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Numerical range and operators on locally convex spaces

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Two main topics are considered. First, a theory of numerical range is expounded for a class of operators on locally convex spaces, generalising in part the normed space theory, some ancillary questions concerning Banach space operators are considered, and some applications to b^*-algebras given. The second topic concerns some recalibration questions for locally convex spaces.

The first chapter is devoted to preliminaries, and summarises the main parts of the theory of numerical range for operators on normed spaces and elements of normed algebras; an expansive account of this theory is presented in [3]. There is also a compendium of notation associated with the notion of a *calibration* (defining family of seminorms) of a locally convex space, and some standard results on locally convex spaces.

The subject of the present theory of numerical range is the algebra of *quotient-bounded* operators with respect to a given calibration (operators bounded with respect to every seminorm). Chapter 2 begins with a review of general properties of these operators, and of the subalgebra of *universally bounded* operators. The spatial numerical range of a quotient-bounded operator is defined; for universally bounded operators this definition differs somewhat in spirit from that of Moore [8]. It is shown that the spatial numerical range has the Williams property: its closure contains the spectrum defined with respect to the algebra of quotient-bounded operators. Further, the closed convex hull of the spectrum may be approximated arbitrarily closely by the spatial numerical ranges corresponding to calibrations satisfying a certain kind of equivalence with

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the given calibration. This result generalises a fundamental property of numerical range in normed spaces. The algebra of quotient-bounded operators is a unital locally m-convex algebra with respect to a natural topology. It is thus possible, following [6], to define the algebra numerical range of a quotient-bounded operator, whose relation to the spatial numerical range generalises the corresponding relation in normed spaces. The class of universally bounded operators is characterised by the property of bounded numerical range. The salient points of this theory may be found in [5].

Chapter 3 is mainly a discussion of numerical ranges of unbounded operators on Banach spaces. It is shown that the Lumer numerical range of an everywhere-defined unbounded linear operator on a Banach space has a density property in the scalar field, and under certain conditions is actually dense. These results have been published in [4]; a stronger result appears in [2].

The results of Chapter 4 all pertain to recalibrations of a locally convex space. Some known results on universally bounded operators are summarised, from which emerges a topological characterisation of the class of all quotient-bounded operators on a locally convex space. Some spectral theory notions of Allan [1] are described for general unital locally m-convex algebras in terms of certain calibration-defined subalgebras, and on a parallel basis for the particular case of algebras of quotient-bounded operators. The underlying result here is a recalibration theorem generalising the classical renorming theorem for operator semigroups on normed spaces. This part of the theory culminates in a collection of results characterising the scalar multiples of the identity in general unital locally m-convex algebras, and likewise in quotient-bounded operator algebras, in terms of certain "boundedness" properties. *Normcomplete* locally convex spaces are characterised by the completeness of certain calibration-defined canonically-normed subspaces, and *pseudocomplete* locally m-convex algebras similarly. These same subspaces are realised as invariant subspaces for universally bounded operators, and used to determine a formula for the norm of such an operator. A large part of the results of this chapter will appear in [7].

The final chapter is concerned with b^*-algebras and locally convex generalisations of *hermitian* operators. Hermitian operators on commutative b^*-algebras are characterised, as in the case of commutative B^*-algebras, as multiplications by hermitian elements, and a representation of an arbitrary b^*-algebra as an algebra of quotient-bounded operators on a product of Hilbert spaces is obtained.

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