3

The main building blocks

3.1 Domain walls

3.1.1 Preliminaries

In four dimensions domain walls are two-dimensional extended objects. In three dimensions they become domain lines, while in two dimensions they reduce to kinks which can be considered as particles since they are localized. Embeddings of bosonic models supporting kinks in $\mathcal{N}=1$ supersymmetric models in two dimensions were first discussed in [1, 7]. Occasional remarks on kinks in models with four supercharges of the type of the Wess–Zumino models [40] can be found in the literature in the 1980s but they went unnoticed. The only issue which caused much interest and debate in the 1980s was the issue of quantum corrections to the BPS kink mass in 2D models with $\mathcal{N}=1$ supersymmetry.

The mass of the BPS saturated kinks in two dimensions must be equal to the central charge Z in Eq. (2.2.2). The simplest two-dimensional model with two supercharges, admitting solitons, was considered in [41]. In components the Lagrangian takes the form

$$\mathcal{L}=\frac{1}{2}\left(\partial_\mu\phi\,\partial^\mu\phi+\bar{\psi}\,i\,\partial\!\!\!/\psi+F^2\right)+\mathcal{W}'(\phi)F-\frac{1}{2}\mathcal{W}''(\phi)\bar{\psi}\psi, \quad (3.1.1)$$

where ϕ is a real field, ψ is a two-component Majorana spinor in two dimensions, and $\mathcal{W}(\phi)$ is a real "superpotential" which in the simplest case takes the form

$$\mathcal{W}(\phi)=\frac{m^2}{\lambda}\,\phi-\frac{\lambda}{3}\,\phi^3. \quad (3.1.2)$$

Moreover, the auxiliary field F can be eliminated by virtue of the classical equation of motion, $F=-\mathcal{W}'$. This is a real reduction (two supercharges) of the

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Wess–Zumino model (Section 3.1.2). The kink (antikink) BPS equation is

$$\partial_z \phi = \pm \frac{d\mathcal{W}}{d\phi}, \tag{3.1.3}$$

with the boundary condition that $\phi(z)$ tends to two distinct vacua, $\phi_{\rm vac} = \pm m/\lambda$ at $z \to \pm\infty$. It can be readily integrated.

The story of kinks in this model is long and dramatic. In the very beginning it was argued [41] that, due to a residual supersymmetry, the mass of the soliton calculated at the classical level remains intact at the one-loop level. A few years later it was noted [42] that the non-renormalization theorem [41] cannot possibly be correct, since the classical soliton mass is proportional to m^3/λ^2 (where m and λ are the bare mass parameter and coupling constant, respectively), and the physical mass of the scalar field gets a logarithmically infinite renormalization. Since the soliton mass is an observable physical parameter, it must stay finite in the limit $M_{\rm uv} \to \infty$, where $M_{\rm uv}$ is the ultraviolet cut off. This implies, in turn, that the quantum corrections cannot vanish – they "dress" m in the classical expression, converting the bare mass parameter into the renormalized one. The one-loop renormalization of the soliton mass was first calculated in [42]. Technically the emergence of the one-loop correction was attributed to a "difference in the density of states in continuum in the boson and fermion operators in the soliton background field." The subsequent work [43] dealt with the renormalization of the central charge, with the conclusion that the central charge is renormalized in just the same way as the kink mass, so that the saturation condition is not violated.

Then many authors repeated one-loop calculations for the kink mass and/or central charge [44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54]. The results reported and the conclusion of saturation/non-saturation oscillated with time, with little sign of convergence. Needless to say, all authors agreed that the logarithmically divergent term in Z matched the renormalization of m. However, the finite (non-logarithmic) term varied from work to work, sometimes even in the successive works of the same authors. Polemics continued unabated through the 1990s. For instance, Nastase *et al.* [53], presenting a perfectly valid calculation of the kink mass, concluded that the BPS saturation was violated at one loop. This assertion reversed the earlier trend [42, 49, 50], according to which the kink mass and the corresponding central charge are renormalized in a concerted way. A somewhat later publication [54] again changed the scene, advocating BPS saturation. However, a dimensionally regularized kink mass determined in [54] was not consistent with that found in [53].

The story culminated in 1998 with the discovery of a quantum anomaly in the central charge [55]. Classically, the kink central charge Z is equal to the difference between the values of the superpotential $\mathcal{W}$ at spatial infinities,

$$Z = \mathcal{W}[\phi(z=\infty)] - \mathcal{W}[\phi(z=-\infty)]. \tag{3.1.4}$$

This is known from the pioneering paper [1]. Due to the anomaly, the central charge gets modified in the following way

$$\mathcal{W} \longrightarrow \mathcal{W} + \frac{\mathcal{W}''}{4\pi}, \tag{3.1.5}$$

where the term proportional to $\mathcal{W}''$ is anomalous [55]. The right-hand side of Eq. (3.1.5) must be substituted in the expression for the central charge (3.1.4) instead of $\mathcal{W}$. Inclusion of the additional anomalous term restores the equality between the kink mass and its central charge. The BPS nature is preserved, which is correlated with the fact that the kink supermultiplet is short in the case at hand [56]. All subsequent investigations confirmed this conclusion (see e.g. the review paper [57] and original papers [58] by van Nieuwenhuizen and collaborators).

Critical domain walls in $\mathcal{N}=1$ four-dimensional theories (four supercharges) started attracting attention in the 1990s. What is the domain wall? It is a two-dimensional object of co-dimension one. It is a field configuration interpolating between vacuum i and vacuum f with some transition domain in the middle. Say, to the left you have vacuum i, to the right you have vacuum f, in the middle you have a transition domain which, for obvious reasons, is referred to as the wall (Fig. 3.1). The most popular model of this time supporting such domain walls was the generalized Wess–Zumino model with the Lagrangian

$$\mathcal{L} = \int d^2\theta\, d^2\bar\theta\, K(\bar\Phi_a, \Phi_a) + \left(\int d^2\theta\, \mathcal{W}(\Phi) + \text{H.c.}\right) \tag{3.1.6}$$

where K is the Kähler potential and Φ_a stands for a set of the chiral superfields. The number of the chiral superfields can be arbitrary, but the superpotential $\mathcal{W}$ must have at least two critical points, two vacua.

(This model can be considered, upon dimensional reduction, in two dimensions as well.) A popular choice was a trivial Kähler potential,

$$K = \sum_a \bar\Phi_a \Phi_a .$$

BPS walls in this system satisfy the first-order differential equations [59, 24, 60, 61, 62]

$$g_{\bar a b}\, \partial_z \Phi^b = e^{i\eta}\, \partial_{\bar a} \bar{\mathcal{W}}, \tag{3.1.7}$$

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where the Kähler metric is given by

$$g_{\bar{a}b} = \frac{\partial^2 K}{\partial \bar{\Phi}^{\bar{a}}\, \partial \Phi^b} \equiv \partial_{\bar{a}} \partial_b K, \tag{3.1.8}$$

and η is the phase of the (1,0) central charge Z as defined in (2.2.6). The phase η depends on the choice of the vacua between which the given domain wall interpolates,

$$Z = 2\left(\mathcal{W}_{\text{vac}_\text{f}} - \mathcal{W}_{\text{vac}_\text{i}}\right). \tag{3.1.9}$$

A useful consequence of the BPS equations is that

$$\partial_z \mathcal{W} = e^{i\eta}\, \|\partial_a \mathcal{W}\|^2, \tag{3.1.10}$$

and thus the domain wall describes a straight line in the $\mathcal{W}$-plane connecting the two vacua. Needless to say, the first-order BPS equation (3.1.7) guarantees the validity of the second-order equation of motion. The opposite is not true, generally speaking. However, if one deals with a *single* chiral field Φ, one can prove [63] that the BPS equation does follow from the second-order equation of motion.

Construction and analysis of BPS saturated domain walls in four dimensions crucially depends on the realization of the fact that the central charges relevant to critical domain walls are not Lorentz scalars; rather they transform as (1,0) + (0,1) under the Lorentz transformations. It was a textbook statement ascending to the pioneering paper [20] that $\mathcal{N} = 1$ superalgebras in four dimensions leave place to no central charges. This statement is correct only with respect to Lorentz-scalar central charges. Townsend was the first to note [64] that "supersymmetric branes," being BPS saturated, require the existence of tensorial central charges antisymmetric in the vectorial Lorentz indices. That the anticommutator $\{Q_\alpha, Q_\beta\}$ in four-dimensional Wess–Zumino model contains the (1,0) central charge is obvious. This anticommutator vanishes, however, in super-Yang–Mills theory at the classical level (Section 3.1.3).

3.1.2 Domain wall in the minimal Wess–Zumino model

The Wess–Zumino model describes interactions of an arbitrary number of the chiral superfields. We will consider the minimal Wess–Zumino model [65] which describes one chiral superfield,

$$\Phi(x_L, \theta) = \phi(x_L) + \sqrt{2}\,\theta^\alpha \psi_\alpha(x_L) + \theta^2 F(x_L), \tag{3.1.11}$$

$$(x_L)_{\alpha\dot{\alpha}} = x_{\alpha\dot{\alpha}} \mp 2i\, \theta_\alpha \bar{\theta}_{\dot{\alpha}}, \tag{3.1.12}$$

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with the canonic kinetic term $K = \bar{\Phi}\Phi$. In components the Lagrangian has the form

$$\mathcal{L} = (\partial^\mu \bar{\phi})(\partial_\mu \phi) + \psi^\alpha i \partial_{\alpha\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} + \bar{F} F + \left\{ F \mathcal{W}'(\phi) - \frac{1}{2} \mathcal{W}''(\phi) \psi^2 + \text{H.c.} \right\}. \tag{3.1.13}$$

From Eq. (3.1.13) it is obvious that F can be eliminated by virtue of the classical equation of motion,

$$\bar{F} = -\frac{\partial \mathcal{W}(\phi)}{\partial \phi}, \tag{3.1.14}$$

so that the scalar potential describing self-interaction of the field ϕ is

$$V(\phi, \bar{\phi}) = \left| \frac{\partial \mathcal{W}(\phi)}{\partial \phi} \right|^2. \tag{3.1.15}$$

In what follows we will often denote the chiral superfield and its lowest (bosonic) component by one and the same letter, making no distinction between capital and small ϕ. Usually it is clear from the context what is meant in each particular case.

If one limits oneself to renormalizable theories, the superpotential $\mathcal{W}$ must be a polynomial function of Φ of power not higher than three. In the model at hand, with one chiral superfield, the generic superpotential can be always reduced to the following "standard" form:

$$\mathcal{W}(\Phi) = \frac{m^2}{\lambda} \Phi - \frac{\lambda}{3} \Phi^3. \tag{3.1.16}$$

The quadratic term can be always eliminated by a redefinition of the field Φ. Moreover, by using symmetries of the model one can always choose the phases of the constants m and λ at will.

The superpotential (3.1.16) implies two degenerate classical vacua,

$$\phi_{\text{vac}} = \pm \frac{m}{\lambda}. \tag{3.1.17}$$

Both vacua are physically equivalent. This equivalence could be explained by the spontaneous breaking of Z_2 symmetry, $\Phi \to -\Phi$, present in the action.

Field configurations interpolating between two degenerate vacua are the domain walls. They have the following properties: (i) the corresponding solutions are static and depend only on one spatial coordinate; (ii) they are topologically stable and indestructible – once a wall is created it cannot disappear. Assume for definiteness that the wall lies in the xy plane. This is the geometry we will always keep in mind.

Then the wall solution $\phi_{\rm w}$ will depend only on z. Since the wall extends indefinitely in the xy plane, its energy $E_{\rm w}$ is infinite. However, the wall tension $T_{\rm w}$ (the energy per unit area $T_{\rm w} = E_{\rm w}/A$) is finite, in principle measurable, and has a clear-cut physical meaning.

The wall solution of the classical equations of motion superficially looks very similar to that of the two-dimensional kink,

$$\phi_{\rm w} = \frac{m}{\lambda} \tanh(|m|z). \tag{3.1.18}$$

Note, however, that the parameters m and λ are not necessarily assumed to be real; the field ϕ is complex in the Wess–Zumino model. A remarkable feature of this solution is that it preserves 1/2 of supersymmetry, much in the same way as the kink of Section 3.1.1. The difference is that 1/2 BPS in the two-dimensional model meant one supercharge, now it means two supercharges.

The SUSY transformations generate the following transformation of the fields:

$$\delta\phi = \sqrt{2}\varepsilon\psi, \quad \delta\psi^{\alpha} = \sqrt{2}\left[\varepsilon^{\alpha} F + i\, \partial_{\mu}\phi\, (\sigma^{\mu})^{\alpha\dot{\alpha}}\, \bar{\varepsilon}_{\dot{\alpha}}\right]. \tag{3.1.19}$$

The domain wall we consider is purely bosonic, $\psi = 0$. Moreover, the BPS equation is

$$F\, |_{\bar{\phi}=\phi_{\rm w}^*} = -e^{-i\eta}\, \partial_z \phi_{\rm w}(z), \tag{3.1.20}$$

where

$$\eta = \arg \frac{m^3}{\lambda^2}, \tag{3.1.21}$$

and $F = -\partial\bar{\mathcal{W}}/\partial\bar{\phi}$. This is a first-order differential equation. The solution quoted above satisfies this condition. The reason for the occurrence of the phase factor $\exp(-i\eta)$ on the right-hand side of Eq. (3.1.20) will become clear shortly. Note that no analog of this phase factor exists in the two-dimensional $\mathcal{N} = 1$ problem on which we dwelled in Section 3.1.1. There was only a sign ambiguity: two possible choices of signs corresponded to kink versus antikink.

If the BPS equation is satisfied, then the second supertransformation in Eq. (3.1.19) reduces to

$$\delta\psi_{\alpha} \propto \varepsilon_{\alpha} + i\, e^{i\eta}\, (\sigma^z)_{\alpha\dot{\alpha}}\, \bar{\varepsilon}^{\dot{\alpha}}. \tag{3.1.22}$$

The right-hand side vanishes provided that

$$\varepsilon_{\alpha} = -i\, e^{i\eta}\, (\sigma^z)_{\alpha\dot{\alpha}}\, \bar{\varepsilon}^{\dot{\alpha}}. \tag{3.1.23}$$

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This picks up two supertransformations (out of four) which do not act on the domain wall (alternatively people often say that they act trivially). *Quod erat demonstrandum*.

Now, let us calculate the wall tension. To this end we rewrite the expression for the energy functional as

$$\begin{aligned}\mathcal{E} &= \int_{-\infty}^{+\infty} dz \left[\partial_z\bar{\phi}\,\partial_z\phi + \bar{F}F\right]\\ &\equiv \int_{-\infty}^{+\infty} dz \left\{\left[e^{-i\eta}\,\partial_z\mathcal{W} + \text{H.c.}\right] + \left|\partial_z\phi + e^{i\eta}\,F\right|^2\right\},\end{aligned} \tag{3.1.24}$$

where ϕ is assumed to depend only on z. In the literature this procedure is called the Bogomol'nyi completion. The second term on the right-hand side is non-negative – its minimal value is zero. The first term, being full derivative, depends only on the boundary conditions on ϕ at $z = \pm\infty$.

Equation (3.1.24) implies that $\mathcal{E} \geq 2\,\text{Re}\left(e^{-i\eta}\,\Delta\mathcal{W}\right)$. The Bogomol'nyi completion can be performed with any η. However, the strongest bound is achieved provided $e^{-i\eta}\,\Delta\mathcal{W}$ is real. This explains the emergence of the phase factor in the BPS equations. In the model at hand, to make $e^{-i\eta}\,\Delta\mathcal{W}$ real, we have to choose η according to Eq. (3.1.21).

When the energy functional is written in the form (3.1.24), it is perfectly obvious that the absolute minimum is achieved provided the BPS equation (3.1.20) is satisfied. In fact, the Bogomol'nyi completion provides us with an alternative way of derivation of the BPS equations. Then, for the minimum of the energy functional – the wall tension T_{w} – we get

$$T_{\text{w}} = |\mathcal{Z}|\,. \tag{3.1.25}$$

Here $\mathcal{Z}$ is the topological charge defined as

$$\mathcal{Z} = 2\left\{\mathcal{W}(\phi(z=\infty)) - \mathcal{W}(\phi(z=-\infty))\right\} = \frac{8\,m^3}{3\,\lambda^2}. \tag{3.1.26}$$

In the problem at hand, the central extension of the superalgebra is tensorial, with the Lorentz structure (1,0)+(0,1),

$$\left\{Q_\alpha, Q_\beta\right\} = -4\,\Sigma_{\alpha\beta}\,\bar{\mathcal{Z}}\,,\qquad \left\{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\right\} = -4\,\bar{\Sigma}_{\dot\alpha\dot\beta}\,\mathcal{Z}. \tag{3.1.27}$$

Here

$$\Sigma_{\alpha\beta} = -\frac{1}{2}\int dx_{[\mu}\mathrm{d}x_{\nu]}\,(\sigma^\mu)_{\alpha\dot\alpha}(\bar{\sigma}^\nu)^{\dot\alpha}_{\beta} \tag{3.1.28}$$

is the wall area tensor.

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The expressions for two supercharges $\tilde{Q}_\alpha$ that do annihilate the wall are

$$\tilde{Q}_\alpha = e^{i\eta/2}\, Q_\alpha - \frac{2}{A}\, e^{-i\eta/2}\, \Sigma_{\alpha\beta}\, n^{\beta}_{\dot\alpha}\, \bar{Q}^{\dot\alpha}, \tag{3.1.29}$$

where

$$n_{\alpha\dot\alpha} = \frac{P_{\alpha\dot\alpha}}{T_{\rm w} A} \tag{3.1.30}$$

is the unit vector proportional to the wall four-momentum $P_{\alpha\dot\alpha}$; it has only the time component in the rest frame. The subalgebra of these "residual" (unbroken) supercharges in the rest frame is

$$\left\{\tilde{Q}_\alpha, \tilde{Q}_\beta\right\} = 8 \sum_{\alpha\beta} \{T_{\rm w} - |\mathcal{Z}|\}\,. \tag{3.1.31}$$

The existence of the subalgebra (3.1.31) immediately proves that the wall tension $T_{\rm w}$ is equal to the central charge $\mathcal{Z}$. Indeed, $\tilde{Q}|\text{wall}\rangle = 0$ implies that $T_{\rm w} - |\mathcal{Z}| = 0$. This equality is valid both to any order in perturbation theory and nonperturbatively.

