, and so

$$a \cup c = a \cup (b \cup c) = (a \cup b) \cup c = b \cup c = c$$

i.e. $a \leq c$. This shows that the relation $\leq$ defines a <u>partial ordering</u> in B.

8.7. For all $a \in B$, $a \leq V$.

For $a \cup V = a \cup (a \cup a') = (a \cup a) \cup a' = a \cup a' = V$.

8.8. For all $a \in B$, $a'' = a$.

Proof. We prove $a \leq a''$ and $a'' \leq a$. Indeed

$a'' \cup a' = a' \cup a'' = a' \cup (a')' = V$ and so

8.9. $\quad a \leq a''$

by the second condition for the relation $\leq$. On the other hand, substituting a' and a'' in turn in 8.9 we obtain the relations

$$a' \leq a''' \text{ and } a'' \leq a''''.$$

Combining the latter relation with 8.8, we obtain $a \leq a''''$, i.e. $a'''' \cup a' = V$, $a' \cup a'''' = V$, $a''' \leq a'$. This yields $a''' = a'$, and further $a \cup a''' = a \quad a' = V$, $a'' \leq a$. Combining the last relation with 8.9, we obtain 8.8.

8.10. For all $a, b \in B$, $a \leq b$ entails $b' \leq a'$, and vice versa.

Proof. The assumption is $b \cup a' = V$. Now by 8.2 and 8.8, this is equivalent to $a' \cup b'' = V$, i.e. $b' \leq a'$. Conversely, if $b' \leq a'$ then by what has already been shown $a'' \leq b''$, i.e. $a \leq b$.

8.11. For all $a, b, c \in B$, $a \leq b$ entails $a \cup c \leq b \cup c$.

For $(a \cup c) \cup (b \cup c) = (a \cup b) \cup c$. It follows that if $a \leq b$, i.e. $a \cup b = b$, then $(a \cup c) \cup (b \cup c) = b \cup c$, $a \cup c \leq b \cup c$.

An example of a Boolean algebra is provided by the set of all subsets, a, b, c, ... of any given set A, if $a \cup b$, a' are interpreted as union and complement in the set theoretical sense. We shall now establish a connection between Boolean algebras and the deductive calculus of the preceding sections.

Let F be the set of formulae of such a calculus. We define a relation $X \simeq Y$ in F by the condition that $X \supset Y$ and $Y \supset X$ be theorems of the calculus (or, which is the same, that $X \supset Y$ and $Y \supset X$ represent tautologies.) It is not difficult to check that $\simeq$ is an equivalence. Moreover, the relation is substitutive with regard to the application of the connectives of the calculus $\vee$ and $\sim$. That is to say, if $X \simeq X'$ and $Y \simeq Y'$, then $\sim X \simeq \sim X'$ and $X \vee Y \simeq X' \vee Y'$. Let B be the set of equivalence classes (a, b, c, ...) of F with respect to the relation $\simeq$. In B, introduce the operations $\cup$ and $'$ by the definitions

$a \cup b = c$ if $X \vee Y \simeq Z$ for some (and hence for all) $X \in a$, $Y \in b$, $Z \in c$; and

$a' = b$ if $\sim X \simeq Y$ for some (and hence for all) $X \in a$, $Y \in b$.

It is not difficult to verify that these operations ($\cup$, $'$) do indeed yield unique results. We claim that they turn B into a Boolean algebra. In view of the associativity and commutativity of the disjunction, it is in fact immediate that 8.1 and 8.2 are satisfied. Coming next to 8.3, let $a, b, c \in B$, $X \in a$, $Y \in b$, $Z \in c$ and suppose that $a \cup b = a$. Then $X \vee Y \simeq X$ and we have to show that $X \vee \sim Y \simeq Z \vee \sim Z$. Now $Z \vee \sim Z$ is a tautology, and for any other formula, W say, we have $Z \vee \sim Z \simeq W$ if and only if W also is a tautology. Thus we only have to show that $X \vee \sim Y$ is a tautology provided $X \vee Y \simeq X$. Now the assumption implies that $(X \vee Y) \supset X$ is a tautology, i.e. $\sim (X \vee Y) \vee X$, and hence, that $(\sim X \wedge \sim Y) \vee X$ and $(\sim X \vee X) \wedge (\sim Y \vee X)$ and $\sim Y \vee X$ and $X \vee \sim Y$ all are tautologies. The last-mentioned fact shows that 8.3 holds.

