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THE INVISCID LIMIT OF THE MODIFIED BENJAMIN–ONO–BURGERS EQUATION

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Abstract

We prove that the modified Benjamin–Ono–Burgers equation is globally well-posed in H^s for $s > 0$. Moreover, we show that the solution of the modified Benjamin–Ono–Burgers equation converges to that of the modified Benjamin–Ono equation in the natural space $C([0, T]; H^s)$, $s \geq 1/2$, as the dissipative coefficient ϵ goes to zero, provided that the L^2 norm of the initial data is sufficiently small.

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1. Introduction

The purpose of this paper is to study the global well-posedness and the inviscid limit behaviour of the Cauchy problem for the modified Benjamin–Ono–Burgers (mBOB) equation

$$\begin{aligned} u_t + \mathcal{H}u_{xx} - \epsilon u_{xx} &= u^2 u_x, \\ u(x, 0) &= \phi(x), \end{aligned} \tag{1.1}$$

where $u(x, t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $0 < \epsilon \leq 1$ and $\mathcal{H}$ is the Hilbert transform:

$$\mathcal{H}u(x) = \frac{1}{\pi} \text{ p.v.} \int_{-\infty}^{+\infty} \frac{u(y)}{x - y}\, dy. \tag{1.2}$$

When the nonlinearity in (1.1) is $-u^2 u_x$, it can also be treated by our method.

Formally, letting $\epsilon = 0$, then (1.1) becomes the modified Benjamin–Ono (mBO) equation:

$$u_t + \mathcal{H}u_{xx} = u^2 u_x, \quad u(x, 0) = \phi(x). \tag{1.3}$$

Thus it is natural to conjecture that the solution of (1.1) converges to that of (1.3) as ϵ tends to zero in the natural space $C([0, T] : H^s)$. The same problem for the Benjamin–Ono–Burgers equation (with quadratic nonlinearity uu_x in (1.1)) was suggested by

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Tao [13], who proved that the Benjamin–Ono equation is globally well-posed in $H^1(\mathbb{R})$. The inviscid limit problems are very interesting from the physical viewpoint and have been studied by many authors [5, 15, 16]. The limit in the low regularity space was first studied by Guo and Wang [5] where they used the l^1-type $X^{s,b}$ structure.

In [2], Guo showed that (1.3) is globally well-posed for $\phi \in H^s$, $s \geq 1/2$, and $\|\phi\|_{L^2}$ sufficiently small. In this paper, we show that (1.1) is globally well-posed for $\phi \in H^s$, $s > 0$. In [14], Vento considered the Cauchy problem for dissipative Benjamin–Ono equations

$$\begin{aligned} u_t + \mathcal{H}u_{xx} + |D|^\alpha u + uu_x &= 0, \quad t > 0,\, x \in \mathbb{R}, \\ u(x,0) &= \phi(x), \end{aligned} \tag{1.4}$$

where $|D|^\alpha$ is the Fourier multiplier with symbol $|\xi|^\alpha$, $0 < \alpha \leq 2$. When $0 \leq \alpha < 1$, the author gave the ill-posedness in $H^s(\mathbb{R})$, $s \in \mathbb{R}$, in the sense that the flow map $u_0 \mapsto u$ (if it exists) fails to be $\mathbb{C}^2$ at the origin. For $1 < \alpha \leq 2$, the author proved the global well-posedness in $H^s(\mathbb{R})$, $s > -\alpha/4$. Comparing to [14], we mainly consider the situation $\alpha = 2$ and with nonlinearity $-u^2 u_x$. In [3], Guo considered the Cauchy problem for the dispersion generalized Benjamin–Ono equation

$$\begin{aligned} \partial_t u + |D|^{1+\alpha}\partial_x u + uu_x &= 0, \quad (x,t) \in \mathbb{R}^2, \\ u(x,0) &= u_0(x), \end{aligned} \tag{1.5}$$

where $0 \leq \alpha \leq 1$, and showed that (1.5) is locally well-posed in H^s for $s > 1 - \alpha$. The $\alpha = 0$ result of [3] follows from our estimates.

The main ingredients of our ideas are the methods in [5] combined with the new observation in [2] for the modified Benjamin–Ono equation. However, there are some new difficulties, since the resolution spaces are different from the one used in [5]. Fortunately, we can overcome these difficulties by using the ideas from [5, 6, 10] and some new techniques.

We now give some notation. Let $\eta_0 : \mathbb{R} \to [0,1]$ denote an even smooth function supported in $[-8/5, 8/5]$ and equal to 1 in $[-5/4, 5/4]$. For $k \in \mathbb{Z}$, let $\chi_k(\xi) = \eta_0(\xi/2^k) - \eta_0(\xi/2^{k-1})$, where χ_k is supported in $\{\xi : |\xi| \in [(5/8)\cdot 2^k, (8/5)\cdot 2^k]\}$ and

$$\chi_{[k_1,k_2]} = \sum_{k=k_1}^{k_2} \chi_k \quad \text{for any } k_1 \leq k_2 \in \mathbb{Z}.$$

For simplicity of notation, let $\eta_k = \chi_k$ if $k \geq 1$ and $\eta_k \equiv 0$ if $k \leq -1$. For $k_1 \leq k_2 \in \mathbb{Z}$, let

$$\eta_{[k_1,k_2]} = \sum_{k=k_1}^{k_2} \eta_k \quad \text{and} \quad \eta_{\leq k_2} = \sum_{k=-\infty}^{k_2} \eta_k.$$

For $k \in \mathbb{Z}$, let P_k denote the operators on $L^2(\mathbb{R})$ defined by

$$\widehat{P_k u}(\xi) = \chi_k(\xi)\widehat{u}(\xi).$$

By a slight abuse of notation, we also define the operators P_k on $L^2(\mathbb{R}\times\mathbb{R})$ by formulas $\mathcal{F}(P_k u)(\xi,\tau)=\chi_k(\xi)\mathcal{F}(u)(\xi,\tau)$. For $l\in\mathbb{Z}$, let

$$P_{\le l}=\sum_{k\le l}P_k,\qquad P_{\ge l}=\sum_{k\ge l}P_k.$$

For $\xi\in\mathbb{R}$, let $\omega(\xi)=-|\xi|\xi$. For $k\in\mathbb{Z}$, let $I_k=\{\xi:|\xi|\in[2^{k-1},2^{k+1}]\}$. For $k\in\mathbb{Z}_+$, let $\widetilde{I}_k=[-2,2]$ if $k=0$ and $\widetilde{I}_k=I_k$ if $k\ge 1$. For $k\in\mathbb{Z}_+$ and $j\ge 0$, let

$$D_{k,j}=\{(\xi,\tau)\in\mathbb{R}\times\mathbb{R}:\xi\in\widetilde{I}_k,\ \tau-\omega(\xi)\in\widetilde{I}_j\}.$$

We introduce the space used in [2, 6]. First we define the $X^{s,b}$-type Banach spaces $X_k(\mathbb{R}\times\mathbb{R})$ for $k\in\mathbb{Z}_+$ as follows:

$$\begin{aligned}X_k=\bigg\{&f\in L^2(\mathbb{R}^2): f \text{ is supported in } \widetilde{I}_k\times\mathbb{R} \text{ and}\\ &\|f\|_{X_k}:=\sum_{j=0}^{\infty}2^{j/2}\beta_{k,j}\|\eta_j(\tau-w(\xi))\cdot f(\xi,\tau)\|_{L^2_{\xi,\tau}}<\infty\bigg\},\end{aligned}\tag{1.6}$$

where

$$\beta_{k,j}=1+2^{2(j-2k)/5}.\tag{1.7}$$

The coefficients $\beta_{k,j}$ are chosen to guarantee the trilinear estimates so that Lemma 4.1 holds. For $k\ge 100$, we also define the Banach spaces $Y_k=Y_k(\mathbb{R}^2)$:

$$\begin{aligned}Y_k=\bigg\{&f\in L^2(\mathbb{R}^2): f \text{ is supported in } \bigcup_{j=0}^{k-1}D_{k,j} \text{ and}\\ &\|f\|_{Y_k}:=2^{-k/2}\|\mathcal{F}^{-1}[(\tau-\omega(\xi)+i)f(\xi,\tau)]\|_{L^1_xL^2_t}<\infty\bigg\}.\end{aligned}\tag{1.8}$$

Then for $k\in\mathbb{Z}_+$, we define

$$Z_k:=X_k \text{ if } k\le 99\quad\text{and}\quad Z_k:=X_k+Y_k \text{ if } k\ge 100.\tag{1.9}$$

The spaces Z_k are our basic Banach spaces.

