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NONEXPANSIVE MAPPINGS AND EXPANSIVE MAPPINGS ON THE UNIT SPHERES OF SOME F-SPACES

DONG-NI TAN

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Abstract

This paper gives a characterization of nonexpansive mappings from the unit sphere of $\ell^\beta(\Gamma)$ onto the unit sphere of $\ell^\beta(\Delta)$ where $0 < \beta \leq 1$. By this result, we prove that such mappings are in fact isometries and give an affirmative answer to Tingley's problem in $\ell^\beta(\Gamma)$ spaces. We also show that the same result holds for expansive mappings between unit spheres of $\ell^\beta(\Gamma)$ spaces without the surjectivity assumption.

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1. Introduction

A mapping V between two metric spaces (X, d_X) and (Y, d_Y) is called nonexpansive if it is a 1-Lipschitz map. That is,

$$d_Y(V(x), V(y)) \leq d_X(x, y) \quad \forall x, y \in X. \tag{1.1}$$

The mapping V is called an isometry if equality holds in (1.1) for all $x, y \in X$, and it is called expansive if '$\leq$' is replaced by the inverse inequality '$\geq$'.

By a direct compactness argument or by Freudenthal and Hurewicz's result [9], every nonexpansive map from a compact metric space onto itself must be an isometry. This does not always hold with the assumption of compactness replaced by boundedness in infinite-dimensional metric linear spaces. For example, a map $T : B(\ell^p) \to B(\ell^p)$ defined by $T(\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots) = (\xi_2, \xi_3, \ldots, \xi_n, \ldots)$ for all $\{\xi_n\}_{n\geq 1}$ in $B(\ell^p)$ where $B(\ell^p)$ denotes the unit ball of ℓ^p and $0 < p \leq \infty$ is such a nonexpansive but not isometric map from $B(\ell^p)$ onto itself. However, what interests us is such maps defined only on the unit sphere, which can be connected with the isometric extension problem raised by Tingley in [12] and described as follows.

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Let E and F be normed spaces with unit spheres $S(E)$ and $S(F)$, respectively. Suppose that $V_0 : S(E) \to S(F)$ is an onto isometry. Is there a linear isometry $V : E \to F$ such that $V|_{S(E)} = V_0$?

In recent years, Ding and his students have been working on this topic and have obtained many important results (see [1–7, 10, 13, 15]).

Ding [2] showed that every onto nonexpansive map between unit spheres of Hilbert spaces is an isometry and answered Tingley's problem affirmatively for Hilbert spaces. In recent work [11], the author proved that the only nonexpansive mappings from the unit sphere of $\mathcal{L}^\infty(\Gamma)$-type spaces (including c_{00}, c, ℓ^∞) onto the unit sphere of $\mathcal{L}^\infty(\Delta)$ are those arising from a bijection between Δ and Γ and a sign pattern. This result yields the fact that such maps are isometries and an affirmative answer to Tingley's problem for $\mathcal{L}^\infty(\Gamma)$-type spaces. A similar result for $\ell^p(\Gamma)$ spaces where $1 < p < \infty$ can be obtained by combining the main result in [3] with that of [8]. For the case $p = 1$, Wang [14] established that every expansive map T from $S(\ell^1(\Gamma))$ onto $S(\ell^1(\Delta))$ with an additional condition $\bigcup_{\gamma\in\Gamma} \operatorname{supp} T(e_\gamma) = \Delta$ is an isometry and can be linearly and isometrically extended to $\ell^1(\Gamma)$. In this paper, we extend these results to F-spaces $\ell^\beta(\Gamma)$ where $0 < \beta \le 1$, and in the $\ell^1(\Gamma)$ case we point out that the condition $\bigcup_{\gamma\in\Gamma} \operatorname{supp} T(e_\gamma) = \Delta$ in [14] can be removed.

Throughout this paper, we consider spaces over the real field. Given a nonempty index set Γ, for every $0 < \beta \le 1$, the space

$$\ell^\beta(\Gamma) = \Big\{x = \{\xi_\gamma\}_{\gamma\in\Gamma} : \sum_{\gamma\in\Gamma} |\xi_\gamma|^\beta < \infty\Big\}$$

is known as an F-space with an F-norm $\|x\| = \sum |\xi_\gamma|^\beta$. As usual, for every $x = \{\xi_\gamma\}_{\gamma\in\Gamma} \in \ell^\beta(\Gamma)$, $\operatorname{supp} x = \{\gamma \in \Gamma : \xi_\gamma \neq 0\}$ and $S(\ell^\beta(\Gamma))$ denotes the unit sphere of $\ell^\beta(\Gamma)$.