In order to prove that 8.4 is satisfied, suppose $a, b, c \in B$, $X \in a$, $Y \in b$, $Z \in c$, as before, and suppose that $X \vee \sim Y \simeq Z \vee \sim Z$, which is to say that $X \vee \sim Y$, i.e. $Y \supset X$, is a tautology. We have to show that in this case, $X \vee Y \simeq X$. But $X \supset (X \vee Y)$ is a tautology and so we only have to show that $(X \vee Y) \supset X$ also is one. But it is easy to check that $(Y \supset X) \supset ((X \vee Y) \supset X)$ is indeed a tautology, and so the fact that the implicans $Y \supset X$, of this formula is a tautology, entails the same for the implicate, $X \vee Y \supset X$. This proves that B is a Boolean algebra.

It will be seen that the relation $\simeq$ of this section is closely connected with the relation $\approx$ of section 3. Indeed, $X \simeq Y$ for the formulae X and Y if and only if $\approx$ holds between the corresponding truth functions. Thus we may say that, in a sense which can easily be made precise, the truth functions of section 3 also constitute a Boolean algebra.

Now let B be an arbitrary Boolean algebra. We consider formal expressions $f(x_1, \ldots, x_n)$ which are obtained from $x_1, \ldots x_n$ by the repeated application of the operations of B.

$$(x_1 \cup x_2)' \cup x_3'$$

is an example of such an expression. $f(x_1, \ldots, x_n)$ represents in an obvious way a function which is defined on B and which takes values on B. Thus, it will be obvious what is meant by $f(a_1, \ldots, a_n)$ where $a_1, \ldots, a_n \varepsilon$ B.

We associate with any given $f(x_1, \ldots, x_n)$ a formula $X = F(p_1, \ldots, p_n)$ of a deductive calculus, where X is obtained from $(x_1, \ldots, x_n)$ by replacing $x_1, \ldots, x_n$ by propositional variables $p_1, \ldots, p_n$ and by replacing $\cup$ and ' everywhere by $\vee$ and $\sim$ respectively. Thus the formula which corresponds to $(x_1 \cup x_2)' \cup x_3'$ is $(p_1 \vee p_2) \vee \sim p_3$. We claim that if X is a theorem of the deductive calculus (i.e. a tautology) then

$$f(a_1, \ldots, a_n) = V \text{ for all } a_1, \ldots, a_n \varepsilon \text{ B}.$$

There are several proofs of this result. We shall derive it here from the axioms and rules of the deductive calculus.

As a first step, we prove that if we replace the variables p, q, r in (1) to (4) of 6.1 by arbitrary elements of B and the connectives $\vee$ and $\sim$ by $\cup$ and ' respectively, then the resulting expressions are equal to V in B. Indeed, we have, with regard to (1)

$$(a \cup a)' \cup a = a' \cup a = a \cup a' = V.$$

Also, with regard to (2),

$$a' \cup (a \cup b) = (a' \cup a) \cup b = (b' \cup b) \cup b = b' \cup b = V$$

while for (3) directly,

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$$(a \cup b)' \cup (a \cup b) = V.$$

The only case which causes some trouble is (4). We prove first, for arbitrary $a, c \in B$.

8.12. $(c \cup a)' \cup c = a' \cup c$.

Indeed, $a \cup (c \cup a) = c \cup a$ and so $a \leq c \cup a$,

and furthermore, $(c \cup a)' \leq a'$ by 8.10. This implies $(c \cup a)' \cup c \leq a' \cup c$ by 8.11.