For $s\ge 0$, we define the Banach spaces $F^s=F^s(\mathbb{R}\times\mathbb{R})$,

$$F^s=\bigg\{u\in\mathcal{S}'(\mathbb{R}\times\mathbb{R}):\|u\|_{F^s}^2=\sum_{k=0}^{\infty}2^{2sk}\|\eta_k(\xi)\mathcal{F}(u)\|_{Z_k}^2<\infty\bigg\};\tag{1.10}$$

and $N^s=N^s(\mathbb{R}\times\mathbb{R})$,

$$\begin{aligned}N^s=\bigg\{&u\in\mathcal{S}'(\mathbb{R}\times\mathbb{R}):\|u\|_{N^s}^2\\ &=\sum_{k=0}^{\infty}2^{2sk}\|\eta_k(\xi)(\tau-\omega(\xi)+i)^{-1}\mathcal{F}(u)\|_{Z_k}^2<\infty\bigg\}.\end{aligned}\tag{1.11}$$

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For $T \geq 0$, we define the time-localized spaces $F^s(T)$ and $N^s(T)$ by

$$\begin{aligned} \|u\|_{F^s(T)} &= \inf_{w \in F^s}\{\|w\|_{F^s},\ w(t) = u(t) \text{ on } [0, T]\}, \\ \|u\|_{N^s(T)} &= \inf_{w \in N^s}\{\|w\|_{N^s},\ w(t) = u(t) \text{ on } [0, T]\}. \end{aligned} \tag{1.12}$$

For $\phi \in L^2(\mathbb{R})$, we denote by W_0 the semigroup associated with the mBO equation

$$\mathcal{F}_x(W_0(t)\phi)(\xi) = \exp[i\omega(\xi)t]\widehat{\phi}(\xi), \quad \forall t \in \mathbb{R},\ \phi \in \mathcal{S}'.$$

For $0 < \epsilon \leq 1$, we denote by W_ϵ the semigroup associated with the free evolution of (1.1),

$$\mathcal{F}_x(W_\epsilon(t)\phi)(\xi) = \exp[-\epsilon\xi^2 t + i\xi|\xi|t]\widehat{\phi}(\xi), \quad \forall t \geq 0,\ \phi \in \mathcal{S}'.$$

We extend W_ϵ to a linear operator defined on the whole real axis by setting

$$\mathcal{F}_x(W_\epsilon(t)\phi)(\xi) = \exp[-\epsilon\xi^2|t| + i\xi|\xi|t]\widehat{\phi}(\xi), \quad \forall t \in \mathbb{R},\ \phi \in \mathcal{S}'.$$

To study the low regularity of (1.1), we introduce a variant version of Bourgain's space with dissipation

$$\|u\|_{X^{b,s,2}} = \|\langle i(\tau - \omega(\xi)) + |\xi|^2\rangle^b \langle\xi\rangle^s \widehat{u}\|_{L^2(\mathbb{R}^2)}, \tag{1.13}$$

where $\langle\cdot\rangle = (1 + |\cdot|^2)^{1/2}$. The time-localized spaces is similar to (1.12). This type of space was introduced by Molinet and Ribaud in [9]. The standard $X^{b,s}$ space used by Bourgain [1] and Kenig *et al.* [7] is defined by

$$\|u\|_{X^{b,s}} = \|\langle\tau - \omega(\xi)\rangle^b \langle\xi\rangle^s \widehat{u}\|_{L^2(\mathbb{R}^2)}.$$

The space $X^{1/2,s,2}$ turns out to be very useful for capturing both dispersive and dissipative effects. For global well-posedness, we follow the methods of Molinet and Ribaud [9], by using $X^{b,s}$-type space combined with the dissipative structures. Similar results were obtained by Vento [14] for the Benjamin–Ono-Burgers equation (with nonlinearity uu_x in (1.1)).

THEOREM 1.1. *Assume that $0 < \epsilon \leq 1$, $s > 0$ and $\phi \in H^s(\mathbb{R})$. For any $T > 0$, there exists a unique solution u_ϵ of (1.1) in*

$$Z_T = C([0, T], H^s) \cap X_T^{1/2,s,2}.$$

Moreover, the solution map $\Phi_T^\epsilon : \phi \to u$ is smooth from $H^s(\mathbb{R})$ to Z_T and u belongs to $C((0, \infty), H^\infty(\mathbb{R}))$.

We show the uniform global well-posedness for Equation (1.1) with respect to ϵ.

THEOREM 1.2. *Assume that $\phi \in H^{1/2}$, $0 < \epsilon \leq 1$ and $\|\phi\|_2 \ll 1$.*

(a) *Existence. For any $T > 0$, there exists a solution u to the Cauchy problem (1.1) satisfying*

$$u \in F^{1/2}(T) \subset C([-T, T] : H^{1/2}).$$

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(b) *Uniqueness.* *The solution mapping* $\Phi_T^\epsilon : \phi \to u$ *is the unique continuous extension of the classical solution* $H^\infty \to C([-T, T] : H^\infty)$.

(c) *Lipschitz continuity.* *For any* $R > 0$, *the mapping* $\Phi_T^\epsilon : \phi \to u$ *is Lipschitz continuous from* $\{\phi \in H^{1/2} : \|\phi\|_{H^{1/2}} < R, \|\phi\|_{L^2} \ll 1\}$ *to* $C([-T, T] : H^{1/2})$.

(d) *Persistence of regularity.* *If in addition* $\phi \in H^s$ *for some* $s > 1/2$, *then the solution* u *belongs to* H^s.

For the limit behaviour, we have the following theorem.

THEOREM 1.3. *Assume that* $\phi \in H^{1/2}$ *and* $\|\phi\|_2 \ll 1$. *Then, for any* $T > 0$, *the solution of (1.1) obtained in Theorem 1.2 converges to that of (1.3) in* $C([0, T]; H^s)$ *for* $s \geq 1/2$ *if* ϵ *goes to* 0.

In Sections 2–4 we give the proofs of Theorems 1.1–1.3.

2. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Comparing the procedure of [14, Section 4], we can easily obtain Theorem 1.1 if the proposition below holds. In particular, the proof that u belongs to $C((0, \infty), H^\infty(\mathbb{R}))$ is parallel to the proof in Section 4 in [14], and so we omit it.

PROPOSITION 2.1. *Let* $s > 0$, $0 < \eta \ll 1$; *then there exists* $C_{s,\eta} > 0$ *such that, for any* u_1, u_2, u_3 *on* $\mathbb{R} \times \mathbb{R}$,

$$\|\partial_x(u_1u_2u_3)\|_{X^{-1/2+\eta,s,2}} \leq C\|u_1\|_{X^{1/2,s,2}}\|u_2\|_{X^{1/2,s,2}}\|u_3\|_{X^{1/2,s,2}}.$$

We now utilize Tao's $[k; Z]$-multiplier from [12] to prove Proposition 2.1. For simplicity, We review some notation Tao used in [12]. We use $A \lesssim B$ to denote the statement that $A \leq CB$ for some large constant C which may vary from line to line and depend on various parameters such as the dimension n, and we use $A \sim B$ to denote the statement that $A \lesssim B \lesssim A$. Let Z be any abelian additive group with an invariant measure $d\xi$. For any integer $k \geq 2$, we let $\Gamma_k(Z)$ denote the hyperplane

$$\Gamma_k(Z) := \{(\xi_1, \ldots, \xi_k) \in Z^k : \xi_1 + \cdots + \xi_k = 0\},$$

which is endowed with the measure

$$\int_{\Gamma_k(Z)} f := \int_{Z^{k-1}} f(\xi_1, \ldots, \xi_{k-1}, -\xi_1 - \cdots - \xi_{k-1})\, d\xi_1 \cdots d\xi_{k-1}.$$