From the non-renormalization theorem for the superpotential [65, 66] we additionally infer that the central charge $\mathcal{Z}$ is not renormalized. This is in contradistinction with the situation in the two-dimensional model of Section 3.1.1. The fact that in four dimensions there are more conserved supercharges than in two turns out crucial. As a consequence, the result

$$T_{\rm w} = \frac{8}{3}\left|\frac{m^3}{\lambda^2}\right| \tag{3.1.32}$$

for the wall tension is exact [62].

The wall tension $T_{\rm w}$ is a physical parameter and, as such, should be expressible in terms of the physical (renormalized) parameters $m_{\rm ren}$ and $\lambda_{\rm ren}$. One can easily verify that this is compatible with the statement of non-renormalization of $T_{\rm w}$. Indeed,

$$m = Z\, m_{\rm ren}, \qquad \lambda = Z^{3/2}\lambda_{\rm ren},$$

where Z is the Z factor coming from the kinetic term. Consequently,

$$\frac{m^3}{\lambda^2} = \frac{m^3_{\rm ren}}{\lambda^2_{\rm ren}}.$$

Thus, the absence of the quantum corrections to Eq. (3.1.32), the renormalizability of the theory, and the non-renormalization theorem for superpotentials – all these three elements are intertwined with each other. In fact, every two elements taken separately imply the third one.

What lessons have we drawn from the example of the domain walls? In the centrally extended SUSY algebras the exact relation $E_{\rm vac} = 0$ is replaced by the exact relation $T_{\rm w} - |\mathcal{Z}| = 0$. Although this statement is valid both perturbatively and nonperturbatively, it is very instructive to visualize it as an explicit cancellation between bosonic and fermionic modes in perturbation theory. The non-renormalization of $\mathcal{Z}$ is a specific feature of four dimensions. We have seen previously that it does not take place in minimally supersymmetric models in two dimensions.

Finding the solution to the BPS equation

In two-dimensional theory integration of the first-order BPS equation (3.1.3) was trivial. Now the BPS equation (3.1.20) presents in fact two equations – one for the real part and one for the imaginary. Nevertheless finding the solution is still trivial. This is due to the existence of an "integral of motion,"

$$\frac{\partial}{\partial z}\left(\mathrm{Im}\, e^{-i\eta}\mathcal{W}\right) = 0. \tag{3.1.33}$$

The proof is straightforward and is valid in the generic Wess–Zumino model with arbitrary number of fields. Indeed, differentiating $\mathcal{W}$ and using the BPS equations we get

$$\frac{\partial}{\partial z}\left(e^{-i\eta}\mathcal{W}\right) = \left|\frac{\partial \mathcal{W}}{\partial \phi}\right|^2, \tag{3.1.34}$$

which immediately entails Eq. (3.1.33). The constraint

$$\mathrm{Im}\, e^{-i\eta}\mathcal{W} = \text{const} \tag{3.1.35}$$

can be interpreted as follows: in the complex $\mathcal{W}$ plane the domain wall trajectory is a straight line (see Section 3.1.1).

Living on a wall

What is the fate of two broken supercharges? As we already know, two out of four supercharges annihilate the wall – these supersymmetries are preserved in the given wall background. Two other supercharges are broken: being applied to the wall solution they create two fermion zero modes. These zero modes correspond to (2+1)-dimensional Majorana (massless) spinor field $\psi(t, x, y)$ localized on the wall.

To elucidate the above assertion it is convenient to turn first to the fate of another symmetry of the original theory which is spontaneously broken for each given wall, namely, translational invariance in the z direction.

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Indeed, each wall solution, e.g. Eq. (3.1.18), breaks this invariance. This means that in fact we must deal with a family of solutions: if $\phi(z)$ is a solution, so is $\phi(z - z_0)$. The parameter z_0 is a collective coordinate – the wall center. People also refer to it as a modulus (in plural, moduli). For the static wall z_0 is a fixed constant.

Assume, however, that the wall is slightly bent. The bending should be negligible compared to the wall thickness (which is of the order of m^{-1}). The bending can be described as an adiabatically slow dependence of the wall center z_0 on t, x, and y. We will write this slightly bent wall field configuration as

$$\phi(t, x, y, z) = \phi_{\rm w}(z - \zeta(t, x, y)). \tag{3.1.36}$$

Substituting this field in the original action we arrive at the following effective (2+1)-dimensional action for the field $\zeta(t, x, y)$:

$$S^{\zeta}_{2+1} = \frac{T_{\rm w}}{2} \int d^3x \, \left(\partial^m \zeta\right) (\partial_m \zeta) \, , \quad m = 0, 1, 2. \tag{3.1.37}$$

It is clear that $\zeta(t, x, y)$ can be viewed as a massless scalar field (called the translational modulus) which lives on the wall. It is nothing but a Goldstone field corresponding to the spontaneous breaking of the translational invariance.

Returning to two broken supercharges, they generate a Majorana (2+1)-dimensional Goldstino field $\psi_\alpha(t, x, y)$, $(\alpha = 1, 2)$ localized on the wall. The total (2+1)-dimensional effective action on the wall world volume takes the form

$$S_{2+1} = \frac{T_{\rm w}}{2} \int d^3x \left\{ \left(\partial^m \zeta\right) (\partial_m \zeta) + \bar{\psi}\, \partial_m \gamma^m \psi \right\} \tag{3.1.38}$$

where γ^m are three-dimensional gamma matrices (in the Majorana representation, see Appendix A, Section A.1).

The effective theory of the moduli fields on the wall worldvolume is supersymmetric, with two conserved supercharges. This is the minimal supersymmetry in 2+1 dimensions. It corresponds to the fact that two out of four supercharges are conserved.

3.1.3 D-branes in gauge field theory

In 1996 Dvali and Shifman found in supersymmetric gluodynamics [11] an anomalous $(1, 0)$ central charge in superalgebra, not seen at the classical level. They argued that this central charge is saturated by domain walls interpolating between vacua with distinct values of the order parameter, the gluino condensate $\langle \lambda\lambda \rangle$, labeling N distinct vacua of super-Yang–Mills theory with the gauge group SU(N).

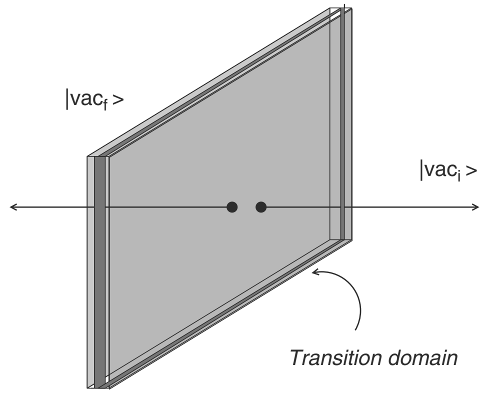

Figure 3.1. A field configuration interpolating between two distinct degenerate vacua.

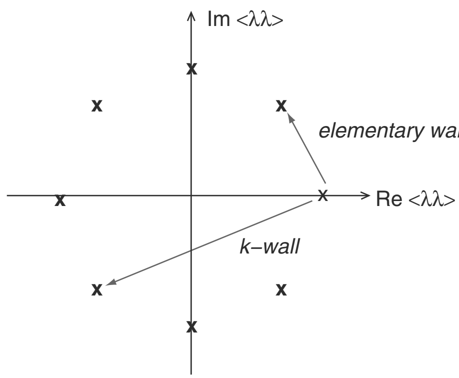

Figure 3.2. N vacua for SU(N). The vacua are labeled by the vacuum expectation value $\langle\lambda\lambda\rangle = -6\,N\,\Lambda^3\,\exp(2\pi\,i\,k/N)$ where $k = 0, 1, \ldots, N-1$. Elementary walls interpolate between two neighboring vacua.

Supersymmetric gluodynamics (it is often referred to as pure super-Yang–Mills theory) is defined by the Lagrangian

$$\mathcal{L} = \frac{1}{g^2}\int d^2\theta\,\mathrm{Tr}\,W^2 + \mathrm{H.c.} = \frac{1}{g^2}\left\{-\frac{1}{4}F^a_{\mu\nu}\,F^{a\mu\nu} + i\lambda^{a\alpha}\mathcal{D}_{\alpha\dot\beta}\bar\lambda^{a\dot\beta}\right\}, \qquad (3.1.39)$$

where $\lambda^{a\alpha}$ is the Weyl spinor in the adjoint representation of SU(N).

The domain wall is a field configuration interpolating between two distinct degenerate vacua (see Fig. 3.1). There is a large variety of walls in supersymmetric gluodynamics. Minimal, or elementary, walls interpolate between vacua n and $n+1$, while k-walls interpolate between n and $n+k$, see Fig. 3.2. In [11] a mechanism was suggested for localizing gauge fields on the wall through bulk confinement.

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Later this mechanism was implemented in models at weak coupling, as we will see below.

Shortly afterwards, Witten interpreted the BPS walls in supersymmetric gluodynamics as analogs of D-branes [12]. This is because their tension scales as $N \sim 1/g_s$ rather than $1/g_s^2$ typical of solitonic objects (here g_s is the string constant). Many promising consequences ensued. One of them was the Acharya–Vafa derivation of the wall worldvolume theory [67]. Using a wrapped D-brane picture and certain dualities they identified the k-wall worldvolume theory as 1+2 dimensional U(k) gauge theory with the field content of $\mathcal{N} = 2$ and the Chern–Simons term at level N breaking $\mathcal{N} = 2$ down to $\mathcal{N} = 1$.

In $\mathcal{N} = 1$ gauge theories with arbitrary matter content and superpotential the general relation (2.2.5) takes the form

$$\left\{Q_\alpha, Q_\beta\right\} = -4\, \Sigma_{\alpha\beta}\, \bar{Z}, \tag{3.1.40}$$

where

$$\Sigma_{\alpha\beta} = -\frac{1}{2}\int dx_{[\mu} dx_{\nu]}\, (\sigma^\mu)_{\alpha\dot\alpha} (\bar\sigma^\nu)^{\dot\alpha}_\beta \tag{3.1.41}$$

is the wall area tensor, and [62, 68]

$$\begin{aligned} Z = \frac{2}{3}\Delta \Bigg\{ &\left[3\mathcal{W} - \sum_f Q_f \frac{\partial \mathcal{W}}{\partial Q_f} \right] \\ &- \left[\frac{3N - \sum_f T(R_f)}{16\pi^2}\, \mathrm{Tr}\, W^2 + \frac{1}{8}\sum_f \gamma_f \bar{D}^2 (\bar{Q}_f e^V Q_f) \right] \Bigg\}_{\theta=0} \end{aligned} \tag{3.1.42}$$

In this expression Δ implies taking the difference at two spatial infinities in the direction perpendicular to the surface of the wall. The first term in the second line presents the gauge anomaly in the central charge. The second term in the second line is a total superderivative. Therefore, it vanishes after averaging over any supersymmetric vacuum state. Hence, it can be safely omitted. The first line presents the classical result, cf. Eq. (3.1.9). At the classical level $Q_f(\partial\mathcal{W}/\partial Q_f)$ is a total superderivative too which can be seen from the Konishi anomaly [69],

$$\bar{D}^2\, (\bar{Q}_f e^V Q_f) = 4\, Q_f \frac{\partial \mathcal{W}}{\partial Q_f} + \frac{T(R_f)}{2\pi^2}\, \mathrm{Tr}\, W^2. \tag{3.1.43}$$

If we discard this total superderivative for a short while (forgetting about quantum effects), we return to $Z = 2\Delta(\mathcal{W})$, the formula obtained in the Wess–Zumino model. At the quantum level $Q_f(\partial\mathcal{W}/\partial Q_f)$ ceases to be a total superderivative because of the Konishi anomaly. It is still convenient to eliminate $Q_f(\partial\mathcal{W}/\partial Q_f)$ in favor of $\mathrm{Tr}W^2$ by virtue of the Konishi relation (3.1.43). In this way one arrives at

$$Z = 2\Delta\left\{\mathcal{W} - \frac{N - \sum_f T(R_f)}{16\pi^2}\,\mathrm{Tr}\,W^2\right\}_{\theta=0}. \tag{3.1.44}$$

We see that the superpotential $\mathcal{W}$ is amended by the anomaly; in the operator form

$$\mathcal{W} \longrightarrow \mathcal{W} - \frac{N - \sum_f T(R_f)}{16\pi^2}\,\mathrm{Tr}\,W^2. \tag{3.1.45}$$

Of course, in pure Yang–Mills theory only the anomaly term survives.

Beginning from 2002 we developed a benchmark $\mathcal{N} = 2$ model, weakly coupled in the bulk (and, thus, fully controllable), which supports both BPS walls and BPS flux tubes. We demonstrated that a gauge field is indeed localized on the wall; for the minimal wall this is a U(1) field while for non-minimal walls the localized gauge field is non-Abelian. We also found a BPS wall-string junction related to the gauge field localization, see Chapter 8. The field-theory string does end on the BPS wall, after all! The end-point of the string on the wall, after Polyakov's dualization, becomes a source of the electric field localized on the wall. In 2005 Norisuke Sakai and David Tong analyzed generic wall-string configurations. Following condensed matter physicists they called them boojums.[1]

Equation (3.1.42) implies that in pure gluodynamics (super-Yang–Mills theory without matter) the domain wall tension is

$$T = \frac{N}{8\pi^2}\left|\langle\mathrm{Tr}\lambda^2\rangle_{\mathrm{vac\ f}} - \langle\mathrm{Tr}\lambda^2\rangle_{\mathrm{vac\ i}}\right| \tag{3.1.46}$$

where $\mathrm{vac}_{\mathrm{i,f}}$ stands for the initial (final) vacuum between which the given wall interpolates. Furthermore, the gluino condensate $\langle\mathrm{Tr}\lambda^2\rangle_{\mathrm{vac}}$ was calculated – exactly – long ago [70], using the very same methods which were later advanced and perfected by Seiberg and Seiberg and Witten in their quest for dualities in $\mathcal{N} = 1$ super-Yang–Mills theories [71] and the dual Meissner effect in $\mathcal{N} = 2$ (see [2, 3]). Namely,

$$2\,\langle\mathrm{Tr}\lambda^2\rangle = \langle\lambda^a_\alpha\lambda^{a,\alpha}\rangle = -6N\Lambda^3\exp\left(\frac{2\pi i k}{N}\right), \quad k = 0, 1, \ldots, N-1. \tag{3.1.47}$$

[1] "Boojum" comes from L. Carroll's children's book *The Hunting of the Snark*. Apparently, it is fun to hunt a snark, but if the snark turns out to be a boojum, you are in trouble! Condensed matter physicists adopted the name to describe solitonic objects of the wall-string junction type in helium-3. Also: The boojum tree (Mexico) is the strangest plant imaginable. For most of the year it is leafless and looks like a giant upturned turnip. G. Sykes, found it in 1922 and said, referring to Carroll, "It must be a boojum!" The common Spanish name for this tree is Cirio, referring to its candle-like appearance.

Here k labels the N distinct vacua of the theory, see Fig. 3.2, and Λ is a dynamical scale, defined in the standard manner (i.e. in accordance with Ref. [72]) in terms of the ultraviolet parameters, M_{uv} (the ultraviolet regulator mass), and g_0^2 (the bare coupling constant),

$$\Lambda^3 = \frac{2}{3}\, M_{\text{uv}}^3 \left(\frac{8\pi^2}{Ng_0^2}\right) \exp\left(-\frac{8\pi^2}{Ng_0^2}\right). \tag{3.1.48}$$

In each given vacuum the gluino condensate scales with the number of colors as N. However, the difference of the values of the gluino condensates in two vacua which lie not too far away from each other scales as N^0. Taking into account Eq. (3.1.46) we conclude that the wall tension in supersymmetric gluodynamics

$$T \sim N.$$

(This statement just rephrases Witten's argument why the above walls should be considered as analogs of D-branes.)

The volume energy density in both vacua, to the left and to the right of the wall, vanish due to supersymmetry. Inside the transition domain, where the order parameter changes its value gradually, the volume energy density is expected to be proportional to N^2, just because there are N^2 excited degrees of freedom. Therefore, $T \sim N$ implies that the wall thickness in supersymmetric gluodynamics must scale as N^{-1}. This is very unusual, because normally we would say: the glueball mass is $O(N^0)$, hence, everything built of regular glueballs should have thickness of order $O(N^0)$.

If the wall thickness is indeed $O(N^{-1})$ the question "what consequences ensue?" immediately comes to one's mind. This issue is far from complete understanding, for relevant discussions see [73, 74, 75].

As was mentioned, there is a large variety of walls in supersymmetric gluodynamics as they can interpolate between vacua with arbitrary values of k. Even if $k_f = k_i + 1$, i.e. the wall is elementary, in fact we deal with several walls, all having one and the same tension – let us call them degenerate walls. The first indication on the wall degeneracy was obtained in Ref. [76], where two degenerate walls were observed in SU(2) theory. Later, Acharya and Vafa calculated the k-wall multiplicity [67] within the framework of D-brane/string formalism,

$$\nu_k = C_N^k = \frac{N!}{k!(N-k)!}. \tag{3.1.49}$$

For $N = 2$ only elementary walls exist, and $\nu = 2$. In the field-theoretic setting Eq. (3.1.49) was derived in [77]. The derivation is based on the fact that the index ν is topologically stable – continuous deformations of the theory do not change ν.