2. Main results

LEMMA 2.1. *Let $x, y \in \ell^\beta(\Gamma)$. Then*

$$\|x + y\| = \|x\| + \|y\|$$

if and only if $\operatorname{supp} x \cap \operatorname{supp} y = \emptyset$ *for* $0 < \beta < 1$ *and* $x \cdot y \ge 0$ *for* $\beta = 1$, *where* $x \cdot y \ge 0$ *means* $x(\gamma) \cdot y(\gamma) \ge 0$ *for every* $\gamma \in \Gamma$.

PROOF. The proof in the case of $\beta = 1$ is trivial. For $0 < \beta < 1$, observe that the function $f(t) = t^\beta$ is strictly concave on $(0, \infty)$. It follows that

$$|\xi + \eta|^\beta \le |\xi|^\beta + |\eta|^\beta$$

for all $\xi, \eta \in \mathbb{R}$ and equality holds if and only if $\xi \cdot \eta = 0$. The desired result is easily obtained from this. □

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LEMMA 2.2. *Let $x \in S(\ell^\beta(\Gamma))$. Then for every $\gamma \in \Gamma$,*

$$\max\{\|x + e_\gamma\|, \|x - e_\gamma\|\} \geq 2^\beta.$$

PROOF. As $\|x\| = 1$, it is easy to see that

$$\max\{\|x + e_\gamma\|, \|x - e_\gamma\|\} = (|x(\gamma)| + 1)^\beta + 1 - |x(\gamma)|^\beta.$$

Since the function $\varphi(t) = (1+t)^\beta - t^\beta$ is decreasing on $[0, \infty)$, it follows that

$$(|x(\gamma)| + 1)^\beta + 1 - |x(\gamma)|^\beta = 1 + \varphi(|x(\gamma)|) \geq 1 + \varphi(1) = 2^\beta,$$

which completes the proof. □

LEMMA 2.3. *Let $T : S(\ell^\beta(\Gamma)) \to S(\ell^\beta(\Delta))$ be a nonexpansive map. For each $\delta \in \Delta$, if $\pm e_\delta \in T(S(\ell^\beta(\Gamma)))$, then there is a unique $\gamma \in \Gamma$ and a sign θ_δ such that*

$$T(\pm e_\gamma) = \pm\theta_\delta e_\delta.$$

PROOF. The hypothesis $\pm e_\delta \in T(S(\ell^\beta(\Gamma)))$ ensures that there exist $x, y \in S(\ell^\beta(\Gamma))$ such that $T(x) = e_\delta$ and $T(y) = -e_\delta$. We first claim that x and y are dependent, that is,

$$x = -y.$$

Assume that the claim is not true. Define a map $f : [0, 1] \to S(\ell^\beta(\Gamma))$ by

$$f(\lambda) = \frac{(1-\lambda)x + \lambda y}{\|(1-\lambda)x + \lambda y\|^{1/\beta}}.$$

It is clear that $\{f(\lambda) : \lambda \in [0, 1]\}$ is a connected path from x to y. Hence the map

$$\phi(\lambda) = \|T(f(\lambda)) + e_\delta\| - \|T(f(\lambda)) - e_\delta\|$$

is continuous on $[0, 1]$. Since $\phi(0) = 2^\beta$ and $\phi(1) = -2^\beta$, we can find $\lambda_0 \in (0, 1)$ such that $\phi(\lambda_0) = 0$, that is,

$$\|T(f(\lambda_0)) + e_\delta\| = \|T(f(\lambda_0)) - e_\delta\|.$$

The definition of the norm in $\ell^\beta(\Delta)$ yields $T(f(\lambda_0))(\delta) = 0$, and thus

$$\|T(f(\lambda_0)) + e_\delta\| = \|T(f(\lambda_0)) - e_\delta\| = 2.$$

This shows that

$$\|f(\lambda_0) - y\| = \|f(\lambda_0) - x\| = 2.$$

By Lemma 2.1 we get that for $0 < \beta < 1$, $\operatorname{supp} f(\lambda_0) \cap (\operatorname{supp} x \cup \operatorname{supp} y) = \emptyset$ and for $\beta = 1$, $f(\lambda_0) \cdot x \leq 0$ and $f(\lambda_0) \cdot y \leq 0$. This is impossible by the definition of f. Therefore the claim is proved.