A $[k; Z]$-multiplier is defined to be any function $m : \Gamma_k(Z) \to \mathbb{C}$, and the multiplier norm $\|m\|_{[k;Z]}$ is defined to be the best constant such that the inequality

$$\left|\int_{\Gamma_k(Z)} m(\xi) \prod_{j=1}^{k} f_j(\xi_j)\right| \leq \|m\|_{[k;Z]} \prod_{j=1}^{k} \|f_j\|_{L^2(Z)}$$

holds for all test functions f_j on Z. For given τ, ξ and $h(\cdot)$, we write

$$\lambda := \tau - h(\xi).$$

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Similarly, we put $\lambda_j := \tau_j - h_j(\xi_j)$. The quantities N_j and L_j measure the spatial frequency of the jth wave and how it resembles a free solution respectively, while the quantity H measures the amount of resonance. In this paper, we consider

$$h(\xi) = -\xi_1|\xi_1| - \xi_2|\xi_2| - \xi_3|\xi_3| = -\lambda_1 - \lambda_2 - \lambda_3,$$

which measures the extent to which the spatial frequencies ξ_1, ξ_2, ξ_3 can resonate with each other. By dyadic decomposition of the variables ξ_j, λ_j, as well as the function $h(\xi)$, one is led to consider

$$\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}\times\mathbb{R}]}, \tag{2.1}$$

where $X_{N_1,N_2,N_3;H;L_1,L_2,L_3}$ is the following multiplier:

$$X_{N_1,N_2,N_3;H;L_1,L_2,L_3}(\xi,\tau) := \chi_{|h(\xi)|\sim H} \prod_{j=1}^{3} \chi_{|\xi_j|\sim N_j}\chi_{|\lambda_j|\sim L_j}.$$

Define the quantities $N_{\max} \geq N_{\text{med}} \geq N_{\min}$ to be the maximum, median, and minimum of N_1, N_2, N_3 respectively. $L_{\max} \geq L_{\text{med}} \geq L_{\min}$ are similar. In this paper, we always assume that N_j, L_j are dyadic numbers. From the identities $\xi_1 + \xi_2 + \xi_3 = 0$ and $\lambda_1 + \lambda_2 + \lambda_3 + h(\xi) = 0$ on the support of the multiplier, we see that $X_{N_1,N_2,N_3;H;L_1,L_2,L_3}$ vanishes unless

$$N_{\max} \sim N_{\text{med}} \quad \text{and} \quad L_{\max} \sim \max(H, L_{\text{med}}). \tag{2.2}$$

From the estimate in [6],

$$|H| \sim |\xi|_{\max} \cdot |\xi|_{\min}, \tag{2.3}$$

where

$$\sum_{j=1}^{3} \xi_j = 0, \qquad |\xi|_{\max} = \max(|\xi_1|, |\xi_2|, |\xi_3|),$$

and

$$|\xi|_{\min} = \min(|\xi_1|, |\xi_2|, |\xi_3|).$$

LEMMA 2.2 [3, Lemma 4.3]. *Let $H, N_1, N_2, N_3, L_1, L_2, L_3 > 0$ obey (2.2) and (2.3). Then:*

(i) *if $N_{\max} \sim N_{\min}$ and $L_{\max} \sim N_{\max}N_{\min}$, then*

$$(2.1) \lesssim L_{\min}^{1/2} L_{\text{med}}^{1/4}; \tag{2.4}$$

(ii) *if $N_2 \sim N_3 \gg N_1$ and $N_{\max}N_{\min} \sim L_1 \gtrsim L_2, L_3$, then*

$$(2.1) \lesssim L_{\min}^{1/2} N_{\max}^{-1/2} \min\left(N_{\max}N_{\min}, \frac{N_{\max}}{N_{\min}} L_{\text{med}}\right)^{1/2}, \tag{2.5}$$

and similarly for permutations;

(iii) *in all other cases,*

$$(2.1) \lesssim L_{\min}^{1/2} N_{\max}^{-1/2} \min(N_{\max}N_{\min}, L_{\text{med}})^{1/2}. \tag{2.6}$$

We now prove Proposition 2.1. By duality and the Plancherel theorem, it suffices to show that

$$\left\| \frac{(\xi_1+\xi_2+\xi_3)\langle\xi_4\rangle^s}{\langle\tau_4-\omega(\xi_4)+i\xi_4^2\rangle^{1/2-\eta}\prod_{j=1}^{3}\langle\xi_j\rangle^s\langle\tau_j-\omega(\xi_j)+i\xi_j^2\rangle^{1/2}} \right\|_{[4,\mathbb{R}\times\mathbb{R}]} \lesssim 1.$$

We estimate $|\xi_1+\xi_2+\xi_3|$ by $\langle\xi_4\rangle$. We then apply the inequality

$$\langle\xi_4\rangle^{s+1} \lesssim \langle\xi_4\rangle^{1/2}\sum_{j=1}^{3}\langle\xi_j\rangle^{s+1/2},$$

where we assume that $s>0$. By symmetry it suffices to show that

$$\left\| \frac{\langle\xi_1\rangle^{-s}\langle\xi_3\rangle^{-s}\langle\xi_2\rangle^{1/2}\langle\xi_4\rangle^{1/2}}{\langle\tau_4-\omega(\xi_4)+i\xi_4^2\rangle^{1/2-\eta}\prod_{j=1}^{3}\langle\tau_j-\omega(\xi_j)+i\xi_j^2\rangle^{1/2}} \right\|_{[4,\mathbb{R}\times\mathbb{R}]} \lesssim 1.$$

We may replace $\langle\tau_2-\omega(\xi_2)+i\xi_2^2\rangle^{1/2}$ by $\langle\tau_2-\omega(\xi_2)+i\xi_2^2\rangle^{1/2-\eta}$. By the TT^* identity [12, Lemma 3.7] this estimate is reduced to the bilinear estimate below.

LEMMA 2.3. *Let $s>0$; for all u, v on $\mathbb{R}\times\mathbb{R}$ and $0<\eta\ll 1$,*

$$\|uv\|_{L^2(\mathbb{R}\times\mathbb{R})} \lesssim \|u\|_{X^{1/2-\eta,-1/2,2}(\mathbb{R}\times\mathbb{R})}\|v\|_{X^{1/2,s,2}(\mathbb{R}\times\mathbb{R})}.$$

PROOF. By the Plancherel identity, it suffices to show that

$$\left\| \frac{\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{1/2}}{\langle\tau_1-\omega(\xi_1)+i\xi_1^2\rangle^{1/2}\langle\tau_2-\omega(\xi_2)+i\xi_2^2\rangle^{1/2-\eta}} \right\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim 1.$$

Observe that, by the translation invariance of the $[k;Z]$-multiplier norm, we can always restrict our estimate on $\lambda_j \gtrsim 1$ and $\max(N_1,N_2,N_3)\gtrsim 1$. The comparison principle and orthogonality [12, Lemmas 3.1, 3.11] reduce our estimate to show that

$$\sum_{N_{\max}\sim N_{\rm med}\sim N}\ \sum_{L_1,L_2,L_3\gtrsim 1} \frac{\langle N_1\rangle^{-s}\langle N_2\rangle^{1/2}}{\max(L_1,\langle N_1\rangle^2)^{1/2}\max(L_2,\langle N_2\rangle^2)^{1/2-\eta}} \times \|X_{N_1,N_2,N_3;L_{\max};L_1,L_2,L_3}\|_{[3;\mathbb{R}\times\mathbb{R}]} \lesssim 1 \tag{2.7}$$

and

$$\sum_{N_{\max}\sim N_{\rm med}\sim N}\ \sum_{L_{\max}\sim L_{\rm med}}\ \sum_{H\ll L_{\max}} \frac{\langle N_1\rangle^{-s}\langle N_2\rangle^{1/2}}{\max(L_1,\langle N_1\rangle^2)^{1/2}\max(L_2,\langle N_2\rangle^2)^{1/2-\eta}} \times \|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3;\mathbb{R}\times\mathbb{R}]} \lesssim 1 \tag{2.8}$$

for all $N\gtrsim 1$.

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First we prove (2.8). We may assume that (2.3) holds. By (2.6), it suffices to prove that

$$\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N}\ \sum_{L_{\max}\sim L_{\mathrm{med}}\gtrsim N_{\min}N_{\max}} L_{\min}^{1/2}N_{\min}^{1/2} \times \frac{\langle N_1\rangle^{-s}\langle N_2\rangle^{1/2}}{\max(L_1,\langle N_1\rangle^2)^{1/2}\max(L_2,\langle N_2\rangle^2)^{1/2-\eta}} \lesssim 1. \tag{2.9}$$

Bounding

$$\langle N_1\rangle^{-s}\langle N_2\rangle^{1/2} \lesssim \frac{N^{1/2}}{\langle N_{\min}\rangle^s},$$

$$\max(L_1,\langle N_1\rangle^2)^{1/2}\max(L_2,\langle N_2\rangle^2)^{1/2-\eta} \gtrsim L_{\min}^{1/2}N^{2(1/2-\eta)}$$

and performing the L summations, it suffices to show that

$$\sum_{N_{\max}\sim N_{\mathrm{med}}\sim N} \frac{\langle N_{\min}\rangle^{1/2-s}}{N^{1/2-2\eta}} \lesssim 1,$$

which is true when $s > 0$.