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Thus, one can add an appropriate set of matter fields sufficient for complete Higgsing of supersymmetric gluodynamics. The domain wall multiplicity in the effective low-energy theory obtained in this way is the same as in supersymmetric gluodynamics albeit the effective low-energy theory, a Wess–Zumino type model, is much simpler.

3.1.4 Domain wall junctions

Two degenerate domain walls can coexist in one plane – a new phenomenon which, to the best of our knowledge, was first discussed in [78]. It is illustrated in Fig. 3.3. Two distinct degenerate domain walls lie on the plane; the transition domain between wall 1 and wall 2 is the domain wall junction (domain line).

Each individual domain wall is 1/2 BPS-saturated. The wall configuration with the junction line (Fig. 3.3) is 1/4 BPS-saturated. We start from $\mathcal{N} = 1$ four-dimensional bulk theory (four supercharges). Naively, the effective theory on the plane must preserve two supercharges, while the domain line must preserve one supercharge. In fact, they have four and two conserved supercharges, respectively. This is another new phenomenon – supersymmetry enhancement – discovered in [78]. One can excite the junction line endowing it with momentum in the direction of the line, without altering its BPS status. A domain line with a plane wave propagating on it (Fig. 3.3) preserves the property of the BPS saturation, see [78].

Let us pass now to more conventional wall junctions. Assume that the theory under consideration has a spontaneously broken Z_N symmetry, with $N \geq 3$, and, correspondingly, N vacua. Then one can have N distinct walls connected in the asterisk-like pattern, see Fig. 3.4. This field configuration possesses an obvious axial symmetry: the vacua are located cyclically.

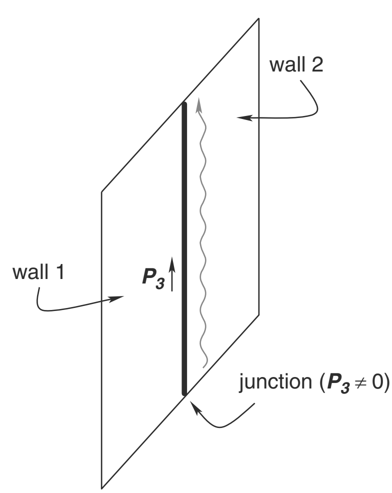

Figure 3.3. Two distinct degenerate domain walls separated by the wall junction.

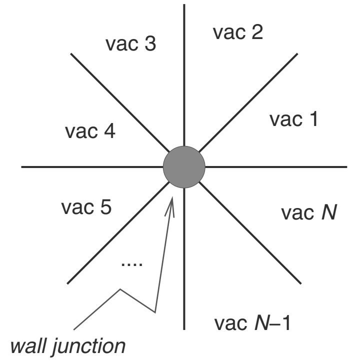

Figure 3.4. The cross section of the wall junction.

This configuration is absolutely topologically stable, as stable as the wall itself. Moreover, it can be 1/4 BPS-saturated for any value of N. It was noted [24] that theories with either U(1) or Z_N global symmetries may contain 1/4-BPS objects with axial geometry. They saturate two central charges simultaneously, (1,0) + (0,1) (the walls) and (1/2, 1/2) (the junction line).

The corresponding Bogomol'nyi equations were derived in [62] and shortly after rediscovered in [79]. Further advances in the issue of the domain wall junctions of the hub-and-spokes type were presented in [80, 81, 82, 83], see also later works [84, 85, 86, 87, 88]. We would like to single out Ref. [81] where the first analytic solution for a BPS wall junction was found in a specific generalized Wess–Zumino model. Among stimulating findings in this work is the fact that the junction tension turned out to be negative in this model. The model has Z_3 symmetry. It is derived from a SU(2) Yang–Mills theory with extended supersymmetry ($\mathcal{N} = 2$) and one matter flavor perturbed by an adjoint scalar mass. The original model contains three pairs of chiral superfields and, in addition, one extra chiral superfield. In fact, the model of [81] can be simplified and adjusted to cover the case of arbitrary N, which was done in [83]. The latter work demonstrates that the tension of the wall junctions is generically negative although exceptional models with the positive tension are possible too. Note that the negative sign of the wall junction tension does not lead to instability since the wall junctions do not exist in isolation. They are always attached to walls which stabilize this field configuration.

Returning to SU(N) supersymmetric gluodynamics ($N \geq 3$) one expects to get in this theory the 1/4 BPS junctions of the type depicted in Fig. 3.4. Of course, this theory is strongly coupled; therefore, the classical Bogomol'nyi equations are irrelevant. However, assuming that such wall junctions do exist, one can find their tension at large N even without solving the theory. To this end one uses [74, 83] the

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expression for the (1/2,1/2) central charge[2] in terms of the contour integral over the axial current [27]. At large N the latter integral is determined by two things: the absolute value of the gluino condensate and the overall change of the phase of the condensate when one makes the 2π rotation around the hub. In this way one arrives at the prediction

$$T_{\text{wall junction}} \sim N^2. \tag{3.1.50}$$

The coefficient in front of the N^2 factor is model dependent.

Can one interpret this N^2 dependence of the hub of the junction? Assume that each wall has thickness $1/N$ and there are N of them. Then it is natural to expect the radius of the intermediate domain where all walls join together to be of the order $(1/N) \times N \sim N^0$. This implies, in turn, that the area of the hub is $O(N^0)$. If the volume energy density inside the junction is N^2 (i.e. the same as inside the walls), one immediately gets Eq. (3.1.50).

3.1.5 Webs of walls

Domain walls can form a network when many junctions are connected together – webs or honeycombs, see Fig. 3.5 borrowed from Ref. [86]. 1/4 BPS solutions of such type were found (in the strong gauge coupling limit) in [86, 87] in four-dimensional $\mathcal{N} = 2$ supersymmetric Yang–Mills theory with the gauge

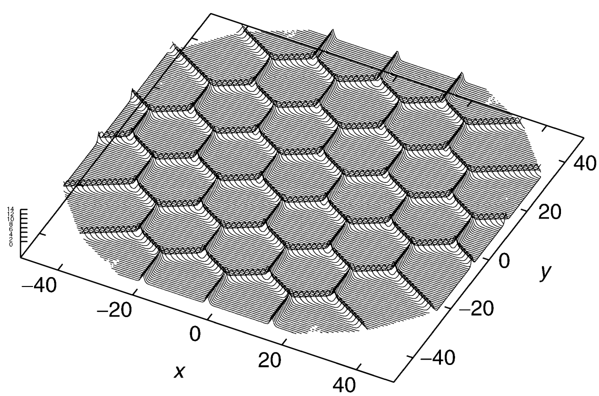

Figure 3.5. Honeycomb web of domain walls. This web in this figure divides 37 vacua and has 18 external legs and 19 internal faces. The moduli space corresponds to CP(36) whose dimension is 72.

[2] There is a subtle point here which must be noted. For the wall type of the hub-and-spokes type the overall tension is the sum of two tensions: the tension of the walls and the tension of the hub. The first is determined by the (1,0) central charge, the second by (1/2,1/2). Each separately is somewhat ambiguous in the case at hand. The ambiguity cancels in the sum [27].

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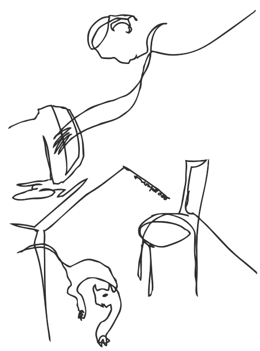
group $U(N_c)$ and N_f flavor hypermultiplets in the fundamental representation ($N_f > N_c$). This model is described in detail in Sections 4.1 and 4.7. The solution saturates two central charges, (1,0) + (0,1) and (1/2,1/2). The moduli space of this particular web of walls is the complex Grassmann manifold $G_{N_f,N_c} = SU(N_f)/[SU(N_f - N_c) \times SU(N_c) \times U(1)]$.

The web of walls can contain several external legs and loops whose maximal numbers are determined by N_f and N_c. If the gauge group is U(1) rather than $U(N_c)$ (with $N_c \geq 2$) the moduli space of the web of walls simplifies and becomes $CP(N_f - 1)$.

Further studies of dynamics of the domain wall loops, as in Fig. 3.5, were carried out in [89]. The authors used the moduli approximation and found that a phase rotation induces a repulsive force which can be interpreted as a Noether charge of Q solitons.



3.2 Vortices in $D = 3$ and flux tubes in $D = 4$

Vortices were among the first examples of topological defects treated in the Bogomol'nyi limit [5, 4, 1] (see also [90]). Explicit embedding of the bosonic sector in supersymmetric models dates back to the 1980s. In [91] a three-dimensional Abelian Higgs model was considered. That model had $\mathcal{N} = 1$ supersymmetry (two supercharges) and thus, according to Section 2.2.2, contained no central charge that could be saturated by vortices. Hence, the vortices discussed in [91] were noncritical. BPS saturated vortices can and do occur in $\mathcal{N} = 2$ three-dimensional models (four supercharges) with a non-vanishing Fayet–Iliopoulos term [92, 93]. Such models can be obtained by dimensional reduction from four-dimensional $\mathcal{N} = 1$ models. We will start from a brief excursion in SQED.

3.2.1 SQED in 3D

The starting point is SQED with the Fayet–Iliopoulos term ξ in four dimensions. The SQED Lagrangian is

$$\mathcal{L} = \left\{ \frac{1}{4\,e^2} \int \mathrm{d}^2\theta\, W^2 + \text{H.c.} \right\} + \int \mathrm{d}^4\theta\, \bar{Q}\, e^{n_e V}\, Q$$
$$+ \int \mathrm{d}^4\theta\, \bar{\tilde{Q}}\, e^{-n_e V}\, \tilde{Q} - n_e\, \xi \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, V(x, \theta, \bar{\theta}), \qquad (3.2.1)$$

where e is the electric coupling constant, Q and $\tilde{Q}$ are chiral matter superfields (with charges n_e and $-n_e$, respectively), and W_α is the supergeneralization of the photon field strength tensor,

$$W_\alpha = \frac{1}{8}\, \bar{D}^2\, D_\alpha V = i\left(\lambda_\alpha + i\theta_\alpha D - \theta^\beta\, F_{\alpha\beta} - i\theta^2 \partial_{\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} \right). \qquad (3.2.2)$$

In four dimensions the absence of the chiral anomaly in SQED requires the matter superfields enter in pairs of the opposite charges, e.g.

$$i\mathcal{D}_\mu \psi = \left(i\,\partial_\mu + n_e A_\mu \right) \psi, \quad i\mathcal{D}_\mu \tilde{\psi} = \left(i\,\partial_\mu - n_e A_\mu \right) \tilde{\psi}. \qquad (3.2.3)$$

Otherwise the theory is anomalous, the chiral anomaly renders it non-invariant under gauge transformations. Thus, the minimal matter sector includes two chiral superfields Q and $\tilde{Q}$, with charges n_e and $-n_e$, respectively. (In the literature a popular choice is $n_e = 1$. In Part II we will use a different normalization, $n_e = 1/2$, which is more convenient in some problems that we address in Part II.)

In three dimensions there is no chirality. Therefore, one can consider 3D SQED with a single matter superfield Q, with charge n_e. Classically it is perfectly fine just to discard the superfield $\tilde{Q}$ from the Lagrangian (3.2.1). However, such "crudely truncated" theory may be inconsistent at the quantum level [94, 95, 96]. Gauge invariance in loops requires, as we will see shortly, simultaneous introduction of a Chern–Simons term in the one matter superfield model [94, 95, 96]. The Chern–Simons term breaks parity. That's the reason why this phenomenon is sometimes referred to as parity anomaly.

A perfectly safe way to get rid of $\tilde{Q}$ is as follows. Let us start from the two-superfield model (3.2.1), which is certainly self-consistent both at the classical and quantum levels. The one-superfield model can be obtained from that with two superfields by making $\tilde{Q}$ heavy and integrating it out. If one manages to introduce a mass $\tilde{m}$ for $\tilde{Q}$ without breaking $\mathcal{N} = 2$ supersymmetry, the large $\tilde{m}$ limit can be viewed as an excellent regularization procedure.

Such mass terms are well known, for a review see [97, 98, 96]. They go under the name of "real masses," are specific to theories with U(1) symmetries dimensionally

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reduced from $D = 4$ to $D = 3$, and present a direct generalization of twisted masses in two dimensions [32]. To introduce a "real mass" one couples matter fields to a background vector field with a non-vanishing component along the reduced direction. For instance, in the case at hand we introduce a background field $V_{\rm b}$ as

$$\Delta\mathcal{L}_m = \int d^4\theta\, \bar{\tilde{Q}}\, e^{V_{\rm b}}\, \tilde{Q}\,, \quad V_{\rm b} = \tilde{m}\,(2\,i)\left(\theta^1\,\bar{\theta}^{\dot{2}} - \theta^2\,\bar{\theta}^{\dot{1}}\right). \tag{3.2.4}$$

The reduced spatial direction is that along the y axis. We couple $V_{\rm b}$ to the U(1) current of $\tilde{Q}$ ascribing to $\tilde{Q}$ charge one with respect to the background field. At the same time Q is assumed to have $V_{\rm b}$ charge zero and, thus, has no coupling to $V_{\rm b}$. Then, the background field generates a mass term only for $\tilde{Q}$, without breaking $\mathcal{N} = 2$. Needless to say, there is no kinetic term for $V_{\rm b}$. Equation (3.2.4) implies that $\tilde{m} = (A_{\rm b})_2$.

After reduction to three dimensions and passing to components (in the Wess–Zumino gauge) we arrive at the action in the following form (in the three-dimensional notation):

$$\begin{aligned} S = \int d^3x\, \Big\{ &-\frac{1}{4e^2}\,F_{\mu\nu}\,F^{\mu\nu} + \frac{1}{2e^2}\,\left(\partial_\mu\, a\right)^2 + \frac{1}{e^2}\,\bar{\lambda}\, i\,\not{\partial}\,\lambda \\ &+ \frac{1}{2e^2}\,D^2 - n_e\,\xi\, D + n_e\, D\left(\bar{q}\, q - \bar{\tilde{q}}\,\tilde{q}\right) \\ &+ \left[\mathcal{D}^\mu \bar{q}\,\mathcal{D}_\mu q + \bar{\psi}\, i \not{\mathcal{D}}\,\psi\right] + \left[\mathcal{D}^\mu \bar{\tilde{q}}\,\mathcal{D}_\mu \tilde{q} + \bar{\tilde{\psi}}\, i\not{\mathcal{D}}\,\tilde{\psi}\right] \\ &- a^2 \bar{q}\, q - (\tilde{m} + a)^2\, \bar{\tilde{q}}\,\tilde{q} + a\,\bar{\psi}\,\psi - (\tilde{m} + a)\,\bar{\tilde{\psi}}\,\tilde{\psi} \\ &+ n_e\left[\sqrt{2}\,(\bar{\lambda}\,\psi)\,\bar{q} + \text{H.c.}\right] - n_e\left[\sqrt{2}\left(\bar{\lambda}\,\tilde{\psi}\right)\bar{\tilde{q}} + \text{H.c.}\right]\Big\}. \end{aligned} \tag{3.2.5}$$

Here a is a real scalar field,

$$a = -n_e\, A_2,$$

λ is the photino field, and q, $\tilde{q}$ and ψ, $\tilde{\psi}$ are matter fields belonging to Q and $\tilde{Q}$, respectively. The covariant derivatives are defined in Eq. (3.2.3). Finally, D is an auxiliary field, the last component of the superfield V. Eliminating D via the equation of motion we get the scalar potential

$$V = \frac{e^2}{2}\, n_e^2\left[\xi - \left(\bar{q}\, q - \bar{\tilde{q}}\,\tilde{q}\right)\right]^2 + a^2 \bar{q}\, q + (\tilde{m} + a)^2\,\bar{\tilde{q}}\,\tilde{q}, \tag{3.2.6}$$

which implies a potentially rather rich vacuum structure. For our purposes – the BPS-saturated vortices – only the Higgs phase is of importance. We will assume that

$$\xi > 0, \quad \tilde{m} \geq 0. \tag{3.2.7}$$

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If $\tilde{\psi}$ and $\tilde{q}$ are viewed as regulators (i.e. $\tilde{m} \to \infty$), they can be integrated out leaving us with the one matter superfield model. It is obvious that integrating them out we get a Chern–Simons term at one loop,[3] with a well-defined coefficient that does not vanish in the limit $\tilde{m} = \infty$. We prefer to keep $\tilde{m}$ as a free parameter, assuming that $\tilde{m} \neq 0$.

From the standpoint of vortex studies, the model (3.2.1) *per se* is not quite satisfactory due to the existence of the flat direction (correspondingly, there is a gapless mode which renders the theory ill-defined in the infrared, see Section 5.1). The flat direction is eliminated at $\tilde{m} \neq 0$. Thus, there are three relevant parameters of dimension of mass,

$$e^2, \ \xi, \ \text{ and } \ \tilde{m}.$$

The weak coupling regime implies that $e^2/\xi \ll 1$.

If $\tilde{m} \neq 0$ the vacuum field configuration is as follows:

$$\tilde{q} = 0, \quad a = 0, \quad \bar{q}q = \xi. \tag{3.2.8}$$

The vanishing of the D term in the vacuum requires $\bar{q}\, q_{\rm vac} = \xi$. Then the term $a^2\bar{q}q$ in (3.2.6) implies that $a = 0$ in the vacuum. Up to gauge transformations the vacuum is unique. The Higgs phase is enforced by our choice $\tilde{m} \neq 0$ and $\xi \neq 0$. The fields $\tilde{q}$, $\tilde{\psi}$ play a role only at the level of quantum corrections, providing a well-defined regularization in loops.