On the other hand, $(c \cup a)' \cup (c \cup a) = V$ and so $a'' \cup ((c \cup a)' \cup c) = V$.

This shows that $a' \leq (c \cup a') \cup c$. Hence, by 8.11 $a' \cup c \leq ((c \cup a') \cup c) \cup c = (c \cup a') \cup c$.

Thus $a' \cup c \leq (c \cup a)' \cup c$ establishing 8.12.

Referring to 6.1 (4), we have to prove that

$$(a' \cup b)' \cup ((c \cup a)' \cup (c \cup b)) = V$$

is true for all $a, b, c \in B$. By 8.2 - 8.4, this is equivalent to

8.13 $(a' \cup b) \cup (c \cup a)' \cup (c \cup b) = (c \cup a)' \cup (c \cup b)$.

Now, from 8.12, 8.11,

$$(a' \cup b) \cup (c \cup a)' \cup (c \cup b) = a' \cup c \cup (c \cup a)' \cup b = (c \cup a)' \cup (c \cup b).$$

This establishes 8.13, and hence 6.1(4).

Any other theorem of the deductive calculus is obtained from 6.1, (1) = (4), by the application of 6.2, 6.3. Now it is clear that the application of a substitution to the left hand side of an identity

$$f(a, b, c, \ldots) = V$$

yields another identity with the same right hand side. On the other hand, if

$$f(a, b, c, \ldots) = V$$

and $(f(a, b, c, \ldots))' \vee g(a, b, c, \ldots) = V$

then $V = f(a, b, c, \ldots) \leq g(a, b, c, \ldots)$. But $g(a, b, c, \ldots) \leq V$ by 8.7 and so $g(a, b, c, \ldots) = V$. This shows that the application of 6.3 also leads to identities of the required type. This completes the proof.

We have as a corollary that if $X \simeq Y$ for two formulae of the deductive calculus then the corresponding expressions in the Boolean algebra define the same function. For if $X \supset Y$ and $Y \supset X$ are theorems (tautologies) then $f(a, b, c, \ldots) \leq g(a, b, c, \ldots)$ and $g(a, b, c, \ldots) \leq f(a, b, c, \ldots)$ in B and hence

$$f(a, b, c, \ldots) = g(a, b, c, \ldots).$$

Let B be a Boolean algebra. For any a, b, ε B we define a new relation $a \cap b$ (read "a cap b").

8.14. $a \cap b = (a' \cup b')'$

It is obvious that this operation is commutative. Moreover,

$$(a \cap b) \cap c = ((a' \cup b')'' \cup c')' = (a' \cup b' \cup c')' = a \cap (b \cap c)$$

so that the operation is associative. Also

8.15. $a \cap a' = (a' \cup a'')'' = (a' \cup a)' = V' = \Lambda$.

Consider now the condition $a \cap b = a$. This is equivalent to $(a' \cup b')' = a$ and hence, to $a' \cup b' = a'$, and to $b' \leq a'$, and, finally, to $a \leq b$. On the other hand, if $a \cap b' = c \cap c'$, for some a, b, c, ε B, then $(a' \cup b'')' = \Lambda$, $a' \cup b = V$, $a \leq b$. Thus, for any a, b, c, ε B, $a \cap b = a$ if and only if $a \cap b' = c \cap c'$.

It follows from the above that B satisfies the axioms 8.1 - 8.4 not only for the operations $\cup$ and $'$ but also if we replace the former operation by $\cap$. Moreover, let us define an operation $\cup^*$ by

$$a \cup^* b = (a' \cap b')'$$

Then $a \cup^* b = (a'' \cup b'')'' = a \cup b$ and so $\cup^*$ coincides with $\cup$. Thus, $\cup$ is obtained from $\cap$, mutatis mutandis, in exactly the same way as $\wedge$ is obtained from $\vee$. We conclude that if a general law or theorem is derived for all Boolean algebras in terms of the operations $\cup$, $\cap$, $'$, then another general law is obtained by replacing $\cup$ everywhere by $\cap$ and vice versa. Moreover, we may include the possibility that the former law includes reference to the element $V = a \cup a'$ in which case this element has to be replaced in the second version by $\Lambda = a \cap a'$. This is the principle of duality for Boolean algebras. In particular, if we have an identity between two expressions produced as above from $\cup$, $\cap$, and $'$, so