We now prove (2.7). First we assume that (2.4) holds. In this case, we have N_1, N_2, $N_3 \sim N \gtrsim 1$. Therefore, it suffices to show that

$$\sum_{L_{\max}\sim N^2} \frac{N^{1/2-s}}{\max(L_1,N^2)^{1/2}\max(L_2,N^2)^{1/2-\eta}} L_{\min}^{1/2}L_{\mathrm{med}}^{1/4} \lesssim 1, \tag{2.10}$$

and this is easily verified when $s > 0$ and $L_{\max} \sim N_{\max}N_{\min}$.

Now we consider the case where (2.5) holds. We do not have perfect symmetry and must consider three cases

$$N \sim N_1 \sim N_2 \gg N_3; \quad H \sim L_3 \gtrsim L_1, L_2, \tag{2.11}$$
$$N \sim N_2 \sim N_3 \gg N_1; \quad H \sim L_1 \gtrsim L_2, L_3, \tag{2.12}$$
$$N \sim N_1 \sim N_3 \gg N_2; \quad H \sim L_2 \gtrsim L_1, L_3, \tag{2.13}$$

separately.

In the first case we reduce by (2.5) to

$$\sum_{N_3\ll N}\ \sum_{1\lesssim L_1,L_2\lesssim NN_3} \frac{N^{1/2-s}}{\max(L_1,N^2)^{1/2}\max(L_2,N^2)^{1/2-\eta}} \times L_{\min}^{1/2}N^{-1/2}\min\Big(NN_3,\frac{N}{N_3}L_{\mathrm{med}}\Big)^{1/2} \lesssim 1.$$

Performing the N_3 summation, we reduce to

$$\sum_{1\lesssim L_1,L_2\lesssim N^2} \frac{N^{1/2-s}}{\max(L_1,N^2)^{1/2}\max(L_2,N^2)^{1/2-\eta}} L_{\min}^{1/2}N^{-1/2}N^{1/2}L_{\mathrm{med}}^{1/4} \lesssim 1,$$

which is similar to (2.10).

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Considering the second and third cases, it suffices to deal with the worst case

$$N \sim N_2 \sim N_3 \gg N_1; \quad H \sim L_1 \gtrsim L_2, L_3.$$

Using the first part of (2.5),

$$\sum_{N_{\min} \ll N} \sum_{1 \lesssim L_{\min}, L_{\mathrm{med}} \ll N N_{\min}} \frac{\langle N_{\min}\rangle^{-s} N^{1/2}}{L_{\min}^{1/2} N^{2(1/2-\eta)}} L_{\min}^{1/2} N_{\min}^{1/2} \lesssim 1.$$

We may assume that $N_{\min} \gtrsim N^{-1}$ since the inner sum vanishes otherwise. Performing the L summation, we reduce to

$$\sum_{N^{-1} \lesssim N_{\min} \ll N} \frac{\langle N_{\min}\rangle^{-s} N^{1/2} N_{\min}^{1/2}}{N^{1-2\eta}} \lesssim 1,$$

which holds when $s > 0$.

To finish the proof of (2.7), it remains to deal with the case where (2.6) holds. This reduces to

$$\sum_{N_{\max} \sim N_{\mathrm{med}} \sim N} \sum_{L_{\max} \sim N_{\max} N_{\min}} \frac{\langle N_1\rangle^{-s} \langle N_2\rangle^{1/2}}{\max(L_1, \langle N_1\rangle^2)^{1/2} N^{2(1/2-\eta)}} L_{\min}^{1/2} N^{-1/2} L_{\mathrm{med}}^{1/2} \lesssim 1.$$

Performing the L summations, we reduce to

$$\sum_{N_{\max} \sim N_{\mathrm{med}} \sim N} \frac{N_{\min}^{1/2}}{\langle N_1\rangle^{s} N^{1/2-2\eta}} \lesssim 1,$$

which is easily verified when $s > 0$. □

3. Proof of Theorem 1.2

Observing that (1.1) is invariant under the scaling

$$u(x,t) \to u_\lambda = \frac{1}{\lambda^{1/2}} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right), \quad \epsilon \to \epsilon \frac{1}{\lambda^{1/2}}, \quad \phi_\lambda = \frac{1}{\lambda^{1/2}} \phi\left(\frac{x}{\lambda}\right), \tag{3.1}$$

we can see that $\|\phi\|_{L^2}$ is invariant under this scaling, and so we require that $\|\phi\|_{L^2} \ll 1$. Before embarking on the proof of Theorem 1.2, we establish two results. Let

$$L(f)(x,t) = W_0(t)\psi(t) \int_{\mathbb{R}^2} e^{ix\xi} \frac{e^{it\tau'} - e^{-\epsilon|t|\xi^2}}{i\tau' + \epsilon\xi^2} \mathcal{F}(W_0(-t) f)(\xi, \tau')\, d\xi\, d\tau'. \tag{3.2}$$

Here we take $\psi = \eta_0$, and it is easy to verify that

$$\chi_{\mathbb{R}_+}(t) L(f)(x,t) = \chi_{\mathbb{R}_+}(t)\psi(t) \int_0^t W_\epsilon(t-\tau) f(\tau)\, d\tau. \tag{3.3}$$

LEMMA 3.1. *If $s \ge 1/2$ and $\phi \in H^s$, there exists $C > 0$ such that, for any $0 < \epsilon \le 1$,*

$$\|\psi(t) \cdot (W_\epsilon(t)\phi)\|_{F^s} \le C \|\phi\|_{H^s}.$$

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PROOF. We use an idea from [5] in our proof. In view of the definition, it suffices to prove that if $k \in \mathbb{Z}_+$, then

$$\|\eta_k(\xi)\mathcal{F}(\psi(t)\cdot(W_\epsilon(t)\phi))\|_{Z_k} \le C\|\eta_k(\xi)\widehat{\phi}(\xi)\|_{L^2}. \tag{3.4}$$

First, we consider the case $k=0$. Observing that $|\xi|\le 2$ in this case, and using Taylor's expansion,

$$\begin{aligned}
&\|\eta_0(\xi)\mathcal{F}(\psi(t)W_\epsilon(t)\phi)\|_{X_0}\\
&\quad\lesssim \sum_{j=0}^{\infty} 2^{j/2}(1+2^{2j/5})\bigg\|\eta_0(\xi)\widehat{\phi}(\xi)\mathcal{F}_t\bigg(\psi(t)\sum_{n\ge 0}\frac{(-1)^n\epsilon^n\xi^2}{n!}|t|^n\bigg)(\tau)\eta_j(\tau)\bigg\|_{L^2_{\xi,\tau}}\\
&\quad\lesssim \sum_{n\ge 0}\frac{4^n}{n!}\|\eta_0(\xi)\widehat{\phi}(\xi)\|_{L^2}\||t|^n\psi(t)\|_{H^1}\\
&\quad\lesssim \|\eta_0(\xi)\widehat{\phi}(\xi)\|_{L^2},
\end{aligned}$$

which is (3.4) as desired.

Secondly, we consider the case $k\ge 1$. Observing that if $|\xi|\sim 2^k$, then for any $j\ge 0$,

$$\|P_j(e^{-\epsilon\xi^2|t|})(t)\|_{L^2} \lesssim \|P_j(e^{-\epsilon 2^{2k}|t|})(t)\|_{L^2},$$

which follows from Plancherel's equality and the fact that

$$\mathcal{F}(e^{-|t|})(\tau) = C\frac{1}{1+|\tau|^2}.$$

It follows from the definition that

$$\begin{aligned}
&\|\eta_k(\xi)\mathcal{F}(\psi(t)W_\epsilon(t)\phi)\|_{X_k}\\
&\quad\lesssim \sum_{j=0}^{\infty} 2^{j/2}\beta_{k,j}\|\eta_k(\xi)\widehat{\phi}(\xi)\eta_j(\tau)\mathcal{F}_t(\psi(t)e^{-\epsilon|t|\xi^2})(\tau)\|_{L^2_{\xi,\tau}}\\
&\quad\lesssim \sum_{j=0}^{\infty} 2^{j/2}\beta_{k,j}\|\eta_k(\xi)\widehat{\phi}(\xi)P_j(\psi(t)e^{-\epsilon|t|\xi^2})(t)\|_{L^2_{\xi,t}}\\
&\quad\lesssim \sum_{j=0}^{\infty} 2^{j/2}\beta_{k,j}\|\eta_k(\xi)\widehat{\phi}(\xi)\|_{L^2}\sup_{|\xi|\sim 2^k}\|P_j(\psi(t)e^{-\epsilon|t|\xi^2})(t)\|_{L^2_t}.
\end{aligned}$$

Therefore, it suffices to show that

$$\sum_{j=0}^{\infty} 2^{j/2}\beta_{k,j}\sup_{|\xi|\sim 2^k}\|P_j(\psi(t)e^{-\epsilon|t|\xi^2})(t)\|_{L^2_t} \lesssim 1. \tag{3.5}$$

We may assume that $j\ge 100$ in the summation. Using the para-product decomposition,

$$u_1u_2 = \sum_{r=0}^{\infty}[(P_{r+1}u_1)(P_{\le r+1}u_2)+(P_{\le r}u_1)(P_{r+1}u_2)] \tag{3.6}$$

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and

$$P_j(u_1u_2) = P_j\Big(\sum_{r\geq j-10}[(P_{r+1}u_1)(P_{\leq r+1}u_2)+(P_{\leq r}u_1)(P_{r+1}u_2)]\Big) \\ = P_j(I+II). \tag{3.7}$$

Now we take $u_1 = \psi(t)$ and $u_2 = e^{-\epsilon|t|\xi^2}$.