Central charge

The general form of the centrally extended $\mathcal{N} = 2$ superalgebra in $D = 3$ was discussed in Section 2.3.2. The central charge relevant in the problem at hand – vortices – is presented by the last term in Eq. (2.3.6). It can be conveniently derived using the complex representation for supercharges and reducing from $D = 4$ to $D = 3$. In four dimensions [27]

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2P_{\alpha\dot{\alpha}} + 2Z_{\alpha\dot{\alpha}} \equiv 2\left(P_\mu + Z_\mu\right)\left(\sigma^\mu\right)_{\alpha\dot{\alpha}}, \tag{3.2.9}$$

where P_μ is the momentum operator, and

$$Z_\mu = n_e\, \xi \int d^3x\, \epsilon_{0\mu\nu\rho}\left(\partial^\nu A^\rho\right) + \cdots \tag{3.2.10}$$

Here ellipses denote full spatial derivatives of currents[4] that fall off exponentially fast at infinity. Such terms are clearly inessential.

[3] In passing from two matter superfields to one, in order to justify integrating out $\tilde{Q}$, one must consider $\tilde{m} \gg e\sqrt{\xi}$. Given that $e^2/\xi \ll 1$, the condition $\tilde{m} \gg e\sqrt{\xi}$ does not necessarily imply that $\tilde{m} \gg \xi$.

[4] Moreover, these currents are not unambiguously defined, see [27].

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In three dimensions the central charge of interest reduces to $P_2 + Z_2$. Thus, in terms of complex supercharges the appropriate centrally extended algebra takes the form[5]

$$\left\{Q,\left(Q^\dagger\right)\gamma^0\right\} = 2\left(P_0\,\gamma^0 + P_1\,\gamma^x + P_3\,\gamma^z\right) + 2\left\{\frac{1}{e^2}\int d^2x\vec{\nabla}\left(\vec{E}\,a\right) + \tilde{m}\,q\, - n_e\,\xi\int d^2x\,B\right\}, \quad (3.2.11)$$

where $\vec{E}$ is the electric field, B is the magnetic field,

$$B = \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\,, \quad (3.2.12)$$

and q is a conserved Noether charge,

$$q = \int d^2x\, j^0, \qquad j^\mu \equiv \bar{\tilde{\psi}}\,\gamma^\mu\,\tilde{\psi} + \bar{\tilde{q}}\,i\,\overset{\leftrightarrow}{\mathcal{D}}_\mu\,\tilde{q}\,. \quad (3.2.13)$$

The second line in Eq. (3.2.11) presents the vortex-related central charge.[6] The term proportional to a gives a vanishing contribution to the central charge. However, the q term (sometimes omitted in the literature) plays an important role. It combines with the ξ term in the expression for the vortex mass converting the bare value of ξ into the renormalized one. In the problem at hand, the vortex mass gets renormalized at one loop, and so does the Fayet–Iliopoulos parameter.

BPS equation for the vortex

At the classical level the fields a and $\tilde{q}$ play no role. They will be set

$$\tilde{q} = 0, \quad a = 0\,. \quad (3.2.14)$$

The first-order equations describing the ANO vortex in the Bogomol'nyi limit [5, 4, 1] take the form

$$\begin{aligned} &B - n_e\,e^2(|q|^2 - \xi) = 0,\\ &(\mathcal{D}_x + i\mathcal{D}_z)\;q = 0, \end{aligned} \quad (3.2.15)$$

with the boundary conditions

$$\begin{aligned} &q \to \sqrt{\xi}\,e^{ik\alpha} \quad \text{at} \quad r \to \infty\,,\\ &q \to 0 \quad \text{at} \quad r \to 0\,, \end{aligned} \quad (3.2.16)$$

[5] In the following expression terms containing equations of motion of the type $a(\vec{\nabla}\vec{E} - J_0)$ are omitted.
[6] The emergence of the U(1) Noether charge $\tilde{m}q$ in the central charge is in one-to-one correspondence with a similar phenomenon taking place in the two-dimensional CP$(N-1)$ models with the twisted mass [34].

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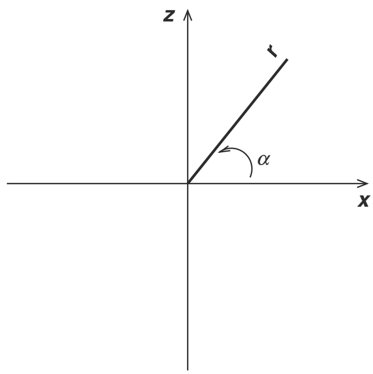

Figure 3.6. Polar coordinates on the x, z plane.

where α is the polar angle on the x, z plane, while r is the distance from the origin in the same plane (Fig. 3.6). Moreover k is an integer, counting the number of windings.

If Eqs. (3.2.15) are satisfied, the flux of the magnetic field is $2\pi k$ (the winding number k determines the quantized magnetic flux), and the vortex mass (string tension) is

$$M = 2\pi\xi\, k, \tag{3.2.17}$$

The linear dependence of the k-vortex mass on k implies the absence of their potential interaction.

For the elementary $k = 1$ vortex it is convenient to introduce two profile functions $\phi(r)$ and $f(r)$ as follows:

$$q(x) = \phi(r)\, e^{i\,\alpha}, \quad A_n(x) = -\frac{1}{n_e}\varepsilon_{nm}\, \frac{x_m}{r^2}\,[1 - f(r)]\,. \tag{3.2.18}$$

The *ansatz* (3.2.18) goes through the set of equations (3.2.15), and we get the following two equations on the profile functions:

$$-\frac{1}{r}\frac{df}{dr} + n_e^2 e^2\left(\phi^2 - \xi\right) = 0, \quad r\,\frac{d\phi}{dr} - f\,\phi = 0. \tag{3.2.19}$$

The boundary conditions for the profile functions are rather obvious from the form of the *ansatz* (3.2.18) and from our previous discussion. At large distances

$$\phi(\infty) = \sqrt{\xi}, \quad f(\infty) = 0\,. \tag{3.2.20}$$

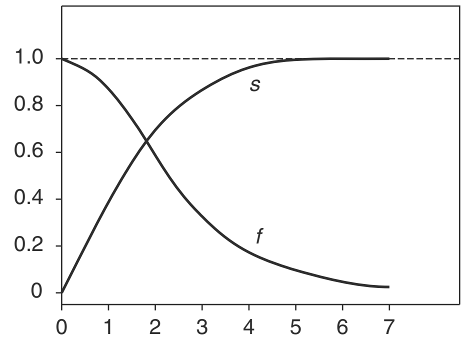

Figure 3.7. Profile functions of the string as functions of the dimensionless variable $m_\gamma r$. The gauge and scalar profile functions are given by f and $s \equiv \phi/\sqrt{\xi}$, respectively.

At the same time, at the origin the smoothness of the field configuration at hand (the absence of singularities) requires

$$\phi(0) = 0, \qquad f(0) = 1\,. \tag{3.2.21}$$

These boundary conditions are such that the scalar field reaches its vacuum value at infinity. Equations (3.2.19) with the above boundary conditions lead to a unique solution for the profile functions, although its analytic form is not known. The vortex size is $\sim e^{-1}\,\xi^{-1/2}$. The solution can be readily obtained numerically. The profile functions ϕ and f which determine the Higgs field and the gauge potential, respectively, are shown in Fig. 3.7.

The fermion zero modes

Quantization of vortices requires the knowledge of the fermion zero modes for the given classical solution. More precisely, since the solution under consideration is static, we are interested in the zero-eigenvalue solutions of the static fermion equations which, thus, effectively become two- rather than three-dimensional,

$$\begin{aligned} &i\left(\gamma^x \mathcal{D}_x + \gamma^z \mathcal{D}_z\right)\psi + n_e\sqrt{2}\,\lambda\, q = 0, \\ &i\left(\gamma^x \partial_x + \gamma^z \partial_z\right)\lambda + e^2 n_e\sqrt{2}\,\psi\,\bar{q} = 0. \end{aligned} \tag{3.2.22}$$

These equations are obtained from (3.2.5) where we dropped the tilded terms (since $\tilde{q} = 0$). The fermion operator is Hermitean implying that every solution for $\{\psi\,,\,\lambda\}$ is accompanied by that for $\{\bar{\psi}\,,\,\bar{\lambda}\}$.

Since the solution to equations (3.2.15) discussed above is 1/2 BPS, two of the four supercharges annihilate it while the other two generate the fermion zero

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modes – superpartners of translational modes. One can show [99] that these are the only normalizable fermion zero modes in the problem at hand.

Short versus long representations

The (1+2)-dimensional model under consideration has four supercharges. The corresponding regular super-representation is four-dimensional (i.e. contains two bosonic and two fermionic states).

The vortex we discuss has two fermion zero modes. Hence, viewed as a particle in 1+2 dimensions it forms a super-doublet (one bosonic state plus one fermionic). Hence, this is a short multiplet. This implies, of course, that the BPS bound must remain saturated when quantum corrections are switched on. Both the central charge and the vortex mass get corrections [100, 99], but they remain equal to each other.

Vortex mass and central charge renormalizations

Assuming that $n_e = 1$ and saturating the central charge in Eq. (3.2.11) by the vortex soliton we get

$$Z_{\text{vortex}} = -\xi \int d^2x\, B + \tilde{m} q = -2\pi\, \xi + \frac{\tilde{m}}{2}. \tag{3.2.23}$$

Here we use the fact that the induced q charge of the vortex is 1/2. This is not difficult to see for any value of $\tilde{m}$ [101]. Proving this assertion becomes especially simple at large $\tilde{m}$ when one can just integrate the tilded fields out in the given vortex field. One then arrives at

$$q = \int d^2x\, \frac{1}{4\pi}\, B = \frac{1}{2}. \tag{3.2.24}$$

Since the renormalized value of the FI parameter ξ is

$$\xi_R = \xi + \frac{m_q - \tilde{m}}{4\pi} \tag{3.2.25}$$

where $m_q = \sqrt{2\xi}\, e$ is the mass of the untilded particles, we can rewrite Eq. (3.2.23) in the form

$$Z_{\text{vortex}} = -2\pi \left(\xi_R - \frac{m_q}{4\pi} \right). \tag{3.2.26}$$

In the very same "physical" regularization scheme outlined above the vortex mass shifts by the same amount [100, 101], and

$$M_{\text{vortex}} = 2\pi \left(\xi_R - \frac{m_q}{4\pi} \right) = |Z_{\text{vortex}}|\, . \tag{3.2.27}$$

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3.2.2 Four-dimensional SQED and the ANO string

In this section we will discuss $\mathcal{N} = 1$ SQED. SQED with extended supersymmetry (i.e. $\mathcal{N} = 2$) is also very interesting. This latter model is presented in Appendix C.

The Lagrangian is the same as in Eq. (3.2.1). We will consider the simplest case: one chiral superfield Q with charge $n_e = 1/2$, and one chiral superfield $\tilde{Q}$ with charge $n_e = -1/2$. The electric charge of matter is chosen to be half-integer to make contact with what follows. This normalization is convenient in the case of non-Abelian models, see Part II. The Lagrangian in components can be obtained from Eq. (3.2.5) by setting $a = \tilde{m} = 0$. The scalar potential obviously takes the form

$$V = \frac{e^2}{2} n_e^2 \left[\xi - \left(\bar{q}\, q - \bar{\tilde{q}}\, \tilde{q} \right) \right]^2 . \tag{3.2.28}$$

The vacuum manifold is a "hyperboloid"

$$\bar{q}\, q - \bar{\tilde{q}}\, \tilde{q} = \xi . \tag{3.2.29}$$

Thus, we deal with the Higgs branch of real dimension two. In fact, the vacuum manifold can be parametrized by a complex modulus $\tilde{q}q$. On this Higgs branch the photon field and superpartners form a massive supermultiplet, while $\tilde{q}q$ and superpartners form a massless one.

As was shown in [102], no finite-thickness vortices exist at a generic point on the vacuum manifold, due to the absence of the mass gap (presence of the massless Higgs excitations). The moduli fields get involved in the solution at the classical level generating a logarithmically divergent tail. An infrared regularization can remove this logarithmic divergence, and vortices become well-defined, see [103] and Chapter 7. One of the possible infrared regularizations is considering a finite-length string instead of an infinite string. Then all infrared divergences are cut off at distances of the order of the string length. The thickness of the string is of the order of logarithm of this length. This is discussed in detail in Chapter 7. Needless to say, such string is not BPS-saturated.

At the base of the Higgs branch, at $\tilde{q} = 0$, the classical solutions of the BPS equations for q and A_μ are well-defined. The form of the solution coincides with that given in Section 3.2.1.

The fact that there is a flat direction and, hence, massless particles in the bulk theory does not disappear, of course. Even though at $\tilde{q} = 0$ the classical string solution is well-defined, infrared problems arise at the loop level. One can avoid massless particles in the spectrum if one embeds the theory (3.2.5) in SQED with eight supercharges, see Section 5.1 and Appendix C. Then the Higgs branch is

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eliminated, and one is left with isolated vacua. After the embedding is done, one can break $\mathcal{N} = 2$ down to $\mathcal{N} = 1$, if one so desires.

A simpler framework is provided by the so-called M model. Its non-Abelian version is considered in Section 5.2. Here we will outline the construction of this model in the context of $\mathcal{N} = 1$ SQED.

We introduce an extra *neutral* chiral superfield M, which interacts with Q and $\tilde{Q}$ through the super-Yukawa coupling,

$$\mathcal{L}_M = \int d^2\theta \, d^2\bar{\theta} \; \frac{1}{h} \, \bar{M} \, M + \left\{ \int d^2\theta \; Q M \tilde{Q} + (\text{H.c}) \right\}. \qquad (3.2.30)$$

Here h is a coupling constant. As we will see momentarily the Higgs branch is lifted. An obvious advantage of this model is that it makes no reference to $\mathcal{N} = 2$. This is probably the simplest $\mathcal{N} = 1$ model which supports BPS-saturated ANO strings without infrared problems.

The scalar potential (3.2.28) is now replaced by

$$V_M = \frac{e^2}{2} \, n_e^2 \left[\xi - \left(\bar{q} \, q - \bar{\tilde{q}} \, \tilde{q} \right) \right]^2 + h \, |q \, \tilde{q}|^2 + |q \, M|^2 + |M \, \tilde{q}|^2 . \qquad (3.2.31)$$

The vacuum is unique modulo gauge transformations,

$$q = \bar{q} = \sqrt{\xi} \, , \quad \tilde{q} = 0 \, , \quad M = 0. \qquad (3.2.32)$$

The classical ANO flux tube solution considered above remains valid as long as we set, additionally, $\tilde{q} = M = 0$. The string tension is the same, $T_{\text{string}} = 2\pi\xi$. (Note that in Eq. (3.2.31) the parameter ξ is defined with n_e^2 factored out. See also Eq. (C.11) and its derivation.) The quantization procedure is straightforward, since one encounters no infrared problems whatsoever – all particles in the bulk are massive. In particular, there are four normalizable fermion zero modes (cf. Ref. [35]).

For further thorough discussions we refer the reader to Section 7.2.

3.2.3 Flux tube junctions

In theories with Z_N symmetry the ANO flux tubes can form junctions of the type depicted in Fig. 3.8. As an example, let us consider a $\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$ gauge theory with three "photons" and three (scalar) matter fields, ϕ, χ, and η,

$$\mathcal{L} = -\frac{1}{4e^2} \sum_{i=1}^{3} (F_i)_{\mu\nu} (F_i)^{\mu\nu} + (D_\mu \bar{\phi})(D^\mu \phi) + (D_\mu \bar{\chi})(D^\mu \chi) + (D_\mu \bar{\eta})(D^\mu \eta) + V(\phi, \chi, \eta), \qquad (3.2.33)$$

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Table 3.1. *Couplings of ϕ, χ, and η with respect to three photons A_1, A_2, and A_3 of the U(1)^3 theory (3.2.33).*

	ϕ	χ	η
A_1	2/3	2/3	$-1/3$
A_2	2/3	$-1/3$	2/3
A_3	$-1/3$	2/3	2/3

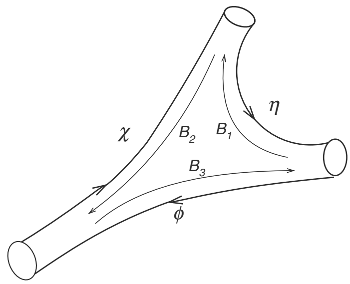
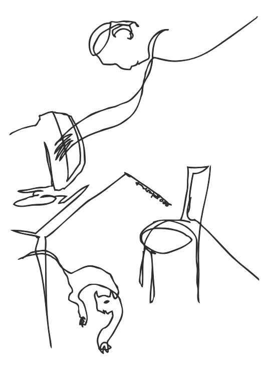


Figure 3.8. A junction of three flux tubes ("Mercedes logo") in the Z_3 invariant theory (3.2.33). The letters ϕ, χ, and η show which fields have windings in three sectors.

whose electric charges with respect to three photons are presented in Table 3.1. The potential $V(\phi, \chi, \eta)$ is assumed to be symmetric under the interchange of ϕ, χ, and η. Another requirement to $V(\phi, \chi, \eta)$ is spontaneous breaking of all three U(1) gauge groups through nonvanishing expectation values $\langle\phi\rangle = \langle\chi\rangle = \langle\eta\rangle \neq 0$.

The three flux tubes form a planar structure of the "Mercedes logo" type, with $2\pi/3$ angles between them. The flux tube in the left-hand side of Fig. 3.8 carries the magnetic fluxes of the third and second photons, the next (clockwise) flux tube the magnetic fluxes of the first and second photons, and the last flux tube of the first and third.