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We next show that supp x is a singleton. If this does not hold, then there is a $\gamma_1 \in \Gamma$ satisfying $0 < |x(\gamma_1)| < 1$. Write $x_1 = x - 2x(\gamma_1)e_{\gamma_1}$. Then by the claim

$$\|T(x_1) - e_\delta\| = \|T(x_1) - T(x)\| \le \|x_1 - x\| = 2^\beta |x(\gamma_1)|^\beta < 2^\beta,$$
$$\|T(x_1) + e_\delta\| = \|T(x_1) - T(-x)\| \le \|x_1 + x\| = 2^\beta (1 - |x(\gamma_1)|^\beta) < 2^\beta.$$

This contradicts Lemma 2.2 and therefore supp x is a singleton.

Let $\{\gamma\} = \operatorname{supp} x$ and $\theta_\delta = x(\gamma)$. Noticing that the uniqueness of γ is easily obtained from the claim, this completes the proof. □

We are now ready to present one of our main results.

THEOREM 2.4. *Let $T : S(\ell^\beta(\Gamma)) \to S(\ell^\beta(\Delta))$ be a surjective nonexpansive map. Then T is an isometry and there is a family of signs $\{\theta_\delta\}_{\delta \in \Delta}$ and a bijection $\sigma : \Delta \to \Gamma$ such that, for any element $x \in S(\ell^\beta(\Gamma))$,*

$$T(x)(\delta) = \theta_\delta x(\sigma(\delta)) \quad \forall \delta \in \Delta. \tag{2.1}$$

PROOF. It is evident that T is an isometry if there is a family of signs $\{\theta_\delta\}_{\delta \in \Delta}$ and a bijection $\sigma : \Delta \to \Gamma$ such that (2.1) holds. Thus it suffices to prove this. By Lemma 2.3 we can define $\sigma : \Delta \to \Gamma$ and $\{\theta_\delta\}_{\delta \in \Delta}$ such that

$$T(\pm e_{\sigma(\delta)}) = \pm \theta_\delta e_\delta \quad \forall \delta \in \Delta. \tag{2.2}$$

It is obvious that σ is injective. To see that σ is surjective and that (2.1) holds, for every $y = \sum \eta_\delta e_\delta \in S(\ell^\beta(\Delta))$, take $x = \sum \xi_\gamma e_\gamma \in S(\ell^\beta(\Gamma))$ such that $T(x) = y$.

For any $\delta \in \Delta$ with $\xi_{\sigma(\delta)} \neq 0$,

$$\|y - \operatorname{sign}(\xi_{\sigma(\delta)})\theta_\delta e_\delta\| = |\eta_\delta - \operatorname{sign}(\xi_{\sigma(\delta)})\theta_\delta|^\beta + 1 - |\eta_\delta|^\beta.$$

On the other hand, clearly,

$$\|x - \operatorname{sign}(\xi_{\sigma(\delta)}) e_{\sigma(\delta)}\| = (1 - |\xi_{\sigma(\delta)}|)^\beta + 1 - |\xi_{\sigma(\delta)}|^\beta.$$

The fact that T is nonexpansive and (2.2) then give

$$\begin{aligned}(1 - |\eta_\delta|)^\beta - |\eta_\delta|^\beta &\le |\eta_\delta - \operatorname{sign}(\xi_{\sigma(\delta)})\theta_\delta|^\beta - |\eta_\delta|^\beta \\ &\le (1 - |\xi_{\sigma(\delta)}|)^\beta - |\xi_{\sigma(\delta)}|^\beta.\end{aligned} \tag{2.3}$$

Noticing that $\phi(t) = (1 - t)^\beta - t^\beta$ is decreasing on $[0, 1]$, we see that

$$|\eta_\delta| \ge |\xi_{\sigma(\delta)}|. \tag{2.4}$$

Thus if supp $x \subset \sigma(\Delta)$, then by (2.4),

$$1 = \sum_{\delta \in \Delta} |\xi_{\sigma(\delta)}| \le \sum_{\delta \in \Delta} |\eta_\delta| = 1.$$