When $j \leq 2k$, we have $\beta_{k,j} \sim 1$, and the situation can be treated as in [5]. When $j > 2k$, it suffices to bound

$$\begin{aligned}
&\sum_{j\geq 100} 2^{j/2}2^{2(j-2k)/5}\|P_j(II)\|_{L^\infty_\xi L^2_t}\\
&\quad\lesssim \sum_{j\geq 100} 2^{j-k}\sum_{r\geq j-10}\|P_{r+1}u_2\|_{L^\infty_\xi L^2_t}\|P_{\leq r+1}u_1\|_{L^\infty_{\xi,t}}\\
&\quad\lesssim \sum_{j\geq 100} 2^{j-r}\sum_{r\geq j-10} 2^{r-k}\|P_{r+1}u_2\|_{L^\infty_\xi L^2_t}\\
&\quad\lesssim \sum_{r} 2^{r-k}\|P_{r+1}(e^{-\epsilon|t|2^{2k}})\|_{L^2_t}\\
&\quad\lesssim 2^{-1-k}\sum_{r} 2^{r+1}\|P_{r+1}(e^{-\epsilon|t|2^{2k}})\|_{L^2_t}\\
&\quad\lesssim 2^{-1-k}\epsilon^{1/2}2^k\|e^{-|t|}\|_{\dot{B}^1_{2,1}} \lesssim 1,
\end{aligned}$$

where we use the fact that $e^{-|t|} \in \dot{B}^1_{2,1}$ and $\|e^{-\epsilon 2^{2k}|t|}\|_{\dot{B}^1_{2,1}} \sim \epsilon^{1/2}2^k\|e^{-|t|}\|_{\dot{B}^1_{2,1}}$.

The first term, $P_j(I)$, in (3.7) can be handled in an easier way. This completes the proof of the proposition. □

The next lemma provides an estimate for the retarded linear term.

LEMMA 3.2. *For $s \geq 1/2$ and $u \in \mathcal{S}(\mathbb{R}\times\mathbb{R})$, there exists $C > 0$ such that*

$$\|\psi(t)L(v)\|_{F^s} \leq C\|v\|_{N^s}.$$

PROOF. In view of the definitions, it suffices to prove that if $k \in \mathbb{Z}_+$, then

$$\|\eta_k(\xi)\mathcal{F}(L(v))\|_{Z_k} \lesssim \|\eta_k(\xi)(i+\tau-\omega(\xi))^{-1}\mathcal{F}(v)\|_{Z_k}.$$

Observe that

$$\begin{aligned}
\mathcal{F}_x(L(v)) &= \psi(t)e^{it\omega(\xi)}\int_{\mathbb{R}}\frac{e^{it\tau'}-e^{-\epsilon|t|\xi^2}}{i\tau'+\epsilon\xi^2}\widehat{v}(\xi,\tau'+\omega(\xi))\,d\tau'\\
&= \psi(t)e^{it\omega(\xi)}\int_{\mathbb{R}}\frac{e^{-it\omega(\xi)}e^{it\tau'}-e^{-\epsilon|t|\xi^2}}{i(\tau'-\omega(\xi))+\epsilon\xi^2}\widehat{v}(\xi,\tau')\,d\tau'\\
&= \psi(t)\int_{\mathbb{R}}\frac{e^{it\tau'}-e^{it\omega(\xi)}e^{-\epsilon|t|\xi^2}}{i(\tau'-\omega(\xi))+\epsilon\xi^2}\widehat{v}(\xi,\tau')\,d\tau'
\end{aligned}$$

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and

$$\mathcal{F}(L(v))(\xi,\tau)=\int_{\mathbb{R}}\frac{\widehat{\psi}(\tau-\tau')-\mathcal{F}_t(\psi(t)e^{-\epsilon|t|\xi^2})(\tau-\omega(\xi))}{i(\tau'-\omega(\xi))+\epsilon\xi^2}\widehat{v}(\xi,\tau')\,d\tau'.$$

For $k\in\mathbb{Z}_+$, let $f_k(\xi,\tau')=\mathcal{F}(v)(\xi,\tau')\eta_k(\xi)(\tau'-\omega(\xi)+i)^{-1}$.

For $f_k\in Z_k$, let

$$\begin{aligned}T(f_k)(\xi,\tau)=\int_{\mathbb{R}} f_k(\xi,\tau')&\frac{\widehat{\psi}(\tau-\tau')-\mathcal{F}_t(\psi(t)e^{-\epsilon|t|\xi^2})(\tau-\omega(\xi))}{i(\tau'-\omega(\xi))+\epsilon\xi^2}\\ &\times(\tau'-\omega(\xi)+i)\,d\tau'.\end{aligned}\tag{3.8}$$

It suffices to show that

$$\|T\|_{Z_k\to Z_k}\le C \text{ uniformly in } k\in Z_+.\tag{3.9}$$

First, we consider the case $k\in[0,99]$, so $f_k=f_{k,j}$ is a function supported in $D_{k,j}$. Let

$$f^{\#}_{k,j}(\xi,\mu')=f_{k,j}(\xi,\mu'+\omega(\xi))\quad\text{and}\quad T^{\#}(f_{k,j})(\xi,\mu)=T(f_{k,j})(\xi,\mu+\omega(\xi)).$$

Thus,

$$T^{\#}(f_k)(\xi,\tau)=\int_{\mathbb{R}} f^{\#}_k(\xi,\tau')\frac{\widehat{\psi}(\tau-\tau')-\mathcal{F}_t(\psi(t)e^{-\epsilon|t|\xi^2})(\tau)}{i\tau'+\epsilon\xi^2}(i+\tau')\,d\tau'.\tag{3.10}$$

Let

$$w(\tau)=W_0(-\tau)v(\tau),\qquad k_\xi(t)=\psi(t)\int_{\mathbb{R}}\frac{e^{it\tau'}-e^{-\epsilon|t|\xi^2}}{i\tau'+\epsilon\xi^2}\widehat{w}(\xi,\tau')\,d\tau'.$$

For (3.9), by definition, it suffices to prove that

$$\begin{aligned}\sum_{j=0}^{\infty}&2^{j/2}\beta_{k,j}\|\eta_k(\xi)\eta_j(\tau)\mathcal{F}_t(k_\xi)(\tau)\|_{L^2_{\xi,\tau}}\\ &\lesssim\sum_{j=0}^{\infty}2^{-j/2}\beta_{k,j}\|\eta_k(\xi)\eta_j(\tau)\widehat{w}(\xi,\tau)\|_{L^2_{\xi,\tau}}.\end{aligned}\tag{3.11}$$

We use an idea from [5] to decompose

$$\begin{aligned}k_\xi(t)&=\psi(t)\int_{|\tau|\le1}\frac{e^{it\tau}-1}{i\tau+\epsilon\xi^2}\widehat{w}(\xi,\tau)\,d\tau+\psi(t)\int_{|\tau|\le1}\frac{1-e^{-\epsilon|t|\xi^2}}{i\tau+\epsilon\xi^2}\widehat{w}(\xi,\tau)\,d\tau\\ &\quad+\psi(t)\int_{|\tau|\ge1}\frac{e^{it\tau}}{i\tau+\epsilon\xi^2}\widehat{w}(\xi,\tau)\,d\tau-\psi(t)\int_{|\tau|\ge1}\frac{e^{-\epsilon|t|\xi^2}}{i\tau+\epsilon\xi^2}\widehat{w}(\xi,\tau)\,d\tau\\ &=I+II+III-IV.\end{aligned}$$

We estimate each of the above four parts.