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3.3 Monopoles

In this section we will discuss magnetic monopoles – very interesting objects which carry magnetic charges. They emerge as free magnetically charged particles in non-Abelian gauge theories in which the gauge symmetry is spontaneously broken down to an Abelian subgroup.[7] The simplest example was found by 't Hooft [105] and Polyakov [106]. The model they considered had been invented by Georgi and Glashow [107] for different purposes. As it often happens, the Georgi–Glashow model turned out to be more valuable than the original purpose, which is long forgotten, while the model itself is alive and well and is being constantly used by theorists.

3.3.1 The Georgi–Glashow model: vacuum and elementary excitations

Let us begin with a brief description of the Georgi–Glashow model. The gauge group is SU(2) and the matter sector consists of one real scalar field ϕ^a in the adjoint representation (i.e. SU(2) triplet). The Lagrangian of the model is

$$L = -\frac{1}{4g^2} F^a_{\mu\nu} F^{\mu\nu,a} + \frac{1}{2}(D_\mu\phi^a)(D^\mu\phi^a) - \frac{1}{8}\lambda(\phi^a\phi^a - v^2)^2, \quad (3.3.1)$$

where the covariant derivative in the adjoint acts as

$$D_\mu\phi^a = \partial_\mu\phi^a + \varepsilon^{abc} A^b_\mu \phi^c. \quad (3.3.2)$$

Below we will focus on the limit of BPS monopoles. This limit corresponds to a vanishing scalar coupling, $\lambda \to 0$. The only role of the last term in Eq. (3.3.1) is to provide a boundary condition for the scalar field. As is clear from Chapter 2 the monopole central charge exists only in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ superalgebras. Therefore, one should understand the theory (3.3.1) (at $\lambda = 0$) as embedded in super-Yang–Mills theories with extended superalgebra. In Part II we will extensively discuss such embeddings in the context of $\mathcal{N} = 2$.

The classical definition of magnetic charges refers to theories that support a long-range (Coulomb) magnetic field. Therefore, in consideration of the isolated monopole the pattern of the symmetry breaking should be such that some of the gauge bosons remain massless. In the Georgi–Glashow model (3.3.1) the pattern is as follows:

$$\mathrm{SU}(2) \to \mathrm{U}(1). \quad (3.3.3)$$

[7] In the confining regime monopoles can be obtained in some theories with no adjoint fields, in which the gauge symmetry is broken completely [104]. This is a recent development.

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To see that this is indeed the case let us note the ϕ^a self-interaction term (the last term in Eq. (3.3.1)) forces ϕ^a to develop a vacuum expectation value,

$$\langle \phi^a \rangle = v\delta^{3a}. \tag{3.3.4}$$

The direction of the vector ϕ^a in the SU(2) space (to be referred to as "color space" or "isospace") can be chosen arbitrarily. One can always reduce it to the form (3.3.4) by a global color rotation. Thus, Eq. (3.3.4) can be viewed as a (unitary) gauge condition on the field ϕ.

This gauge is very convenient for discussing the particle content of the theory, elementary excitations. Since the color rotation around the third axis does not change the vacuum expectation value of ϕ^a,

$$\exp\left\{i\alpha\frac{\tau_3}{2}\right\}\phi_{\rm vac}\exp\left\{-i\alpha\frac{\tau_3}{2}\right\} = \phi_{\rm vac}, \quad \phi_{\rm vac} = v\frac{\tau_3}{2}, \tag{3.3.5}$$

the third component of the gauge field remains massless – we will call it a "photon,"

$$A^3_\mu \equiv A_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.3.6}$$

The first and the second components form massive vector bosons,

$$W^\pm_\mu = \frac{1}{\sqrt{2}\,g}\left(A^1_\mu \pm A^2_\mu\right). \tag{3.3.7}$$

As usual in the Higgs mechanism, the massive vector bosons eat up the first and the second components of the scalar field ϕ^a. The third component, the physical Higgs field, can be parametrized as

$$\phi^3 = v + \varphi, \tag{3.3.8}$$

where φ is the physical Higgs field. In terms of these fields the Lagrangian (3.3.1) can be readily rewritten as

$$\begin{aligned} L = &-\frac{1}{4g^2}F_{\mu\nu}\,F_{\mu\nu} + \frac{1}{2}(\partial_\mu\varphi)^2 \\ &-\left(D_\alpha W^+_\mu\right)\left(D_\alpha W^-_\mu\right) + \left(D_\mu W^+_\mu\right)\left(D_\nu W^-_\nu\right) + g^2(v+\phi)^2\,W^+_\mu W^-_\mu \\ &-2\,W^+_\mu\,F_{\mu\nu}\,W^-_\nu + \frac{g^2}{4}\left(W^+_\mu\,W^-_\nu - W^+_\nu\,W^-_\mu\right)^2, \end{aligned} \tag{3.3.9}$$

where the covariant derivative now includes only the photon field,

$$D_\alpha\,W^\pm = (\partial_\alpha \pm i\,A_\alpha)\,W^\pm. \tag{3.3.10}$$

The last line presents the magnetic moment of the charged (massive) vector bosons and their self-interaction. In the limit $\lambda \to 0$ the physical Higgs field is massless. The mass of the $W^{\pm}$ bosons is

$$M_W = g\,v. \tag{3.3.11}$$

3.3.2 Monopoles – topological argument

Let us explain why this model has a topologically stable soliton.

Assume that the monopole's center is at the origin and consider a large sphere $\mathcal{S}_R$ of radius R with the center at the origin. Since the mass of the monopole is finite, by definition, $\phi^a\phi^a = v^2$ on this sphere. ϕ^a is a three-component vector in the isospace subject to the constraint $\phi^a\phi^a = v^2$ which gives us a two-dimensional sphere $\mathcal{S}_G$. Thus, we deal here with mappings of $\mathcal{S}_R$ into $\mathcal{S}_G$. Such mappings split in distinct classes labeled by an integer n, counting how many times the sphere $\mathcal{S}_G$ is swept when we sweep once the sphere $\mathcal{S}_R$, since

$$\pi_2(\mathrm{SU}(2)/\mathrm{U}(1)) = Z. \tag{3.3.12}$$

$\mathcal{S}_G$ = SU(2)/U(1) because for each given vector ϕ^a there is a U(1) subgroup which does *not* rotate it. The SU(2) group space is a three-dimensional sphere while that of SU(2)/U(1) is a two-dimensional sphere.

An isolated monopole field configuration (the 't Hooft–Polyakov monopole) corresponds to a mapping with $n = 1$. Since it is impossible to continuously deform it to the topologically trivial mapping, the monopoles are topologically stable.

3.3.3 Mass and magnetic charge

Classically the monopole mass is given by the energy functional

$$E = \int d^3x \left\{ \frac{1}{2\,g^2}\, B_i^a B_i^a + \frac{1}{2}\, \left(D_i\phi^a\right)\left(D_i\phi^a\right) \right\}, \tag{3.3.13}$$

where

$$B_i^a = -\frac{1}{2}\,\varepsilon_{ijk} F_{jk}^a. \tag{3.3.14}$$

The fields are assumed to be time-independent, $B_i^a = B_i^a(\vec{x})$, $\phi^a = \phi^a(\vec{x})$. For static fields it is natural to assume that $A_0^a = 0$. This assumption will be verified a posteriori, after we find the field configuration minimizing the functional (3.3.13). Equation (3.3.13) assumes the limit $\lambda \to 0$. However, in performing minimization we should keep in mind the boundary condition $\phi^a(\vec{x})\phi^a(\vec{x}) \to v^2$ at $|\vec{x}| \to \infty$.

Equation (3.3.13) can be identically rewritten as follows:

$$E=\int d^3x\left\{\frac{1}{2}\left(\frac{1}{g}\,B_i^a-D_i\phi^a\right)\left(\frac{1}{g}\,B_i^a-D_i\phi^a\right)+\frac{1}{g}\,B_i^a D_i\phi^a\right\}. \tag{3.3.15}$$

The last term on the right-hand side is a full derivative. Indeed, after integrating by parts and using the equation of motion $D_i B_i^a=0$ we get

$$\begin{aligned}\int d^3x\left\{\frac{1}{g}\,B_i^a D_i\phi^a\right\}&=\frac{1}{g}\int d^3x\,\partial_i\left(B_i^a\phi^a\right)\\ &=\frac{1}{g}\int_{\mathcal{S}_R} d^2S_i\,\left(B_i^a\phi^a\right).\end{aligned} \tag{3.3.16}$$

In the last line we made use of Gauss' theorem and passed from the volume integration to that over the surface of the large sphere. Thus, the last term in Eq. (3.3.15) is topological.

The combination $B_i^a\phi^a$ can be viewed as a gauge invariant definition of the magnetic field $\vec{\mathcal{B}}$. More exactly,

$$\mathcal{B}_i=\frac{1}{v}\,B_i^a\phi^a. \tag{3.3.17}$$

Indeed, far away from the monopole core one can always assume ϕ^a to be aligned in the same way as in the vacuum (in an appropriate gauge), $\phi^a = v\delta^{3a}$. Then $\mathcal{B}_i=B_i^3$. The advantage of the definition (3.3.17) is that it is gauge independent.

Furthermore, the magnetic charge Q_M inside a sphere $\mathcal{S}_R$ can be defined through the flux of the magnetic field through the surface of the sphere,[8]

$$Q_M=\int_{\mathcal{S}_R} d^2S_i\,\frac{1}{g}\mathcal{B}_i. \tag{3.3.18}$$

From Eq. (3.3.30) (see below) we will see that

$$\mathcal{B}_i\equiv\frac{1}{v}\,B_i^a\phi^a\longrightarrow n^i\,\frac{1}{r^2}\quad\text{at } r\to\infty, \tag{3.3.19}$$

and, hence,

$$Q_M=\frac{4\pi}{g}. \tag{3.3.20}$$

[8] A remark: Conventions for the charge normalization used in different books and papers may vary. In his original paper on the magnetic monopole [108], Dirac uses the convention $e^2=\alpha$ and the electromagnetic Hamiltonian $\mathcal{H}=(8\pi)^{-1}(\vec{E}^2+\vec{B}^2)$. Then, the electric charge is defined through the flux of the electric field as $e=(4\pi)^{-1}\int_{\mathcal{S}_R} d^2S_i\,E_i$, and analogously for the magnetic charge. We use the convention according to which $e^2=4\pi\alpha$, and the electromagnetic Hamiltonian $\mathcal{H}=(2g^2)^{-1}(\vec{E}^2+\vec{B}^2)$. Then $e=g^{-1}\int_{\mathcal{S}_R} d^2S_i\,E_i$ while $Q_M=g^{-1}\int_{\mathcal{S}_R} d^2S_i\,B_i$.

Combining Eqs. (3.3.18), (3.3.17) and (3.3.16) we conclude that

$$E = v\, Q_M + \int d^3x \left\{ \frac{1}{2} \left(\frac{1}{g}\, B_i^a - D_i \phi^a \right) \left(\frac{1}{g}\, B_i^a - D_i \phi^a \right) \right\}. \quad (3.3.21)$$

The minimum of the energy functional is attained at

$$\frac{1}{g}\, B_i^a - D_i \phi^a = 0. \quad (3.3.22)$$

The mass of the field configuration realizing this minimum – the monopole mass — is obviously equal

$$M_M = \frac{4\pi\, v}{g}\,. \quad (3.3.23)$$

Thus, the mass of the critical monopole is in one-to-one relation with its magnetic charge. Equation (3.3.22) is nothing but the Bogomol'nyi equation in the monopole problem. If it is satisfied, the second-order differential equations of motion are satisfied too.

3.3.4 Solution of the Bogomol'nyi equation for monopoles

To solve the Bogomol'nyi equations we need to find an appropriate *ansatz* for ϕ^a. As one sweeps S_R the vector ϕ^a must sweep the group space sphere. The simplest choice is to identify these two spheres point-by-point,

$$\phi^a = v\, \frac{x^a}{r} = v n^a, \quad r \to \infty. \quad (3.3.24)$$

where $n^i \equiv x^i/r$. This field configuration obviously belongs to the class with $n = 1$. The SU(2) group index a got entangled with the coordinate $\vec{x}$. Polyakov proposed to refer to such fields as "hedgehogs."

Next, observe that finiteness of the monopole energy requires the covariant derivative $D_i\phi^a$ to fall off faster than $r^{-3/2}$ at large r, cf. Eq. (3.3.13). Since

$$\partial_i \phi^a = v\, \frac{1}{r} \left\{ \delta^{ai} - n^a n^i \right\} \sim \frac{1}{r} \quad (3.3.25)$$

one must choose A_i^b in such a way as to cancel (3.3.25). It is not difficult to see that

$$A_i^a = \varepsilon^{aij}\, \frac{1}{r}\, n^j, \quad r \to \infty. \quad (3.3.26)$$

Then the term $1/r$ is canceled in $D_i\phi^a$.

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Equations (3.3.24) and (3.3.26) determine the index structure of the field configuration we are going to deal with. The appropriate *ansatz* is perfectly clear now,

$$\phi^a = v\, n^a H(r), \quad A_i^a = \varepsilon^{aij}\, \frac{1}{r}\, n^j\, F(r), \tag{3.3.27}$$

where H and F are functions of r with the boundary conditions

$$H(r) \to 1, \quad F(r) \to 1 \quad \text{at} \;\; r \to \infty, \tag{3.3.28}$$

and

$$H(r) \to 0, \quad F(r) \to 0 \quad \text{at} \;\; r \to 0. \tag{3.3.29}$$

The boundary condition (3.3.28) is equivalent to Eqs. (3.3.24) and (3.3.26), while the boundary condition (3.3.29) guarantees that our solution is nonsingular at $r \to 0$.

After some straightforward algebra we get

$$\begin{aligned} B_i^a &= \left(\delta^{ai} - n^a n^i\right) \frac{1}{r}\, F' + n^a n^i\, \frac{1}{r^2} \left(2F - F^2\right), \\ D_i \phi^a &= v \left\{ \left(\delta^{ai} - n^a n^i\right) \frac{1}{r}\, H(1 - F) + n^a n^i\, H' \right\}, \end{aligned} \tag{3.3.30}$$

where prime denotes differentiation with respect to r.

Let us return now to the Bogomol'nyi equations (3.3.22). This is a set of nine first-order differential equations. Our *ansatz* has only two unknown functions. The fact that the *ansatz* goes through and we get two scalar equations on two unknown functions from the Bogomol'nyi equations is a highly nontrivial check. Comparing Eqs. (3.3.22) and (3.3.30) we get

$$\begin{aligned} \frac{1}{g}\, F' &= v\, H(1 - F), \\ H' &= \frac{1}{g\, v}\, \frac{1}{r^2} \left(2F - F^2\right). \end{aligned} \tag{3.3.31}$$

The functions H and F are dimensionless. It is convenient to make the radius r dimensionless too. A natural unit of length in the problem at hand is $(gv)^{-1}$. From now on we will measure r in these units,

$$\rho = r\,(gv). \tag{3.3.32}$$

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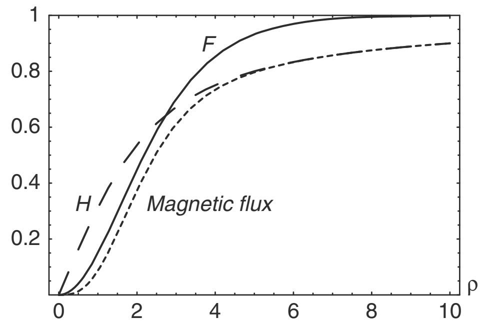

Figure 3.9. The functions F (solid line) and H (long dashes) in the critical monopole solution, vs. ρ. The short-dashed line shows the flux of the magnetic field $\mathcal{B}_i$ (in the units 4π) through the sphere of radius ρ.

The functions H and F are to be considered as functions of ρ, while the prime will denote differentiation over ρ. Then the system (3.3.31) takes the form

$$F' = H(1 - F),$$
$$H' = \frac{1}{\rho^2}\left(2F - F^2\right). \tag{3.3.33}$$

These equations have known analytical solutions,

$$F = 1 - \frac{\rho}{\sinh\rho},$$
$$H = \frac{\cosh\rho}{\sinh\rho} - \frac{1}{\rho}. \tag{3.3.34}$$

At large ρ the functions H and F tend to unity (cf. Eq. (3.3.28)) while at $\rho \to 0$

$$F = \mathrm{O}(\rho^2), \quad H = \mathrm{O}(\rho).$$

They are plotted in Fig. 3.9. Calculating the flux of the magnetic field through the large sphere we verify that for the solution at hand $Q_M = 4\pi/g$.

3.3.5 Collective coordinates (moduli)

The monopole solution presented in the previous section breaks a number of valid symmetries of the theory, for instance, translational invariance. As usual, the symmetries are restored after the introduction of the collective coordinates (moduli), which convert a given solution into a family of solutions.

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Our first task is to count the number of moduli in the monopole problem. A straightforward way to count this number is counting linearly independent zero modes. To this end, one represents the fields A_μ and ϕ as a sum of the monopole background plus small deviations,

$$A^a_\mu = A^{a(0)}_\mu + a^a_\mu, \quad \phi^a = \phi^{a(0)} + (\delta\phi)^a, \tag{3.3.35}$$

where the superscript (0) marks the monopole solution. At this point it is necessary to impose a gauge-fixing condition. A convenient condition is

$$\frac{1}{g} D_i a^a_i - \varepsilon^{abc}\phi^b(\delta\phi)^c = 0, \tag{3.3.36}$$

where the covariant derivative in the first term contains only the background field.

Substituting the decomposition (3.3.35) in the Lagrangian one finds the quadratic form for $\{a,\ (\delta\phi)\}$, and determines the zero modes of this form (subject to the condition (3.3.36)).

We will not trace this procedure in detail, referring the reader to the original literature [109]. Instead, we suggest a simple heuristic consideration.

Let us ask ourselves what are the valid symmetries of the model at hand? They are: (i) three translations; (ii) three spatial rotations; (iii) three rotations in the SU(2) group. Not all these symmetries are independent. It is not difficult to check that the spatial rotations are equivalent to the SU(2) group rotations for the monopole solution. Thus, we should not count them independently. This leaves us with six symmetry transformations.