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As a result,

$$|\eta_\delta| = |\xi_{\sigma(\delta)}| \tag{2.5}$$

and inequality (2.3) turning out to be an equality obviously implies that

$$\operatorname{sign}(\eta_\delta) = \operatorname{sign}(\xi_{\sigma(\delta)})\theta_\delta. \tag{2.6}$$

Since Equations (2.5) and (2.6) have already established that (2.1) holds for all $x \in S(\ell^\beta(\Gamma))$ satisfying $\operatorname{supp} x \subset \sigma(\Delta)$, to finish the proof we only need to show that σ is surjective. Suppose to the contrary that there is a $\gamma_0 \in \Gamma\backslash\sigma(\Delta)$. Choose $\delta_0 \in \operatorname{supp} T(e_{\gamma_0})$ and put

$$x_0^\pm = \frac{1}{2^{1/\beta}} e_{\gamma_0} \pm \frac{1}{2^{1/\beta}} e_{\sigma(\delta_0)} \quad \text{and} \quad \eta_{\delta_0}^\pm = T(x_0^\pm)(\delta_0).$$

It is easy to see from Lemma 2.3 that $\operatorname{supp} T(x_0^+)$ cannot be a singleton. Thus we can let $\delta_1 \in \operatorname{supp} T(x_0^+)$ satisfy $\delta_1 \neq \delta_0$. Then write

$$\eta_{\delta_1} = T(x_0^+)(\delta_1) \quad \text{and} \quad x_1 = \frac{1}{2^{1/\beta}} e_{\sigma(\delta_0)} - \frac{1}{2^{1/\beta}} \operatorname{sign}(\eta_{\delta_1})\theta_{\delta_1} e_{\sigma(\delta_1)}.$$

Note from the above argument that

$$T(x_1) = \frac{1}{2^{1/\beta}} \theta_{\delta_0} e_{\delta_0} - \frac{1}{2^{1/\beta}} \operatorname{sign}(\eta_{\delta_1}) e_{\delta_1}.$$

It follows that

$$\begin{aligned} 1 = \|x_0^+ - x_1\| &\geq \|T(x_0^+) - T(x_1)\| \\ &\geq \left|\eta_{\delta_0}^+ - \frac{1}{2^{1/\beta}}\theta_{\delta_0}\right|^\beta + \left|\eta_{\delta_1} + \frac{1}{2^{1/\beta}} \operatorname{sign}(\eta_{\delta_1})\right|^\beta \\ &> \left|\eta_{\delta_0}^+ - \frac{1}{2^{1/\beta}}\theta_{\delta_0}\right|^\beta + \frac{1}{2}. \end{aligned}$$

Thus $\operatorname{sign}(\eta_{\delta_0}^+) = \theta_{\delta_0}$. Similarly, we can also obtain $\operatorname{sign}(\eta_{\delta_0}^-) = -\theta_{\delta_0}$.

By (2.4), we have $|\eta_{\delta_0}^\pm| \geq (1/2)^{1/\beta}$ and observe that

$$\begin{aligned} 2^{\beta-1} = \|x_0^+ - x_0^-\| &\geq \|T(x_0^+) - T(x_0^-)\| \\ &\geq |\eta_{\delta_0}^+ - \eta_{\delta_0}^-|^\beta = (|\eta_{\delta_0}^+| + |\eta_{\delta_0}^-|)^\beta. \end{aligned} \tag{2.7}$$

Consequently,

$$\eta_{\delta_0}^+ = \frac{1}{2^{1/\beta}}\theta_{\delta_0} \quad \text{and} \quad \eta_{\delta_0}^- = -\frac{1}{2^{1/\beta}}\theta_{\delta_0}. \tag{2.8}$$

Moreover, this and the inequality becoming an equality in (2.7) imply that

$$T(x_0^+)(\delta) = T(x_0^-)(\delta) \quad \forall \delta \neq \delta_0. \tag{2.9}$$

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Now using the same technique as in Lemma 2.3, we define

$$\phi(\lambda) = \|T(f(\lambda)) - T(x_0^+)\| - \|T(f(\lambda)) - T(x_0^-)\|$$

for all $\lambda \in [0, 1]$ where $f(\lambda) = ((1-\lambda)x_0^+ + \lambda x_0^-)/(\|(1-\lambda)x_0^+ + \lambda x_0^-\|^{1/\beta})$.