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First, we consider the contribution of IV. Using the Taylor expansion for $k=0$ and (3.5) for $k \geq 1$, we get

$$\sum_{j=0}^{\infty} 2^{j/2}\beta_{k,j}\|\eta_k(\xi)P_j(IV)(t)\|_{L^2_{\xi,t}} \leq \sum_{j=0}^{\infty} 2^{j/2}\beta_{k,j}\int_{|\tau|\geq 1}\frac{\|\eta_k(\xi)\widehat{w}(\xi,\tau)\|_{L^2_\xi}}{|\tau|}\,d\tau$$
$$\times \sup_{\xi\in I_k}\|\eta_k(\xi)P_j(\psi(t)e^{-\epsilon|t|\xi^2})(t)\|_{L^2_t}$$
$$\lesssim \sum_{j=0}^{\infty} 2^{-j/2}\beta_{k,j}\|\eta_k(\xi)\eta_j(\tau)\widehat{w}(\xi,\tau)\|_{L^2_{\xi,\tau}}.$$

Secondly, we consider the contribution of III. Let

$$g(\xi,\tau)=\frac{|\widehat{w}(\xi,\tau)|}{|i\tau+\epsilon\xi^2|}\chi_{|\tau|\geq 1}.$$

When $j>2k$,

$$\sum_{j=0}^{\infty} 2^{j/2}\beta_{k,j}\|\eta_k(\xi)P_j(III)(t)\|_{L^2_{\xi,t}}$$
$$\lesssim \sum_{j=0}^{\infty} 2^{j/2}2^{2(j-2k)/5}\|\eta_k(\xi)\eta_j(\tau)\widehat{\psi}*g(\xi,\tau)\|_{L^2_{\xi,\tau}}$$
$$\lesssim \sum_{j\geq 1}^{\infty} 2^{9j/10}2^{-4k/5}\left\|\frac{\eta_j(\tau')\|\eta_k(\xi)\widehat{w}(\xi,\tau')\|_{L^2_\xi}}{|i\tau'|}\chi_{|\tau'|\geq 1}\right\|_{L^2_{\tau'}}$$
$$\lesssim \sum_{j=0}^{\infty} 2^{-j/2}\beta_{k,j}\|\eta_k(\xi)\eta_j(\tau)\widehat{w}(\xi,\tau)\|_{L^2_{\xi,\tau}},$$

where we used the fact that $B^{9/10}_{2,1}$ is a multiplication algebra and $\mathcal{F}^{-1}(|\widehat{\psi}|)\in B^{9/10}_{2,1}$. When $j\leq 2k$, we can get the desired result by the same estimate as in [5].

Thirdly, we consider the contribution of II. For $\epsilon\xi^2\geq 1$, as for IV, we get

$$\sum_{j=0} 2^{j/2}\beta_{k,j}\|\eta_k(\xi)P_j(II)(t)\|_{L^2_{\xi,t}} \lesssim \sum_{j=0} 2^{j/2}\beta_{k,j}\int\frac{\|\widehat{w}(\xi,\tau)\|_{L^2_\xi}}{\langle\tau\rangle}\,d\tau$$
$$\times \sup_{\xi\in I_k}\|\eta_k(\xi)P_j(\psi(1-e^{-\epsilon|t|\xi^2}))(t)\|_{L^2_t}$$
$$\lesssim \sum_{j=0} 2^{-j/2}\beta_{k,j}\|\eta_k(\xi)\eta_j(\tau)\widehat{w}(\xi,\tau)\|_{L^2_{\xi,\tau}}.$$

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For $\epsilon\xi^2 \leq 1$, using Taylor's expansion,

$$\begin{aligned}
&\sum_{j=0} 2^{j/2}\beta_{k,j}\|\eta_k(\xi)P_j(II)(t)\|_{L^2_{\xi,t}} \\
&\quad\lesssim \sum_{n\geq 1}\sum_{j=0} 2^{j/2}\beta_{k,j}\left\|\eta_k(\xi)P_j(|t|^n\psi(t))\frac{\epsilon^n\xi^{2n}}{n!}\int_{|\tau|\leq 1}\frac{\widehat{w}(\xi,\tau)}{i\tau+\epsilon\xi^2}\,d\tau\right\|_{L^2_{\xi,t}} \\
&\quad\lesssim \left\|\int_{|\tau|\leq 1}\frac{\epsilon\xi^2|\eta_k(\xi)\widehat{w}(\xi,\tau)|}{|i\tau+\epsilon\xi^2|}\,d\tau\right\|_{L^2_\xi} \\
&\quad\lesssim \sum_{j=0} 2^{-j/2}\beta_{k,j}\|\eta_k(\xi)\eta_j(\tau)\widehat{w}(\xi,\tau)\|_{L^2_{\xi,\tau}},
\end{aligned}$$

where we used the fact that

$$\||t|^n\psi(t)\|_{B^{9/10}_{2,1}} \lesssim \||t|^n\psi(t)\|_{H^1} \leq C2^n.$$

Finally, we consider the contribution of I. Using Taylor's expansion,

$$I = \psi(t)\int_{|\tau|\leq 1}\sum_{n\geq 1}\frac{(it\tau)^n}{n!(i\tau+\epsilon\xi^2)}\widehat{w}(\tau)\,d\tau.$$

Thus, we get

$$\begin{aligned}
&\sum_{j=0} 2^{j/2}\beta_{k,j}\|\eta_k(\xi)P_j(I)(t)\|_{L^2_{\xi,t}} \\
&\quad\lesssim \sum_{n\geq 1}\left\|\frac{t^n\psi(t)}{n!}\right\|_{B^{9/10}_{2,1}}\left\|\int_{|\tau|\leq 1}\frac{|\tau|}{|i\tau+\epsilon\xi^2|}|\eta_k(\xi)\widehat{w}(\xi,\tau)|\,d\tau\right\|_{L^2_\xi} \\
&\quad\lesssim \sum_{j=0} 2^{-j/2}\beta_{k,j}\|\eta_k(\xi)\eta_j(\tau)\widehat{w}(\xi,\tau)\|_{L^2_{\xi,\tau}}.
\end{aligned}$$

From the definition of the spaces X_k, we get

$$\|T\|_{X_k\to X_k} \leq C \text{ uniformly in } k \geq 1, \tag{3.12}$$

as desired.

We now consider $f_k \in Y_k$, $k \geq 100$. As in [6], we can assume that f_k is supported in the set $\{(\xi,\tau') : |\tau'-\omega(\xi)| \leq 2^{k-20}\}$. We decompose

$$g_k(\xi,\tau') = \frac{\tau'-\omega(\xi)}{\tau'-\omega(\xi)+i}g_k(\xi,\tau') + \frac{i}{\tau'-\omega(\xi)+i}g_k(\xi,\tau').$$

By (3.12) and the fact that the result

$$\|i(\tau'-\omega(\xi)+i)^{-1}g_k(\xi,\tau')\|_{X_k} \leq C\|g_k\|_{Y_k}$$

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in [6] also holds for our choice $\beta_{k,j}$, it suffices to show that

$$\begin{aligned}&\left\|\mathcal{F}_t(\psi(t)e^{-\epsilon|t|\xi^2})(\tau-\omega(\xi))\int_{\mathbb{R}} g_k(\xi,\tau')\frac{\tau'-\omega(\xi)}{\tau'-\omega(\xi)+i\epsilon\xi^2}\,d\tau'\right\|_{X_k}\\&\quad+\left\|\int_{\mathbb{R}} g_k(\xi,\tau')\frac{\tau'-\omega(\xi)}{\tau'-\omega(\xi)+i\epsilon\xi^2}\widehat{\psi}(\tau-\tau')\,d\tau'\right\|_{Z_k}\le C\|g_k\|_{Y_k}.\end{aligned}\tag{3.13}$$

The first term on the left-hand side of (3.13) can be treated by ideas similar to those in [6]. For the second term, we decompose

$$g_k(\xi,\tau')=\frac{\tau'-\omega(\xi)+i}{\tau'-\omega(\xi)+i}g_k(\xi,\tau')+\frac{\tau-\tau'}{\tau'-\omega(\xi)+i}g_k(\xi,\tau').$$