One should not forget, however, that two of those six act non-trivially in the "trivial vacuum." Indeed, the latter is characterized by the condensate (3.3.4). While rotations around the third axis in the isospace leave the condensate intact (see Eq. (3.3.5)), the rotations around the first and second axes do not. Thus, the number of moduli in the monopole problem is $6 - 2 = 4$. These four collective coordinates have a very transparent physical interpretation. Three of them correspond to translations. They are introduced in the solution through the substitution

$$\vec{x} \to \vec{x} - \vec{x}_0. \tag{3.3.37}$$

The vector $\vec{x}_0$ now plays the role of the monopole center. The unit vector $\vec{n}$ is now defined as $\vec{n} = (\vec{x} - \vec{x}_0)/|\vec{x} - \vec{x}_0|$.

The fourth collective coordinate is related to the unbroken U(1) symmetry of the model. This is the rotation around the direction of alignment of the field ϕ. In the

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"trivial vacuum" ϕ^a is aligned along the third axis. The monopole generalization of Eq. (3.3.5) is

$$\begin{aligned} A^{(0)} &\to U^{-1} A^{(0)} U - i U^{-1} \partial U, \\ \phi^{(0)} &\to U^{-1} \phi^{(0)} U = \phi^{(0)}, \\ U &= \exp\left\{ i\alpha\phi^{(0)}/v \right\}, \end{aligned} \tag{3.3.38}$$

where the fields $A^{(0)}$ and $\phi^{(0)}$ are understood here in the matrix form,

$$A^{(0)} = A^{a(0)}\,(\tau^a/2), \quad \phi^{(0)} = \phi^{a(0)}\,(\tau^a/2)\,.$$

Unlike the vacuum field, which is not changed under (3.3.5), the monopole solution for the vector field changes its form. The change looks as a gauge transformation. Note, however, that the gauge matrix U does not tend to unity at $r \to \infty$. Thus, this transformation is in fact a global U(1) rotation. The physical meaning of the collective coordinate α will become clear shortly. Now let us note that (i) for small α Eq. (3.3.38) reduces to

$$\delta A_i^a = \alpha\,\frac{1}{v}\,(D_i\phi^{(0)})^a, \quad \delta\phi = 0, \tag{3.3.39}$$

and this is compatible with the gauge condition (3.3.36); (ii) the variable α is compact, since the points α and $\alpha + 2\pi$ can be identified (the transformation of $A^{(0)}$ is identically the same for α and $\alpha + 2\pi$). In other words, α is an angle variable.

Having identified all four moduli relevant to the problem we can proceed to the quasiclassical quantization. The task is to obtain quantum mechanics of the moduli. Let us start from the monopole center coordinate $\vec{x}_0$. To this end, as usual, we assume that $\vec{x}_0$ weakly depends on time t, so that the only time dependence of the solution enters through $\vec{x}_0(t)$. The time dependence is important only in time derivatives, so that the quantum-mechanical Lagrangian of the moduli can be obtained from the following expression:

$$\begin{aligned} \mathcal{L}_{\rm QM} = -\,M_M + \frac{1}{2}\,(\dot{x}_0)_k(\dot{x}_0)_j \int d^3x \left\{ \left[\frac{1}{g}\,F_{ik}^{a(0)}\right]\left[\frac{1}{g}\,F_{ij}^{a(0)}\right] \right. \\ \left. + \left[D_k\phi^{a(0)}\right]\left[D_j\phi^{a(0)}\right] \right\}, \end{aligned} \tag{3.3.40}$$

where $\partial_k A$ and $\partial_k \phi$ where supplemented by appropriate gauge transformations to satisfy the gauge condition (3.3.36).

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Averaging over the angular orientations of $\vec{x}$ yields

$$\mathcal{L}_{\rm QM} = -M_M + \frac{1}{2}\,(\dot{\vec{x}}_0)^2 \int d^3x \left\{ \frac{2}{3}\,\frac{1}{g^2}\,B_i^{a(0)} B_i^{a(0)} + \frac{1}{3}\,D_i\phi^{a(0)} D_i\phi^{a(0)} \right\}$$
$$= -M_M + \frac{M_M}{2}\,(\dot{\vec{x}}_0)^2. \tag{3.3.41}$$

This last result readily follows if one combines Eqs. (3.3.13) and (3.3.22). Of course, this final answer could have been guessed from the very beginning since this is nothing but the Lagrangian describing free non-relativistic motion of a particle of mass M_M endowed with the coordinate $\vec{x}_0$.

Now, having tested the method in the case where the answer was obvious, let us apply it to the fourth collective coordinate α. Using Eq. (3.3.39) we get

$$\mathcal{L}_{\alpha{\rm QM}} = \frac{1}{2}\,\frac{M_M}{M_W^2}\,\dot{\alpha}^2, \tag{3.3.42}$$

or, equivalently,

$$\mathcal{H}_\alpha = \frac{1}{2}\,\frac{M_W^2}{M_M}\,p_\alpha^2, \quad p_\alpha \equiv -i\,\frac{d}{d\alpha}, \tag{3.3.43}$$

where $\mathcal{H}_\alpha$ is the part of the Hamiltonian relevant to α. The full quantum-mechanical Hamiltonian describing the moduli dynamics is, thus,

$$\mathcal{H} = M_M + \frac{p^2}{2M_M} + \frac{1}{2}\,\frac{M_W^2}{M_M}\,p_\alpha^2\,, \quad p \equiv -i\,\frac{d}{dx_0}\,. \tag{3.3.44}$$

It describes free motion of a spinless particle endowed with an internal (compact) variable α. While the spatial part of $\mathcal{H}$ does not raise any questions, the α dynamics deserves additional discussion.

The α motion is free, but one should not forget that α is an angle. Because of the 2π periodicity, the corresponding wave functions must have the form

$$\Psi(\alpha) = e^{ik\alpha}, \tag{3.3.45}$$

where k is an integer, $k = 0, \pm 1, \pm 2, \ldots$. Strictly speaking, only the ground state, $k = 0$, describes the monopole – a particle with the magnetic charge $4\pi/g$ and vanishing electric charge. Excitations with $k \neq 0$ correspond to a particle with the magnetic charge $4\pi/g$ *and* the electric charge kg, the dyon.

To see that this is indeed the case, let us note that for $k \neq 0$ the expectation value of p_α is k and, hence, the expectation value of $\dot{\alpha} = (M_W^2/M_M)\,p_\alpha$ is $M_W^2 k/M_M$.

Moreover, let us define a gauge-invariant electric field $\mathcal{E}_i$ (analogous to $\mathcal{B}_i$ of Eq. (3.3.17)) as

$$\mathcal{E}_i \equiv \frac{1}{v}\, E_i^a \phi^a = \frac{1}{v}\, \phi^{a(0)}\, \dot{A}_i^{a(0)} = \frac{1}{v^2}\, \dot{\alpha}\, \phi^{a(0)}\, (D_i \phi^{a(0)}). \tag{3.3.46}$$

Since for the critical monopole $D_i\phi^{a(0)} = (1/g) B_i^{a(0)}$ we see that

$$\mathcal{E}_i = \dot{\alpha}\, \frac{1}{M_W} \mathcal{B}_i, \tag{3.3.47}$$

and the flux of the gauge-invariant electric field over the large sphere is

$$\frac{1}{g} \int_{\mathcal{S}_R} d^2 S_i\, \mathcal{E}_i = \frac{M_W^2 k}{M_M}\, \frac{1}{M_W}\, \frac{1}{g} \int_{\mathcal{S}_R} d^2 S_i \mathcal{B}_i \tag{3.3.48}$$

where we replaced $\dot{\alpha}$ by its expectation value. Thus, the flux of the electric field reduces to

$$\frac{1}{g} \int_{\mathcal{S}_R} d^2 S_i\, \mathcal{E}_i = kg, \tag{3.3.49}$$

which proves the above assertion of the electric charge kg.

It is interesting to note that the mass of the dyon can be written as

$$M_D = M_M + \frac{1}{2}\, \frac{M_W^2}{M_M}\, k^2 \approx \sqrt{M_M^2 + M_W^2\, k^2} = v\sqrt{Q_M^2 + Q_E^2}\,. \tag{3.3.50}$$

Although from our derivation it might seem that the square root formula is approximate, in fact, the prediction for the dyon mass $M_D = v(Q_M^2 + Q_E^2)^{1/2}$ is exact; it follows from the BPS saturation and the central charges in $\mathcal{N} = 2$ model (see Chapter 2).

Magnetic monopoles were introduced in theory by Dirac in 1931 [108]. He considered macroscopic electrodynamics and derived a self-consistency condition for the product of the magnetic charge of the monopole Q_M and the elementary electric charge e,[9]

$$Q_M\, e = 2\pi\,. \tag{3.3.51}$$

This is known as the Dirac quantization condition. For the 't Hooft–Polyakov monopole we have just derived that $Q_M g = 4\pi$, twice larger than in the Dirac

[9] In Dirac's original convention the charge quantization condition is, in fact, $Q_M e = (1/2)$.

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quantization condition. Note, however, that g is the electric charge of the W bosons. It is *not* the minimal possible electric charge that can be present in the theory at hand. If quarks in the fundamental (doublet) representation of SU(2) were introduced in the Georgi–Glashow model, their U(1) charge would be $e = g/2$, and the Dirac quantization condition would be satisfied for such elementary charges.

3.3.6 Singular gauge, or how to comb a hedgehog

The *ansatz* (3.3.27) for the monopole solution we used so far is very convenient for revealing a nontrivial topology lying behind this solution, i.e. the fact that SU(2)/U(1) $\sim S_2$ in the group space is mapped onto the spatial S_2. However, it is often useful to gauge-transform it in such a way that the scalar field becomes oriented along the third axis in the color space, $\phi^a \sim \delta^{3a}$, in all space (i.e. at all x), repeating the pattern of the "plane" vacuum (3.3.4). Polyakov suggested to refer to this gauge transformation as "combing the hedgehog." Comparison of Figs. 3.10a and 3.10b shows that this gauge transformation cannot be nonsingular. Indeed, the matrix which combs the hedgehog,

$$U^\dagger \left(n^a \tau^a\right) U = \tau^3, \tag{3.3.52}$$

has the form

$$U = \frac{1}{\sqrt{2}} \left(\sqrt{1+n^3} + i\, \frac{\nu^a \tau^a}{\sqrt{1+n^3}} \right), \tag{3.3.53}$$

where

$$\nu^a = \varepsilon^{3ab}\, n^b, \quad \nu^a \nu^a = 1 - (n^3)^2, \tag{3.3.54}$$

and $\vec{n}$ is the unit vector in the direction of $\vec{x}$. The matrix U is obviously singular at $n^3 = -1$ (see Fig. 3.10). This is a gauge artifact since all physically measurable quantities are nonsingular and well-defined. In the "old" Dirac description of the monopole [110] the singularity of U at $n^3 = -1$ would correspond to the Dirac string.

In the singular gauge the monopole magnetic field at large $|\vec{x}|$ takes the "color-combed" form

$$B_i \to \frac{\tau^3}{2} \frac{n^i}{r^2} = 4\pi\, \frac{\tau^3}{2}\, \frac{n^i}{4\pi\, r^2}. \tag{3.3.55}$$

The latter equation naturally implies the same magnetic charge $Q_M = 4\pi/g$, as was derived in Section 3.3.2.

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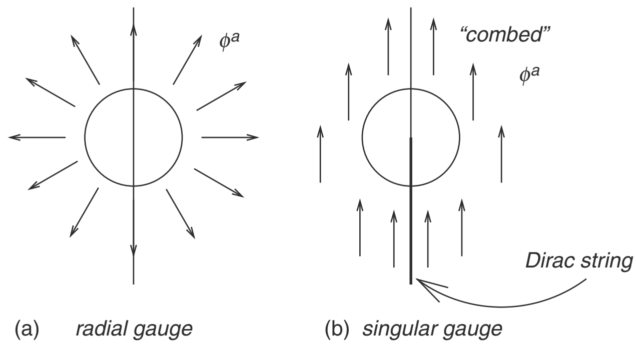

Figure 3.10. Transition from the radial to singular gauge or combing the hedgehog.

3.3.7 Monopoles in SU(N)

Let us now extend the construction presented above from SU(2) to SU(N) [111, 112]. The starting Lagrangian is the same as in Eq. (3.3.1), with the replacement of the structure constants ε^{abc} of SU(2) by the SU(N) structure constants f^{abc}. The potential of the scalar-field self-interaction can be of a more general form than in Eq. (3.3.1). Details of this potential are unimportant for our purposes since in the critical limit the potential tends to zero; its only role is to fix the vacuum value of the field ϕ at infinity.

Recall that all generators of the Lie algebra can be always divided into two groups – the Cartan generators H_i, which all commute with each other, and a set of raising (lowering) operators $E_{\boldsymbol{\alpha}}$,

$$E_{\boldsymbol{\alpha}}^{\dagger} = E_{-\boldsymbol{\alpha}}. \tag{3.3.56}$$

For SU(N) – and we will not discuss other groups – there are $N-1$ Cartan generators which can be chosen as

$$\begin{aligned}
H^1 &= \frac{1}{2}\,\mathrm{diag}\,\{1,-1,0,\ldots,0\}\,,\\
H^2 &= \frac{1}{2\sqrt{3}}\,\mathrm{diag}\,\{1,1,-2,0,\ldots,0\}\,,\\
&\ldots\\
H^m &= \frac{1}{\sqrt{2m(m+1)}}\,\mathrm{diag}\,\{1,1,1,\ldots,-m,\ldots,0\}\,,\\
&\ldots\\
H^{N-1} &= \frac{1}{\sqrt{2N(N-1)}}\,\mathrm{diag}\,\{1,1,1,\ldots,1,-(N-1)\}\,,
\end{aligned} \tag{3.3.57}$$

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$N(N-1)/2$ raising generators $E_{\boldsymbol{\alpha}}$, and $N(N-1)/2$ lowering generators $E_{-\boldsymbol{\alpha}}$. The Cartan generators are analogs of $\tau_3/2$ while $E_{\pm\boldsymbol{\alpha}}$ are analogs of $\tau_\pm/2$. Moreover, $N(N-1)$ vectors $\boldsymbol{\alpha}$, $-\boldsymbol{\alpha}$ are called root vectors. They are $(N-1)$-dimensional.

By making an appropriate choice of basis, any element of SU(N) algebra can be brought to the Cartan subalgebra. Correspondingly, the vacuum value of the (matrix) field $\phi \equiv \phi^a T^a$ can always be chosen to be of the form

$$\phi_{\text{vac}} = \boldsymbol{h}\boldsymbol{H}, \tag{3.3.58}$$

where $\boldsymbol{h}$ is an $(N-1)$-component vector,

$$\boldsymbol{h} = \{h_1, h_2, \ldots, h_{N-1}\}. \tag{3.3.59}$$

For simplicity we will assume that for all simple roots $\boldsymbol{h}\boldsymbol{\gamma} > 0$ (otherwise, we will just change the condition defining positive roots to meet this constraint).

Depending on the form of the self-interaction potential distinct patterns of gauge symmetry breaking can take place. We will discuss here the case when the gauge symmetry is maximally broken,

$$\text{SU}(N) \to \text{U}(1)^{N-1}. \tag{3.3.60}$$

The unbroken subgroup is Abelian. This situation is general. In special cases, when $\boldsymbol{h}$ is orthogonal to $\boldsymbol{\alpha}^m$ for some m (or a set of m's) the unbroken subgroup will contain non-Abelian factors, as will be explained momentarily. These cases will not be considered here.

The topological argument proving the existence of a variety of topologically stable monopoles in the above set-up parallels that of Section 3.3.2, except that Eq. (3.3.12) is replaced by

$$\pi_2\left(\text{SU}(N)/\text{U}(1)^{N-1}\right) = \pi_1\left(\text{U}(1)^{N-1}\right) = Z^{N-1}. \tag{3.3.61}$$

There are $N-1$ independent windings in the SU(N) case.