Since ϕ is continuous on $[0, 1]$ and $\phi(0)\phi(1) < 0$, there is a $\lambda_0 \in (0, 1)$ such that

$$\|T(f(\lambda_0)) - T(x_0^+)\| = \|T(f(\lambda_0)) - T(x_0^-)\|.$$

Hence by the form of $T(x_0^{\pm})$ given by (2.8) and (2.9) we see that

$$T(f(\lambda_0))(\delta_0) = 0.$$

So

$$\|f(\lambda_0) - e_{\sigma(\delta_0)}\| \geq \|T(f(\lambda_0)) - T(e_{\sigma(\delta_0)})\| = 2$$

yields $f(\lambda_0)(\sigma(\delta_0)) = 0$.

It follows that $f(\lambda_0) = e_{\gamma_0}$, that is, $T(e_{\gamma_0})(\delta_0) = 0$. This contradicts the choice of δ_0. Thus the proof is complete. □

REMARK 2.5. In the case where $\dim(\ell^\beta(\Gamma)) < \infty$, that is, the cardinality of Γ is finite, the above conclusion that T is an isometry cannot be simply obtained by a compactness argument or Freudenthal and Hurewicz's result [9] which states that every nonexpansive map from a totally bounded metric space onto itself must be an isometry since the nonexpansive map is not assumed to be from $S(\ell^\beta(\Gamma))$ onto itself. The statement of Theorem 2.4 remains valid if we consider the quasi-Banach space consisting of the all the points $x = \{\xi_\gamma\}_{\gamma\in\Gamma} \in \ell^\beta(\Gamma)$ with the quasi-norm $\|x\|_\beta = (\sum |\xi_\gamma|^\beta)^{1/\beta}$ for $0 < \beta < 1$.

COROLLARY 2.6. *Every surjective nonexpansive mapping $T : S(\ell^\beta(\Gamma)) \to S(\ell^\beta(\Delta))$ can be extended to a linear surjective isometry on $\ell^\beta(\Gamma)$.*

REMARK 2.7. We can see from Lemma 2.3 that the surjection assumption of T in Theorem 2.4 and Corollary 2.6 in fact can reduce to $\{\pm e_\delta\}_{\delta\in\Delta} \subset T(S(\ell^\beta(\Gamma)))$. On the other hand, by Theorem 2.4, we have in fact shown that every nonexpansive map T from $S(\ell^\beta(\Gamma))$ onto $S(\ell^\beta(\Delta))$ ensures that for every $\gamma \in \Gamma$, supp $T(e_\gamma)$ is a singleton. However, without the assumption of surjectivity or $\{\pm e_\delta\}_{\delta\in\Delta} \subset T(S(\ell^\beta(\Gamma)))$ this is not always true. For example, let $T : S(\ell^\beta_{(2)}) \to S(\ell^\beta_{(3)})$ be defined by

$$T(\xi_1 e_1 + \xi_2 e_2) = \xi_1(1/2^{1/\beta} e_1 + 1/2^{1/\beta} e_2) + \xi_2 e_3,$$

where $\{\xi_1, \xi_2\} \subset \mathbb{R}$ satisfies $|\xi_1|^\beta + |\xi_2|^\beta = 1$. Then T is an isometry, but $e_1, e_2 \notin T(S(\ell^\beta_{(2)}))$ and supp $T(e_1) = \{1, 2\}$. Considering this example, we give a more general result for expansive maps on $S(\ell^\beta(\Gamma))$.

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THEOREM 2.8. *Let T be an expansive map from $S(\ell^\beta(\Gamma))$ to $S(\ell^\beta(\Delta))$ such that $T(S(\ell^\beta(\Gamma))) = S(F)$, where F is a linear closed subspace of $\ell^\beta(\Delta)$. Then T is an isometry and can be extended to a linear isometry on $\ell^\beta(\Gamma)$.*

PROOF. Since $\|T(x) - T(y)\| \geq \|x - y\|$ for all $x, y \in S(\ell^\beta(\Gamma))$, we see that T is injective and its inverse T^{-1} is nonexpansive. Note that $T^{-1}(T(e_\gamma)) = e_\gamma$ and $T^{-1}(T(-e_\gamma)) = -e_\gamma$ holds for all $\gamma \in \Gamma$. By the same argument as in Lemma 2.3, we deduce that

$$T(-e_\gamma) = -T(e_\gamma). \tag{2.10}$$

It follows that, for every $\gamma_1 \neq \gamma_2$,

$$\|T(e_{\gamma_1}) + T(e_{\gamma_2})\| \geq \|e_{\gamma_1} + e_{\gamma_2}\| = 2.$$