The second term on the left-hand side of (3.13) is dominated by

$$\begin{aligned}&C\left\|\frac{\eta_{[0,k-1]}(\tau-\omega(\xi))}{\tau-\omega(\xi)+i}\int_{\mathbb{R}} g_k(\xi,\tau')(\tau'-\omega(\xi)+i)\right.\\&\qquad\left.\times\,\widehat{\psi}(\tau-\tau')\frac{\tau'-\omega(\xi)}{\tau'-\omega(\xi)+i\epsilon\xi^2}\,d\tau'\right\|_{Y_k}\\&\quad+C\sum_{j\ge k-1}2^{j/2}\beta_{k,j}\left\|\frac{\eta_j(\tau-\omega(\xi))}{\tau-\omega(\xi)+i}\int_{\mathbb{R}} g_k(\xi,\tau')\right.\\&\qquad\left.\times\,\widehat{\psi}(\tau-\tau')\frac{\tau'-\omega(\xi)}{\tau'-\omega(\xi)+i\epsilon\xi^2}\,d\tau'\right\|_{L^2}\\&\quad+C\sum_{j\le k}2^{j/2}\left\|\frac{\eta_j(\tau-\omega(\xi))}{\tau-\omega(\xi)+i}\int_{\mathbb{R}} g_k(\xi,\tau')\right.\\&\qquad\left.\times\,\widehat{\psi}(\tau-\tau')\frac{(\tau-\tau')(\tau'-\omega(\xi))}{\tau'-\omega(\xi)+i\epsilon\xi^2}\,d\tau'\right\|_{L^2}.\end{aligned}\tag{3.14}$$

This concludes the proof. □

We use the following lemma to bound the first term in (3.14); other terms are similarly treated by the method in [6].

LEMMA 3.3. *If $k\ge 1$, $0\le j\le k$ and g_k is supported in $I_k\times\mathbb{R}$, then*

$$\left\|\mathcal{F}^{-1}\left[\frac{\tau-\omega(\xi)}{\tau-\omega(\xi)+i\epsilon\xi^2}\eta_{\le j}(\tau-\omega(\xi))g_k(\xi,\tau)\right]\right\|_{L^1_xL^2_t}\lesssim\|\mathcal{F}^{-1}[g_k(\xi,\tau)]\|_{L^1_xL^2_t}.$$

PROOF. Using Plancherel's theorem, it suffices to prove that

$$\left\|\int_{\mathbb{R}} e^{ix\xi}\frac{\tau-\omega(\xi)}{\tau-\omega(\xi)+i\epsilon\xi^2}\chi_{[k-1,k+1]}(\xi)\eta_{\le j}(\tau-\omega(\xi))\,d\xi\right\|_{L^1_xL^\infty_\tau}\le C.\tag{3.15}$$

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In proving (3.15), we may assume that $k \geq 100$. Observe that the function on the left-hand side of (3.15) is not zero only if $\tau \approx 2^{2k}$. By symmetry, it suffices to consider the case $\xi \in [2^{k-2}, 2^{k+2}]$. Hence we have $\tau - \omega(\xi) = \tau + \xi^2$. Changing variable $\tau + \xi^2 = m$, it suffices to show that

$$\left| \int_{\mathbb{R}} e^{ix\xi} \frac{m}{m + i\epsilon\xi^2} \chi_{[k-1,k+1]}(\xi) \eta_{\leq j}(m)\, d\xi \right| \leq C. \tag{3.16}$$

On twice integrating by parts the left-hand side of (3.16),

$$\begin{aligned} &\left| \int_{\mathbb{R}} e^{ix\xi} \frac{m}{m + i\epsilon\xi^2} \chi_{[k-1,k+1]}(\xi) \eta_{\leq j}(m)\, d\xi \right| \\ &\quad = \left| \int_{\mathbb{R}} e^{ix\xi} \frac{m}{m + i\epsilon\xi^2} \chi_{[k-1,k+1]}(\xi) \eta_{\leq j}(m) \frac{1}{2\xi}\, dm \right| \\ &\quad = \left| \frac{1}{x^2} \int_{\mathbb{R}} e^{ix\xi} \frac{d}{dm}\left[\frac{1}{\xi'} \frac{d}{dm} \left(\frac{m}{m + i\epsilon\xi^2} \chi_{[k-1,k+1]}(\xi) \eta_{\leq j}(m) \right) \right] dm \right|, \end{aligned} \tag{3.17}$$

where we use the notation $\xi' = d\xi/dm$ and the fact that $\xi' = 1/2\xi$. To bound the right-hand side of (3.17), it suffices to estimate

$$\frac{d}{dm}\left[\frac{1}{\xi'} \frac{d}{dm} \left(\frac{m}{m + i\epsilon\xi^2} \chi_{[k-1,k+1]}(\xi) \eta_{\leq j}(m) \right) \right].$$

Let $I = m/(m + i\epsilon\xi^2)$ and $II = \chi_{[k-1,k+1]}(\xi)\eta_{\leq j}(m)$. It suffices to estimate

$$\begin{aligned} &\frac{1}{\xi'} II \frac{d^2 I}{dm} + \frac{1}{\xi'} \frac{dI}{dm} \frac{dII}{dm} + I \frac{d}{dm}\left(\frac{1}{\xi'} \frac{dII}{dm} \right) + \frac{dI}{dm} \frac{d}{dm}\left(\frac{1}{\xi'} II \right) \\ &\quad = L_1 + L_2 + L_3 + L_4. \end{aligned}$$

Now we obtain an estimate for L_1. After some calculation, we obtain

$$\begin{aligned} L_1 &\lesssim \frac{1}{\xi'} II \times \frac{2i\epsilon m(\xi')^2 + 2i\epsilon m\xi\xi'' + 4i\epsilon m\xi\xi' - 8m\epsilon^2\xi^2(\xi')^2 - 2i\epsilon\xi^2 + 4\epsilon^2\xi^3\xi'}{(m + i\epsilon\xi^2)^3} \\ &\lesssim II, \end{aligned}$$

where we use the notation $\xi'' = d^2\xi/dm^2$ and the fact that $\xi'' = 1/4\xi^3$ and $\xi' = 1/2\xi$. Similarly, for L_2,

$$L_2 \lesssim \frac{1}{\xi'} \frac{dII}{dm} \times \frac{i\epsilon\xi^2 - 2i\epsilon m\xi\xi'}{m + i\epsilon\xi^2} \lesssim II.$$

Observing the uniform boundedness of I, dI/dm, the contributions of L_3 and L_4 have been controlled in [6].

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Collecting the estimates above and noticing the support of $\chi_{[k-1,k+1]}$, $\eta_{\le j}(m)$,

$$\begin{aligned}\left|\int_{\mathbb{R}} e^{ix\xi}\frac{m}{m+i\epsilon\xi^2}\chi_{[k-1,k+1]}(\xi)\eta_{\le j}(m)\,d\xi\right| &\lesssim \left|\frac{1}{x^2}\int_{\mathbb{R}} e^{ix\xi}\left(\frac{1}{2\xi}+1+\frac{1}{4\xi^3}+2\xi\right)\chi_{[k-1,k+1]}(\xi)\eta_{\le j}(m)\,dm\right| \\ &\lesssim \left|\frac{1}{x^2}\int_{\mathbb{R}} e^{ix\xi}\left(\frac{1}{2\xi}+1+\frac{1}{4\xi^3}+2\xi\right)\chi_{[k-1,k+1]}(\xi)\eta_{\le j}(\tau-\omega(\xi))2\xi\,d\xi\right| \\ &\lesssim \frac{2^{j-k}}{1+(2^{j-k}x)^2},\end{aligned}$$

where we make a change of variable to $m=\tau-\omega(\xi)$. If $\tau=2^{2k}$, we get the desired result. □

For later use, we recall the following trilinear estimate.

LEMMA 3.4 [2, Proposition 6.3]. *For $s\ge 1/2$,*

$$\begin{aligned}\|\partial_x(\psi(t)^3uvw)\|_{N^s} \lesssim\ & \|u\|_{F^s}\|v\|_{F^{1/2}}\|w\|_{F^{1/2}}+\|u\|_{F^{1/2}}\|v\|_{F^s}\|w\|_{F^{1/2}} \\ &+\|u\|_{F^{1/2}}\|v\|_{F^{1/2}}\|w\|_{F^s}.\end{aligned}$$

REMARK 3.5. In [2], the coefficients are $\beta_{k,j}=1+2^{(j-2k)/2}$. Lemma 3.4 also holds for our choice $\beta_{k,j}=1+2^{2(j-2k)/5}$; see [4] for details.

Noticing the assumption $\|\phi\|_{L^2}\ll 1$ and the scaling (3.1), it suffices to consider (1.1) with data ϕ satisfying

$$\|\phi\|_{H^s}=r\ll 1.$$

Notice that $F^s\subseteq C(\mathbb{R};H^s)$ for any $s\ge 0$; see [2]. Collecting (4.3), Lemmas 3.1, 3.2, 3.4 and standard fixed-point machinery, we obtain part (a) of Theorem 1.2. The rest of Theorem 1.2 follows from a standard argument.