The gauge field A_μ (in the matrix form, $A_\mu \equiv A_\mu^a T^a$) can be represented as

$$A_\mu^a T^a = \sum_{m=1}^{N-1} A_\mu^m H^m + \sum_{\boldsymbol{\alpha}} A_\mu^{\boldsymbol{\alpha}} E_{\boldsymbol{\alpha}}, \tag{3.3.62}$$

where A_μ^m's ($m = 1, \ldots, N-1$) can be viewed as "photons," while $A_\mu^{\boldsymbol{\alpha}}$'s as "$W$ bosons." The mass terms are obtained from the term

$$\text{Tr}\left(\left[A_\mu, \phi\right]\right)^2$$

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in the Lagrangian. Substituting here Eqs. (3.3.58) and (3.3.62) it is easy to see that the W-boson masses are

$$M_{\boldsymbol{\alpha}} = g\,\boldsymbol{h}\boldsymbol{\alpha}. \tag{3.3.63}$$

$N-1$ massive bosons corresponding to simple roots $\boldsymbol{\gamma}$ play a special role: they can be thought of as fundamental, in the sense that the quantum numbers and masses of all other W bosons can be obtained as linear combinations (with non-negative integer coefficients) of those of the fundamental W bosons. With regards to the masses this is immediately seen from Eq. (3.3.63) in conjunction with

$$\boldsymbol{\alpha} = \sum_{\boldsymbol{\gamma}} k_{\boldsymbol{\gamma}}\,\boldsymbol{\gamma}. \tag{3.3.64}$$

Construction of SU(N) monopoles reduces, in essence, to that of a SU(2) monopole through various embeddings of SU(2) in SU(N). Note that each simple root $\boldsymbol{\gamma}$ defines an SU(2) subgroup[10] of SU(N) with the following three generators:

$$\begin{aligned} t^1 &= \frac{1}{\sqrt{2}}\left(E_{\boldsymbol{\gamma}} + E_{-\boldsymbol{\gamma}}\right), \\ t^2 &= \frac{1}{\sqrt{2}\,i}\left(E_{\boldsymbol{\gamma}} - E_{-\boldsymbol{\gamma}}\right), \\ t^3 &= \boldsymbol{\gamma}\,\boldsymbol{H}, \end{aligned} \tag{3.3.65}$$

with the standard algebra $[t^i, t^j] = i\varepsilon^{ijk}t^k$. If the basic SU(2) monopole solution corresponding to the Higgs vacuum expectation value v is denoted as $\{\phi^a(\boldsymbol{r};\, v),\ A_i^a(\boldsymbol{r};\, v)\}$, see Eq. (3.3.27), the construction of a specific SU(N) monopole proceeds in three steps: (i) choose a simple root $\boldsymbol{\gamma}$; (ii) decompose the vector $\boldsymbol{h}$ in two components, parallel and perpendicular with respect to $\boldsymbol{\gamma}$,

$$\begin{aligned} \boldsymbol{h} &= \boldsymbol{h}_{\parallel} + \boldsymbol{h}_{\perp}\,, \\ \boldsymbol{h}_{\parallel} &= \tilde{v}\boldsymbol{\gamma}\,, \quad \boldsymbol{h}_{\perp}\boldsymbol{\gamma} = 0\,, \\ \tilde{v} &\equiv \boldsymbol{\gamma}\,\boldsymbol{h} > 0\,; \end{aligned} \tag{3.3.66}$$

(iii) replace $A_i^a(\boldsymbol{r};\, v)$ by $A_i^a(\boldsymbol{r};\, \tilde{v})$ and add a covariantly constant term to the field $\phi^a(\boldsymbol{r};\, \tilde{v})$ to ensure that at $r \to \infty$ it has the correct asymptotic behavior, namely, $2\,\mathrm{Tr}\,\phi^2 = \boldsymbol{h}^2$. Algebraically the SU($N$) monopole solution takes the form

$$\phi = \phi^a(\boldsymbol{r};\, \tilde{v})\,t^a + \boldsymbol{h}_{\perp}\boldsymbol{H}, \quad A_i = A_i^a(\boldsymbol{r};\, \tilde{v})\,t^a. \tag{3.3.67}$$

[10] Generally speaking, each root $\boldsymbol{\alpha}$ defines an SU(2) subalgebra according to Eq. (3.3.65), but we will deal only with the simple roots for reasons which will become clear momentarily.

Note that the mass of the corresponding W boson $M_{\boldsymbol{\gamma}} = g\tilde{v}$, in full parallel with the SU(2) monopole.

It is instructive to verify that (3.3.67) satisfies the BPS equation (3.3.22). To this end it is sufficient to note that $[\boldsymbol{h}_{\perp}\boldsymbol{H},\ A_i] = 0$, which in turn implies

$$D_i\ (\boldsymbol{h}_{\perp}\boldsymbol{H}) = 0.$$

What remains to be done? We must analyze the magnetic charges of the SU(N) monopoles and their masses. In the singular gauge (Section 3.3.6) the Higgs field is aligned in the Cartan subalgebra, $\phi \sim \boldsymbol{h}\,\boldsymbol{H}$. The magnetic field at large distances from the monopole core, being commutative with ϕ, also lies in the Cartan subalgebra. In fact, from Eq. (3.3.65) we infer that combing of the SU(N) monopole leads to

$$B_i \to 4\pi\ \boldsymbol{\gamma}\boldsymbol{H}\ \frac{n^i}{4\pi\ r^2}, \tag{3.3.68}$$

which implies, in turn, that the set of $N-1$ magnetic charges of the SU(N) monopole is given by the components of the $(N-1)$-vector

$$\boldsymbol{Q}_M = \frac{4\pi}{g}\,\boldsymbol{\gamma}. \tag{3.3.69}$$

Of course, the very same result is obtained in a gauge invariant manner from a defining formula

$$2\mathrm{Tr}\,(B_i\phi)\ \underset{r\to\infty}{\longrightarrow}\ \left(\boldsymbol{Q}_M\boldsymbol{h}\right)\frac{g}{4\pi}\,\frac{n_i}{r^2}. \tag{3.3.70}$$

Equation (3.3.15) implies that the mass of this monopole is

$$M_{M\boldsymbol{\gamma}} = \boldsymbol{Q}_M\boldsymbol{h} = \frac{4\pi\ \tilde{v}}{g}, \tag{3.3.71}$$

to be compared with the mass of the corresponding W bosons,

$$M_{\boldsymbol{\gamma}} = g\boldsymbol{\gamma}\boldsymbol{h} = g\tilde{v}, \tag{3.3.72}$$

in perfect parallel with the SU(2) monopole results of Section 3.3.3. The general magnetic charge quantization condition takes the form

$$\exp\left\{ig\,\boldsymbol{Q}_M\boldsymbol{H}\right\} = 1. \tag{3.3.73}$$

Let us ask ourselves what happens if one builds monopoles on non-simple roots. Such solutions are in fact composite: they consist of the basic “simple-root”

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monopoles – the masses and quantum numbers (magnetic charges) of the composite monopoles can be obtained by summing up the masses and quantum numbers of the basic monopoles, according to Eq. (3.3.64).

3.3.8 The θ term induces a fractional electric charge for the monopole (the Witten effect)

There is a P- and T-odd term, the θ term, which can be added to the Lagrangian for the Yang–Mills theory without spoiling renormalizability. It is given by

$$\mathcal{L}_\theta = \frac{\theta}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} = -\frac{\theta}{8\pi^2}\ \vec{E}^a \cdot \vec{B}^a. \tag{3.3.74}$$

This interaction violates P and CP but not C. As is well known, this term is a surface term and does not affect the classical equations of motion. There is, however, a θ dependence in instanton effects which involve nontrivial long-range behavior of the gauge fields. As was realized by Witten [113], in the presence of magnetic monopoles θ also has a nontrivial effect, it shifts the allowed values of electric charge in the monopole sector of the theory.

Since the equations of motions do not change, the monopole solution obtained above stays intact. What changes is the effective quantum-mechanical Lagrangian. As usual, we assume an adiabatic time dependence of moduli. In the case at hand we must replace the constant phase modulus α by $\alpha(t)$. This generates the electric field

$$E^a_i = \dot{\alpha}\,(\delta A^a_i/\delta\alpha) = \frac{\dot{\alpha}}{v}\left(D_i\phi^{(0)}\right)^a,$$

where Eq. (3.3.39) is used. The magnetic field does not change, and can be expressed through $\left(D_i\phi^{(0)}\right)^a$ using Eq. (3.3.22). As a result, the quantum-mechanical Lagrangian for α acquires a full derivative term,

$$\mathcal{L}_{\alpha\mathrm{QM}} = \frac{1}{2\mu}\,\dot{\alpha}^2 - \frac{\theta}{2\pi}\,\dot{\alpha}, \quad \mu = \frac{M_W^2}{M_M}. \tag{3.3.75}$$

This changes the expression for the canonic momentum conjugated to α. If previously p_α was $\dot{\alpha}/\mu$, now

$$p_\alpha = \frac{\dot{\alpha}}{\mu} - \frac{\theta}{2\pi}. \tag{3.3.76}$$

Correspondingly,

$$\dot{\alpha} = \mu\left(p_\alpha + \frac{\theta}{2\pi}\right). \tag{3.3.77}$$

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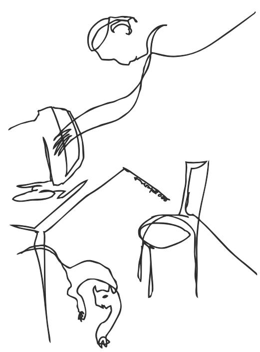
From Sect. 3.3.5 we know that the electric charge of the field configuration at hand is (see Eq. (3.3.49))

$$Q_E = \frac{1}{M_W\, g}\ \langle \dot{\alpha} \rangle\ \int_{\mathcal{S}_R} d^2 S_i\, \mathcal{B}_i\,. \tag{3.3.78}$$

Substituting Eq. (3.3.77) and $\langle\, p_\alpha \rangle = k$ we arrive at

$$Q_E = \left(k + \frac{\theta}{2\pi} \right) g. \tag{3.3.79}$$

We see that at $\theta \neq 0$ the electric charge of the dyon is non-integer. As θ changes from zero to the physically equivalent point $\theta = 2\pi$ the dyon charges shift by one unit. The dyon spectrum as a whole remains intact.



3.4 Monopoles and fermions

The critical 't Hooft–Polyakov monopoles we have just discussed can be embedded in $\mathcal{N} = 2$ super-Yang–Mills. There are no $\mathcal{N} = 1$ models with the 't Hooft–Polyakov monopoles (albeit $\mathcal{N} = 1$ theories supporting confined monopoles are found [104]). The minimal model with the BPS-saturated 't Hooft–Polyakov monopole is the $\mathcal{N} = 2$ generalization of supersymmetric gluodynamics, with the gauge group SU(2). In terms of $\mathcal{N} = 1$ superfields it contains one vector superfield in the adjoint describing gluon and gluino, plus one chiral superfield in the adjoint describing a scalar $\mathcal{N} = 2$ superpartner for gluon and a Weyl spinor, an $\mathcal{N} = 2$ superpartner for gluino.

The couplings of the fermion fields to the boson fields are of a special form, they are fixed by $\mathcal{N} = 2$ supersymmetry. In this section we will first present the Lagrangian of $\mathcal{N} = 2$ supersymmetric gluodynamics, including the part with the adjoint fermions, and then consider effects due to the adjoint fermions. We conclude

Section 3.4 with a comment on fermions in the fundamental representation in the monopole background.

3.4.1 $\mathcal{N} = 2$ super-Yang–Mills (without matter)

Two $\mathcal{N} = 1$ superfields are used to build the model,

$$W_\alpha = i\left(\lambda_\alpha + i\theta_\alpha\, D - \theta^\beta\, F_{\alpha\beta} - i\theta^2 D_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}\right), \tag{3.4.1}$$

and

$$\mathcal{A} = a + \sqrt{2}\,\psi\,\theta + \theta^2 F. \tag{3.4.2}$$

Here the notation is spinorial, and all fields are in the adjoint representation of SU(2). The corresponding generators are

$$\left(T^a\right)_{bd} = i\,\varepsilon_{bad}. \tag{3.4.3}$$

The Lagrangian contains kinetic terms and their supergeneralizations. In components

$$\begin{aligned}\mathcal{L} = \frac{1}{g^2}\Big\{ & -\frac{1}{4}F^{a\,\mu\nu}F^a_{\mu\nu} + \lambda^{\alpha,a}\, i\, D_{\alpha\dot\alpha}\,\bar\lambda^{\dot\alpha,a} + \frac{1}{2}\,D^a D^a \\ & + \psi^{\alpha,a}\, i\, D_{\alpha\dot\alpha}\,\bar\psi^{\dot\alpha,a} + (D^\mu\,\bar a)(D_\mu\, a) \\ & - \sqrt{2}\,\varepsilon_{abc}\left(\bar a^a\,\lambda^{\alpha,b}\,\psi^c_\alpha + a^a\,\bar\lambda^b_{\dot\alpha}\,\bar\psi^{\dot\alpha,c}\right) - i\,\varepsilon_{abc}\,D^a\,\bar a^b\,a^c\Big\}.\end{aligned} \tag{3.4.4}$$

As usual, the D field is auxiliary and can be eliminated via the equation of motion,

$$D^a = i\,\varepsilon_{abc}\,\bar a^b a^c. \tag{3.4.5}$$

There is a flat direction: if the field a is real all D terms vanish. If a is chosen to be purely real or purely imaginary and the fermion fields ignored we obviously return to the Georgi–Glashow model discussed above.

Let us perform the Bogomol'nyi completion of the bosonic part of the Lagrangian (3.4.4) for static field configurations. Neglecting all time derivatives and, as usual, setting $A_0 = 0$, one can write the energy functional as follows:

$$\begin{aligned}\mathcal{E} = & \sum_{i=1,2,3;\,a=1,2,3}\int d^3x\left[\frac{1}{\sqrt{2}g}F_i^{*a} \pm \frac{1}{g}D_i a^a\right]^2 \\ & \mp \frac{\sqrt{2}}{g^2}\int d^3x\,\partial_i\left(F_i^{*a}\,a^a\right),\end{aligned} \tag{3.4.6}$$

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where

$$F_m^* = \frac{1}{2}\,\varepsilon_{mnk}\,F_{nk},$$

and the square of the D term (3.4.5) is omitted – the D term vanishes provided a is real, which we will assume. This assumption also allows us to replace the absolute value in the first line by the square brackets. The term in the second line can be written as an integral over a large sphere,

$$\frac{\sqrt{2}}{g^2}\int d^3x\,\partial_i\left(F_i^{*a}\,a^a\right) = \frac{\sqrt{2}}{g^2}\int dS_i\left(a^a\,F_i^{*a}\right). \tag{3.4.7}$$

The Bogomol'nyi equations for the monopole are

$$F_i^{*a} \pm \sqrt{2}D_i\,a^a = 0. \tag{3.4.8}$$

This coincides with Eq. (3.3.22) in the Georgi–Glashow model, up to a normalization. (The field a is complex, generally speaking, and its kinetic term is normalized differently.) If the Bogomol'nyi equations are satisfied, the monopole mass is determined by the surface term (classically). Assuming that in the "flat" vacuum a^a is aligned along the third direction and taking into account that in our normalization the magnetic flux is 4π we get

$$M_M = \frac{\sqrt{2}\,a^3_{\rm vac}}{g^2}\,4\pi, \tag{3.4.9}$$

where – we recall – $a^3_{\rm vac}$ is assumed to be positive. This is in full agreement with Eq. (3.3.23).

3.4.2 Supercurrents and the monopole central charge

The general classification of central charges in $\mathcal{N}=2$ theories in four dimensions is presented in Section 2.3.3. Here we will briefly discuss the Lorentz-scalar central charge Z in the theory (3.4.4). It is this central charge that is saturated by critical monopoles.

The model, being $\mathcal{N}=2$, possesses two conserved supercurrents,

$$\begin{aligned} J^I_{\alpha\beta\dot\beta} &= \frac{2}{g^2}\left\{i\,F^a_{\beta\alpha}\bar\lambda^a_{\dot\beta} + \varepsilon_{\beta\alpha}D^a\bar\lambda^a_{\dot\beta} + \sqrt{2}\left(D_{\alpha\dot\beta}\bar a^a\right)\psi^a_\beta\right\} + \text{f.d.},\\ J^{II}_{\alpha\beta\dot\beta} &= \frac{2}{g^2}\left\{i\,F^a_{\beta\alpha}\bar\psi^a_{\dot\beta} + \varepsilon_{\beta\alpha}D^a\bar\psi^a_{\dot\beta} - \sqrt{2}\left(D_{\alpha\dot\beta}\bar a^a\right)\lambda^a_\beta\right\} + \text{f.d.}, \end{aligned} \tag{3.4.10}$$

where f.d. stands for full derivatives. Both expressions can be combined in one compact formula if we introduce an SU(2) index f ($f = 1, 2$) (to be repeatedly used in Part II) in the following way:

$$\lambda^f = \begin{cases} \lambda, & f = 1 \\ \psi, & f = 2. \end{cases} \tag{3.4.11}$$

Then $\lambda_1 = -\psi$ and $\lambda_2 = \lambda$. The supercurrent takes the form ($f = 1, 2$)

$$\begin{aligned} J_{\alpha\beta\dot\beta,\,f} = \frac{2}{g^2}\Big\{ & i F^a_{\beta\alpha}\bar\lambda^a_{\dot\beta,\,f} + \varepsilon_{\beta\alpha} D^a \bar\lambda^a_{\dot\beta,\,f} - \sqrt{2}\left(D_{\alpha\dot\beta}\bar a^a\right)\lambda^a_{\beta,\,f} \\ & + \frac{\sqrt{2}}{6}\left[\partial_{\alpha\dot\beta}(\lambda_{\beta,\,f}\,\bar a) + \partial_{\beta\dot\beta}(\lambda_{\alpha,\,f}\,\bar a) - 3\varepsilon_{\beta\alpha}\partial^{\gamma}_{\dot\beta}(\lambda_{\gamma,\,f}\,\bar a)\right]\Big\}. \end{aligned} \tag{3.4.12}$$

Classically the commutator of the corresponding supercharges is

$$\begin{aligned} \{Q^I_\alpha,\, Q^{II}_\beta\} = 2\,Z\,\varepsilon_{\alpha\beta} &= -\frac{2\sqrt{2}}{g^2}\,\varepsilon_{\alpha\beta}\int d^3x\,\mathrm{div}\left(\bar a^a\left(\vec E^a - i\,\vec B^a\right)\right) \\ &= -\frac{2\sqrt{2}}{g^2}\,\varepsilon_{\alpha\beta}\int dS_j\left(\bar a^a\left(E^a_j - i\,B^a_j\right)\right). \end{aligned} \tag{3.4.13}$$

Z in Eq. (3.4.13) is sometimes referred to as the monopole central charge. For the BPS-saturated monopoles $M_M = Z$.

Quantum corrections in the monopole central charge and in the mass of the BPS saturated monopoles were first discussed in Refs. [8, 114, 43] two decades ago. The monopole central charge is renormalized at one-loop level. This is obvious due to the fact that the corresponding quantum correction must convert the bare coupling constant in Eq. (3.4.13) into the renormalized one. The fact that the logarithmic renormalizations of the monopole mass and the gauge coupling constant match was established long ago. However, there is a residual non-logarithmic effect which cannot be obtained from Eq. (3.4.13). It was not until 2004 that people realized that the monopole central charge (3.4.13) must be supplemented by an anomalous term [39].