8.16. $f(a_1, \ldots, a_n, \cup, \cap, ', V) = g(a_1, \ldots, a_n, \cup, \cap, ', V)$

which holds for all $a_1, \ldots, a_n \in B$, then we obtain another identity by replacing $\cup$, $\cap$, $'$, V on both sides by $\cap$, $\cup$, $'$, Λ respectively, in symbols

8.17. $f(a_1, \ldots, a_n, \cap, \cup, ', \Lambda) = g(a_1, \ldots, a_n, \cap, \cup, ', \Lambda)$

In view of the connection between Boolean algebras and the propositional calculus which was explained above, this leads to a corresponding result known as the principle of duality of the propositional calculus, which we may formulate either for the axiomatic approach or in terms of truth functions. In view of $p \wedge q \approx (\sim p \vee \sim q)$, we see that the connective of conjunction corresponds to the cup operation. Thus we have, in terms of truth functions -

Let $f(p_1, \ldots, p_n, \vee, \wedge, \sim)$ and $g(p_1, \ldots, p_n, \vee, \wedge, \sim)$ be two truth functions in which we have displayed the connectives. Then the equivalence

$$f(p_1, \ldots, p_n, \vee, \wedge, \sim) \approx g(p_1, \ldots, p_n, \vee, \wedge, \sim)$$

entails that at the same time

$$f(p_1, \ldots, p_n, \wedge, \vee, \sim) \approx g(p_1, \ldots, p_n, \wedge, \vee, \sim).$$

9. Ideals in Boolean algebras. Let B, B^* be two Boolean algebras and let $a^* = f(a)$, $a \to a^*$ be a mapping of all elements of B into B^* such that the following conditions are satisfied for all a, b in B.

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9.1 $f(a \cup b) = f(a) \cup f(b)$

$f(a') = (f(a))'$

Such a mapping is said to be homomorphic, in agreement with the terminology current elsewhere in algebra.

Then

$$f(V) = f(a \cup a') = f(a) \cup (f(a))' = \overline{V}$$

where $\overline{V} = c \cup c'$ for all $c \in B^*$. Thus, $V \rightarrow \overline{V}$ in the homomorphism.

Let J be the set of all elements $a \in B$ such that $a \rightarrow \overline{V}$. J is not empty since $V \in J$. Also, if $f(a) = \overline{V}$, $f(b) = \overline{V}$ then

$$f(a \cap b) = f((a' \cup b')') = (f(a') \cup f(b'))'$$

$$= ((f(a))' \cup (f(b))')' = (\overline{V}' \cup \overline{V}')' = \overline{V}.$$

Hence,

9.2. For any $a, b \in B$, if $a \in J$ and $b \in J$ then $a \cap b \in J$. Also, if $f(a) = \overline{V}$, then for any $b \in B$,

$$f(a \cup b) = f(a) \cup f(b) = \overline{V} \cup f(b) = \overline{V}.$$ Hence,

9.3. For any $a, b, \in B$, if $a \in J$ then $a \cup b \in J$.

A non-empty subset J of a Boolean algebra B is said to be an <u>ideal</u> if it satisfies conditions 9.2 and 9.3. 9.3 is equivalent to the following:

9.4. For any $a, c \in B$, if $a \in J$ and $a \leq c$, then $c \in J$.

Proof. Suppose that 9.3 is satisfied, and $a \in J$. Then $a \cup c = c$ if $a \leq c$, and so $a \cup c = c \in J$, by 9.3. On the other hand $a \leq a \cup b$. Hence, if $a \in J$, and 9.4 is satisfied, then 9.3 also is satisfied, as we can see readily by taking $c = a \cup b$ in 9.4.

(to be continued)

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