Hence

$$\|T(e_{\gamma_1}) + T(e_{\gamma_2})\| = \|T(e_{\gamma_1}) - T(e_{\gamma_2})\| = 2,$$

which together with Lemma 2.1 guarantees that

$$\operatorname{supp} T(e_{\gamma_1}) \cap \operatorname{supp} T(e_{\gamma_2}) = \emptyset. \tag{2.11}$$

Thus $y = \sum \xi_\gamma T(e_\gamma)$ has norm one for every $\sum \xi_\gamma e_\gamma \in S(\ell^\beta(\Gamma))$. Since F is a linear closed subspace and $T(S(\ell^\beta(\Gamma))) = S(F)$, it follows that $y \in T(S(\ell^\beta(\Gamma)))$. Hence there is an element $x = \sum \alpha_\gamma e_\gamma \in S(\ell^\beta(\Gamma))$ such that $T(x) = y$.

For any $\xi_\gamma \neq 0$, by (2.10) and (2.11) we get

$$\|T(x) - \operatorname{sign}(\xi_\gamma) T(e_\gamma)\| = (1 - |\xi_\gamma|)^\beta + 1 - |\xi_\gamma|^\beta.$$

Furthermore,

$$\|x - \operatorname{sign}(\xi_\gamma) e_\gamma\| = |\operatorname{sign}(\xi_\gamma) - \alpha_\gamma|^\beta + 1 - |\alpha_\gamma|^\beta.$$

Thus by the fact that T is expansive,

$$|\operatorname{sign}(\xi_\gamma) - \alpha_\gamma|^\beta - |\alpha_\gamma|^\beta \leq (1 - |\xi_\gamma|)^\beta - |\xi_\gamma|^\beta. \tag{2.12}$$

It follows that $|\alpha_\gamma| \geq |\xi_\gamma|$. This yields $1 = \sum |\alpha_\gamma|^\beta \geq \sum |\xi_\gamma|^\beta = 1$, which combined with (2.12) ensures that for every γ, $\alpha_\gamma = \xi_\gamma$ even if $\xi_\gamma = 0$. That is,

$$T\Big(\sum \xi_\gamma e_\gamma\Big) = \sum \xi_\gamma T(e_\gamma) \tag{2.13}$$

for every $\sum \xi_\gamma e_\gamma \in S(\ell^\beta(\Gamma))$.

Finally, by its property given by (2.13), T is clearly an isometry and the desired extension $\widetilde{T}$ defined by

$$\widetilde{T}\Big(\sum \overline{\xi}_\gamma e_\gamma\Big) = \sum \overline{\xi}_\gamma T(e_\gamma) \quad \forall \sum \overline{\xi}_\gamma e_\gamma \in \ell^\beta(\Gamma).$$

It is plain that $\widetilde{T}$ is a linear isometry on $\ell^\beta(\Gamma)$ and its restriction to $S(\ell^\beta(\Gamma))$ is just T. The proof is complete. □

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REMARK 2.9. If $\beta = 1$, then some minor modifications of the previous example give a counterexample showing that there is an expansive map or, to be precise, an isometry between $S(\ell^1(\Gamma))$ and $S(\ell^1(\Delta))$ which cannot be linearly extended to the whole space. In fact, let $T : S(\ell^1_{(2)}) \to S(\ell^1_{(3)})$ be defined by

$$T(\xi_1 e_1 + \xi_2 e_2) = \begin{cases} \xi_1(1/4e_1 + 3/4e_2) + \xi_2 e_3 & \text{if } \xi_1 \geq 0, \\ \xi_1(1/2e_1 + 1/2e_2) + \xi_2 e_3 & \text{otherwise,} \end{cases}$$

where $\{\xi_1, \xi_2\} \subset \mathbb{R}$ satisfies $|\xi_1| + |\xi_2| = 1$. It is easy to check that T does not satisfy the condition of Theorem 2.8 since $-T(S(\ell^1_{(2)})) \nsubseteq T(S(\ell^1_{(2)}))$, and that T is an isometry which cannot be linearly extended to $\ell^1_{(2)}$ because it is not even an odd operator.

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DONG-NI TAN, School of Mathematical Science and LPMC, Nankai University, Tianjin 300071, PR China
e-mail: 0110127@mail.nankai.edu.cn

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