4. Proof of Theorem 1.3

4.1. Uniform global well-posedness for mBOB. We now extend the local solution obtained above to a global one. We use a conservation law to obtain our goal. From [8, 11], we know that there are two conservation laws for the real-valued mBO equation (1.3):

$$\frac{d}{dt}\int_{\mathbb{R}} u^2\,dx=0, \tag{4.1}$$

$$\frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}u\mathcal{H}u_x-\frac{1}{12}u^4(x,t)\,dx=0. \tag{4.2}$$

Let u be a smooth solution of (1.1). Multiply by u and use partial integration to obtain

$$\frac{1}{2}\|u(t)\|_2^2+\epsilon\int_0^t\|\Lambda u(\tau)\|_2^2\,d\tau=\frac{1}{2}\|\phi\|_2^2,$$

where we use the notation $\Lambda=|\partial_x|$.

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Turning to the conservation law for (1.1), let

$$H[u]=\int_{\mathbb{R}}\frac{1}{2}u\mathcal{H}u_x-\frac{1}{12}u^4\,dx.$$

Noticing that (4.2) is a conserved quantity of (1.3),

$$\begin{aligned}\frac{d}{dt}H[u]&=\int_{\mathbb{R}}\partial_t u\mathcal{H}u_x-\frac{1}{3}u^3u_t\,dx\\&=\epsilon\int_{\mathbb{R}}u_{xx}\mathcal{H}u_x-\frac{1}{3}u^3u_{xx}\,dx\\&=-\epsilon\|\Lambda^{3/2}u\|_2^2+\epsilon\int_{\mathbb{R}}u^2u_x^2\,dx\\&\le-\epsilon\|\Lambda^{3/2}u\|_2^2+\epsilon\|u_x\|_2^2\|u\|_\infty^2\\&\lesssim-\epsilon\|\Lambda^{3/2}u\|_2^2+\frac{\epsilon}{2}\|\Lambda^{3/2}u\|_2^2,\end{aligned}$$

where we use $\|u\|_2\le\|\phi\|_2\ll 1$, the Gagliardo–Nirenberg inequality and the interpolation inequality

$$\|u\|_\infty\lesssim\|u\|_2^{1/2}\|u_x\|_2^{1/2},\quad \|u_x\|_2\lesssim\|u\|_2^{1/3}\|\Lambda^{3/2}u\|_2^{2/3}.$$

Therefore,

$$\sup_{[0,T]}\|u(t)\|_{H^{1/2}}+\epsilon^{1/2}\bigg(\int_0^T\|\Lambda^{3/2}u(\tau)\|_2^2\,d\tau\bigg)^{1/2}\le C(T,\|\phi\|_{H^{1/2}}),\quad \forall T>0. \tag{4.3}$$

Hence the solution is global.

4.2. Limit behaviour. From persistence of regularity of Theorem 1.2, it suffices to show that $s=1/2$. We reprise some ideas from [5, 15, 16] to obtain our result.

LEMMA 4.1 [4, Lemma 8.1]. *Assume that $\delta>0$. If $s\in\mathbb{R}$ and $u\in L_t^2H_x^s$, then*

$$\|u\|_{N^s}\lesssim\|u\|_{L_t^2H_x^s}. \tag{4.4}$$

Assume that u is an $H^{1/2}$-strong solution of (1.1) obtained above, and that v is an $H^{1/2}$-strong solution to (1.3) in [2], with initial data $\phi_1,\phi_2\in H^{1/2}$ respectively. From the scaling (3.1) and the assumption that $\|\phi_i(x)\|_2\ll 1$, $i=1,2$, we may suppose that $\|\phi_1\|_{H^{1/2}},\|\phi_2\|_{H^{1/2}}\ll 1$. Let $w=u-v$ and $\phi=\phi_1-\phi_2$. Then w solves

$$\begin{aligned}w_t+\mathcal{H}w_{xx}-\epsilon w_{xx}&=u^2u_x-v^2v_x,\quad (x,t)\in\mathbb{R}^2,\\ w(0)&=\phi(x).\end{aligned} \tag{4.5}$$

We first view ϵu_{xx} as a perturbation to the difference equation of the mBO equation. Consider the integral equation of (4.5):

$$w(x,t)=W_0(t)\phi-\int_0^t W_0(t-\tau)[\epsilon u_{xx}+u^2u_x-v^2v_x]\,d\tau,\quad t\ge 0.$$

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For technical reasons, let

$$\Phi_\phi^\epsilon(w(x,t)) = \psi(t)\bigg[W_0(t)\phi - \epsilon\int_0^t W_0(t-\tau)\psi(\tau)u_{xx}(\tau)\,d\tau \\ - \frac{1}{3}\int_0^t W_0(t-\tau)(w(v^2+u^2+vu))_x(\tau)\,d\tau\bigg].$$

Then $\Phi_\phi^\epsilon(w)$ solves the integral equation on $t \in [0,1]$. By Lemmas 3.1, 3.2, 3.4 and 4.1,

$$\|\Phi_\phi^\epsilon(w)\|_{F^{1/2}} \lesssim \|\phi\|_{H^{1/2}} + \|w\|_{F^{1/2}}\|u\|_{F^{1/2}}(\|v\|_{F^{1/2}} + \|u\|_{F^{1/2}}) \\ + \epsilon\|u\|_{L^2_{[0,1]}\dot{H}^{5/2}_x} + \|w\|_{F^{1/2}}\|u\|_{F^{1/2}}\|v\|_{F^{1/2}}.$$

Since from (3.1) and the assumption that $\|\phi_i\|_2 \ll 1$, $i = 1, 2$,

$$\|v\|_{F^{1/2}} \lesssim \|\phi_2\|_{H^{1/2}} \ll 1, \quad \|u\|_{F^{1/2}} \lesssim \|\phi_1\|_{H^{1/2}} \ll 1,$$

we obtain

$$\|w\|_{F^{1/2}} \lesssim \|\phi\|_{H^{1/2}} + \epsilon\|u\|_{L^2_{[0,1]}\dot{H}^{5/2}_x}.$$

From the persistence of regularity of Theorem 1.2, we obtain

$$\|u - v\|_{C([0,1],H^{1/2})} \lesssim \|\phi_1 - \phi_2\|_{H^{1/2}} + \epsilon^{1/2}C(\|\phi_1\|_{H^{5/2}}, \|\phi_2\|_{H^{1/2}}).$$

For general $\phi_1 \in H^{5/2}$, $\phi_2 \in H^{1/2}$, using the scaling (3.1), we can show that there exists $T = T(\|\phi_1\|_{H^{5/2}}, \|\phi_2\|_{H^{1/2}}) > 0$ such that

$$\|u - v\|_{C([0,T],H^{1/2})} \lesssim \|\phi_1 - \phi_2\|_{H^{1/2}} + \epsilon^{1/2}C(T, \|\phi_1\|_{H^{5/2}}, \|\phi_2\|_{H^{1/2}}). \tag{4.6}$$

Therefore, (4.6) automatically holds for any $T > 0$, due to (4.1) and (4.2). Let $S_T(\phi)$ be the solution mapping of (1.3) with initial data ϕ. For fixed $T > 0$, we need to prove that for any $\eta > 0$, there exists $\sigma > 0$ such that if $0 < \epsilon < \sigma$, then

$$\|\Phi_T^\epsilon(\phi) - S_T(\phi)\|_{C([0,T];H^{1/2})} < \eta. \tag{4.7}$$

Denoting $\phi_K = P_{\le K}\phi$, we obtain

$$\|\Phi_T^\epsilon(\phi) - S_T(\phi)\|_{C([0,T];H^{1/2})} \le \|\Phi_T^\epsilon(\phi) - \Phi_T^\epsilon(\phi_K)\|_{C([0,T];H^{1/2})} \\ + \|\Phi_T^\epsilon(\phi_K) - S_T(\phi_K)\|_{C([0,T];H^{1/2})} \\ + \|S_T(\phi_K) - S_T(\phi)\|_{C([0,T];H^{1/2})}.$$

From Theorem 1.3 and (4.6), we get

$$\|\Phi_T^\epsilon(\phi) - S_T(\phi)\|_{C([0,T];H^{1/2})} \lesssim \|\phi_K - \phi\|_{H^{1/2}} + \epsilon^{1/2}C(T, K, \|\phi_K\|_{H^{5/2}}).$$

If we fix K large enough, then let ϵ go to zero, we get (4.7).

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