To elucidate the point, let us consider (following [38]) the formula for the monopole/dyon mass obtained in the exact Seiberg–Witten solution [2],

$$M_{n_e,n_m} = \sqrt{2}\left|a\left(n_e - \frac{a_D}{a}\,n_m\right)\right|, \tag{3.4.14}$$

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where $n_{e,m}$ are integer electric and magnetic numbers (we will consider here only a particular case when either $n_e = 0, 1$ or $n_m = 0, 1$) and

$$a_D = i\, a \left(\frac{4\pi}{g_0^2} - \frac{2}{\pi} \ln \frac{M_0}{a} \right). \tag{3.4.15}$$

The subscript 0 is introduced for clarity, it marks the bare charge. The renormalized coupling constant is defined in terms of the ultraviolet parameters as follows:

$$\frac{\partial a_D}{\partial a} \equiv \frac{4\pi i}{g^2}. \tag{3.4.16}$$

Because of the $a \ln a$ dependence, $\partial a_D/\partial a$ differs from a_D/a by a constant (non-logarithmic) term, namely,

$$\frac{a_D}{a} = i \left(\frac{4\pi}{g^2} - \frac{2}{\pi} \right). \tag{3.4.17}$$

Combining Eq. (3.4.14) and (3.4.17) we get

$$M_{n_e,n_m} = \sqrt{2} \left| a \left(n_e - i \left(\frac{4\pi}{g^2} - \frac{2}{\pi} \right) n_m \right) \right|, \tag{3.4.18}$$

This does not match Eq. (3.4.13) in the non-logarithmic part (i.e. the term $2\sqrt{2}\, n_m/\pi$). Since the relative weight of the electric and magnetic parts in Eq. (3.4.13) is fixed to be g^2, the presence of the above non-logarithmic term implies that, in fact, the chiral structure $E_j^a - i\, B_j^a$ obtained at the canonic commutator level cannot be maintained once quantum corrections are switched on. This is a quantum anomaly.

So far no direct calculation of the anomalous contribution in $\{Q_\alpha^I,\, Q_\beta^{II}\}$ in the operator form has been carried out. However, it is not difficult to reconstruct it indirectly, using Eq. (3.4.18) and a close parallel between $\mathcal{N} = 2$ super-Yang–Mills theory and $\mathcal{N} = 2$ CP$(N-1)$ model with twisted mass in two dimensions in which a similar problem was solved [34],

$$\left\{ Q_\alpha^I,\, Q_\beta^{II} \right\}_{\text{anom}} = 2\, \varepsilon_{\alpha\beta}\, \delta Z_{\text{anom}} = -\left(\varepsilon_{\alpha\beta} \right) 2\sqrt{2}\, \frac{1}{4\pi^2} \int dS_j\, \Sigma^j \tag{3.4.19}$$

where

$$\begin{aligned} \Sigma^j &= \frac{i}{2} \frac{\partial}{\partial \bar\theta^{\dot\beta}} \left(\bar{\mathcal{A}}^a\, \bar W^a_{\dot\alpha} \right) \left(\sigma^j \right)^{\dot\alpha\dot\beta} \Big|_{\bar\theta=0} \\ &= \bar a^a \left(\vec E^a + i\, \vec B^a \right)^j - \frac{\sqrt{2}}{2}\, \bar\lambda^a_{\dot\alpha} \left(\sigma^j \right)^{\dot\alpha\dot\beta}\, \bar\psi^a_{\dot\beta}, \end{aligned} \tag{3.4.20}$$

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to be added to Eq. (3.4.13). The (1,0) conversion matrix $\left(\sigma^j\right)^{\dot\alpha\dot\beta}$ is defined in Section A.5. Equation (3.4.20) is to be compared with that obtained at the end of Section 4.5.3. We hasten to note that the bifermion term $\bar\lambda\bar\psi$ in $\delta Z_{\rm anom}$ was calculated in Ref. [39].

In the SU(N) theory we would have $N/8\pi^2$ instead of $1/4\pi^2$ in Eq. (3.4.19).

Adding the canonic and the anomalous terms in $\{Q^I_\alpha,\ Q^{II}_\beta\}$ together we see that the fluxes generated by color-electric and color-magnetic terms are now shifted, untied from each other, by a non-logarithmic term in the magnetic part. Normalizing to the electric term, $M_W = \sqrt{2}a$, we get for the magnetic term

$$M_M = \sqrt{2}\, a \left(\frac{4\pi}{g^2} - \frac{2}{\pi}\right), \tag{3.4.21}$$

as it is necessary for the consistency with the exact Seiberg–Witten solution.

3.4.3 Zero modes for adjoint fermions

Equations for the fermion zero modes can be readily derived from the Lagrangian (3.4.4),

$$\begin{aligned} i\,D_{\alpha\dot\alpha}\lambda^{\alpha,c} - \sqrt{2}\,\varepsilon_{abc}\,a^a\,\bar\psi^b_{\dot\alpha} = 0 \\ i\,D_{\alpha\dot\alpha}\psi^{\alpha,c} + \sqrt{2}\,\varepsilon_{abc}\,a^a\,\bar\lambda^b_{\dot\alpha} = 0, \end{aligned} \tag{3.4.22}$$

plus Hermitean conjugate. After a brief reflection we can get two complex (four real) zero modes.[11] Two of them are obtained if we substitute

$$\lambda^\alpha = F^{\alpha\beta}, \quad \bar\psi_{\dot\alpha} = \sqrt{2}\,D_{\alpha\dot\alpha}\,\bar a. \tag{3.4.23}$$

The other two solutions correspond to the following substitution:

$$\psi^\alpha = F^{\alpha\beta}, \quad \bar\lambda_{\dot\alpha} = \sqrt{2}\,D_{\alpha\dot\alpha}\,\bar a. \tag{3.4.24}$$

This result is easy to understand. Our starting theory has eight supercharges. The classical monopole solution is BPS-saturated, implying that four of these eight supercharges annihilate the solution (these are the Bogomol'nyi equations), while the action of the other four supercharges produces the fermion zero modes.

With four real fermion collective coordinates, the monopole supermultiplet is four-dimensional: it includes two bosonic states and two fermionic. (The above counting refers just to monopole, without its antimonopole partner. The anti-monopole supermultiplet also includes two bosonic and two fermionic states.) From

[11] This means that the monopole is described by two complex fermion collective coordinates, or four real.

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the standpoint of $\mathcal{N} = 2$ supersymmetry in four dimensions this is a short multiplet. Hence, the monopole states remain BPS saturated to all orders in perturbation theory (in fact, the criticality of the monopole supermultiplet is valid beyond perturbation theory [2, 3]).

3.4.4 Zero modes for fermions in the fundamental representation

This topic, being related to an interesting phenomenon of charge fractionalization, is marginal for this review. Therefore, we will limit ourselves to a brief comment. The interested reader is referred to [16, 115, 17] for further details. The fermion part of the Lagrangian can be obtained from (3.4.4) with the obvious replacement of the adjoint Dirac fermion by the fundamental one, which we will denote by χ,

$$\mathcal{L} = \frac{1}{g^2}\left\{-\frac{1}{4}F^{a\,\mu\nu}F^a_{\mu\nu} + \frac{1}{2}(D^\mu\,\phi)(D_\mu\,\phi) + \bar{\chi}\,i\,\not{D}\chi - \bar{\chi}\,\phi\chi\right\}. \tag{3.4.25}$$

The Dirac equation then takes the form

$$(i\gamma^\mu D_\mu - \phi)\chi = 0. \tag{3.4.26}$$

Gamma matrices can be chosen in any representation. The one which is most convenient here is

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} -i\sigma^i & 0 \\ 0 & i\sigma^i \end{pmatrix}. \tag{3.4.27}$$

For the static 't Hooft–Polyakov monopole configuration (with $A_0 = 0$) the zero mode equations reduce to two decoupled equations

$$\begin{aligned} \not{\mathcal{D}}\chi^- &\equiv (\sigma^i D_i + \phi)\chi^- = 0. \\ \not{\mathcal{D}}^\dagger\chi^+ &\equiv (\sigma^i D_i - \phi)\chi^+ = 0, \quad i = 1, 2, 3. \end{aligned} \tag{3.4.28}$$

provided we parametrize $\chi(\vec{x})$ in terms of the following two-component spinors:

$$\chi = \begin{pmatrix} \chi^+ \\ \chi^- \end{pmatrix}. \tag{3.4.29}$$

Now we can use the Callias theorem [116] which says

$$\dim\ker\not{\mathcal{D}} - \dim\ker\not{\mathcal{D}}^\dagger = n_m\,, \tag{3.4.30}$$

where n_m is the topological number, $n_m = 1$ for the monopole and $n_m = -1$ for the antimonopole. This implies, in turn, that Eq. (3.4.28) has one complex zero mode,

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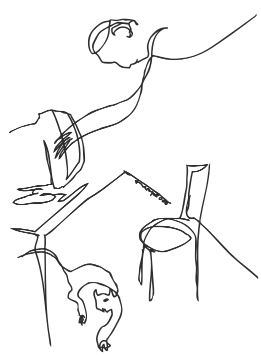
i.e. in the case at hand we characterize the monopole by one complex fermion collective coordinate (and a conjugate, of course). This fact leads to a drastic consequence: the monopole acquires a half-integer electric charge. It becomes a dyon with charge 1/2 even in the absence of the θ term. This phenomenon – the charge fractionalization in the cases with a single complex fermion collective coordinate – is well known in the literature [115, 16, 34, 17] and dates back to Jackiw and Rebbi [117].

3.4.5 The monopole supermultiplet: dimension of the BPS representations

As was first noted by Montonen and Olive [118], all states in $\mathcal{N} = 2$ model – W bosons and monopoles alike – are BPS saturated. This results in the fact that supermultiplets of this model are short. Regular (long) supermultiplet would contain $2^{2\mathcal{N}} = 16$ helicity states, while the short ones contain $2^{\mathcal{N}} = 4$ helicity states – two bosonic and two fermionic. This is in full accord with the fact that the number of the fermion zero modes on the given monopole solution is four, resulting in dim-4 representation of the supersymmetry algebra. If we combine particles and antiparticles together, as is customary in field theory, we will have one Dirac spinor on the fermion side of the supermultiplet. This statement is valid in both cases, the monopole supermultiplet and that of W-bosons.



3.5 More on kinks (in $\mathcal{N} = 2$ CP(1) model)

Kinks in two-dimensional $\mathcal{N} = 2$ CP($N - 1$) models will play a crucial role in our subsequent studies of confined monopoles in Part II of this book (see e.g. Sections 4.4.1, 4.4.3, 4.4.4, and 4.5). Here we will review basic features of such kinks using CP(1) as the simplest example.

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The Lagrangian of the CP(1) model with the twisted mass has the form [32]

$$\begin{aligned}\mathcal{L}_{CP(1)} = G\Big\{&\partial_\mu\phi^\dagger\,\partial^\mu\phi - |m|^2\phi^\dagger\,\phi + \frac{i}{2}\big(\psi_L^\dagger\,\overleftrightarrow{\partial_R}\,\psi_L + \psi_R^\dagger\,\overleftrightarrow{\partial_L}\,\psi_R\big)\\ &- i\,\frac{1-\phi^\dagger\,\phi}{\chi}\,\big(m\,\psi_L^\dagger\psi_R + \bar{m}\psi_R^\dagger\psi_L\big)\\ &- \frac{i}{\chi}\,\big[\psi_L^\dagger\psi_L\big(\phi^\dagger\,\overleftrightarrow{\partial_R}\,\phi\big) + \psi_R^\dagger\,\psi_R\big(\phi^\dagger\,\overleftrightarrow{\partial_L}\,\phi\big)\big]\\ &- \frac{2}{\chi^2}\,\psi_L^\dagger\,\psi_L\,\psi_R^\dagger\,\psi_R\Big\} + \frac{i g^2\,\theta}{4\pi}\,G\,\varepsilon^{\mu\nu}\partial_\mu\phi^\dagger\,\partial_\nu\phi,\end{aligned} \tag{3.5.1}$$

where

$$\partial_L = \frac{\partial}{\partial t} + \frac{\partial}{\partial z}, \qquad \partial_R = \frac{\partial}{\partial t} - \frac{\partial}{\partial z}, \tag{3.5.2}$$

and

$$G = \frac{2}{g^2\,\chi^2}, \qquad \chi = 1 + \phi\phi^\dagger. \tag{3.5.3}$$

Moreover, m is a complex mass parameter and θ is the vacuum angle. The above Lagrangian has an obvious U(1) symmetry. At $m = 0$ it describes the $\mathcal{N} = 2$ supergeneralization of the σ model on the sphere S_2 (see Appendix B). The metric of the sphere G is chosen in the Fubini–Study form.

It is not difficult to derive the supercurrent,

$$J_\alpha^\mu = \sqrt{2}\,G\big[\partial_\nu\phi^\dagger\gamma^\nu\gamma^\mu\psi + i\phi^\dagger\gamma^\mu\mu\,\psi\big]_\alpha, \tag{3.5.4}$$

where

$$\mu = m\,\frac{1+\gamma_5}{2} + \bar{m}\,\frac{1-\gamma_5}{2}. \tag{3.5.5}$$

The superalgebra is centrally extended, as in Eq. (2.3.4) with $Z' = 0$ and

$$Z = m q_{\mathrm{U}(1)} - i\int dz\,\partial_z\Big\{m\,D - \frac{1}{2\pi}\big(m g_0^2 D - i\,R\,\psi_R^\dagger\,\psi_L\big)\Big\}, \tag{3.5.6}$$

where

$$R = \frac{2}{\chi^2}, \qquad D = \frac{2}{g^2}\,\frac{\phi^\dagger\phi}{\chi}, \tag{3.5.7}$$

and $q_{\mathrm{U}(1)}$ is the Noether charge corresponding to the U(1) current

$$\mathcal{J}_\mu = G\Big[\phi^\dagger i\,\overleftrightarrow{\partial_\mu}\,\phi + \bar{\psi}\gamma_\mu\big(\psi + \Gamma\phi\,\psi\big)\Big], \qquad \Gamma = -2\,\frac{\phi^\dagger}{\chi}. \tag{3.5.8}$$

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Thus, the term $m\, q_{\mathrm{U}(1)}$ in Z represents the Noether charge while the integral represents the topological charge.

The first two terms are classical, while the term in parentheses is the quantum anomaly derived in [33, 34].[12] At large $|m|$ the theory is at weak coupling.

3.5.1 BPS solitons at the classical level

The U(1) invariant scalar potential term

$$V = |m|^2\, G\, \bar{\phi}\phi \tag{3.5.9}$$

lifts the vacuum degeneracy leaving us with two discrete vacua: at the south and north poles of the sphere (Fig. 3.11) i.e. $\phi = 0$ and $\phi = \infty$.

The kink solutions interpolate between these two vacua. Let us focus, for definiteness, on the kink with the boundary conditions

$$\phi \to 0 \quad \text{at} \quad z \to -\infty, \qquad \phi \to \infty \quad \text{at} \quad z \to \infty\,. \tag{3.5.10}$$

Consider the following linear combinations of supercharges

$$\mathcal{Q} = Q_R - i\, e^{-i\beta} Q_L, \quad \bar{\mathcal{Q}} = \bar{Q}_R + i\, e^{i\beta} \bar{Q}_L, \tag{3.5.11}$$

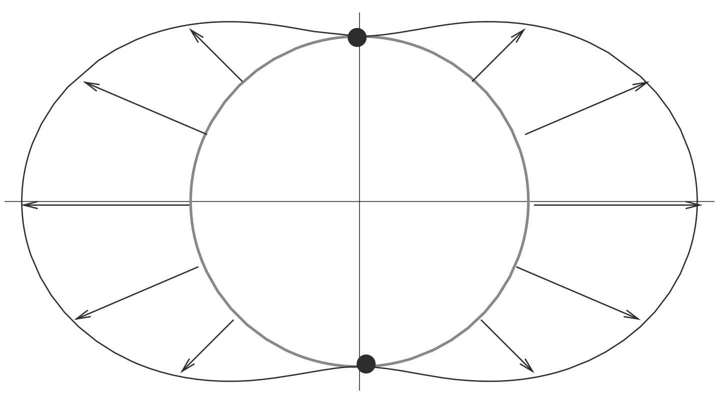

Figure 3.11. Meridian slice of the target space sphere (thick solid line). Arrows present the scalar potential in (3.5.1), their length being the strength of the potential. Two vacua of the model are denoted by closed circles.

[12] In the first of these papers only the bifermion part of the anomaly was obtained. The full anomalous term in the central charge (3.5.6) was found in [34]; later it was confirmed in [119].

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where β is the argument of the mass parameter,

$$m = |m|\, e^{i\beta}. \tag{3.5.12}$$

Then

$$\{\mathcal{Q}\bar{\mathcal{Q}}\} = 2H - 2Z\,, \quad \{\mathcal{Q}\mathcal{Q}\} = \{\bar{\mathcal{Q}}\bar{\mathcal{Q}}\} = 0. \tag{3.5.13}$$

Now, let us require $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ to vanish on the classical solution. Since for static field configurations

$$\mathcal{Q} = -\left(\partial_z\bar{\phi} - |m|\bar{\phi}\right)\left(\Psi_R + i e^{-i\beta}\Psi_L\right),$$

the vanishing of these two supercharges implies

$$\partial_z\bar{\phi} = |m|\bar{\phi} \quad \text{or} \quad \partial_z\phi = |m|\phi\,. \tag{3.5.14}$$

This is the BPS equation in the CP(1) model with the twisted mass.

The BPS equation (3.5.14) has a number of peculiarities compared to those in more familiar Wess–Zumino models. The most important feature is its complexification, i.e. the fact that Eq. (3.5.14) is holomorphic in ϕ.

The solution of this equation is, of course, trivial, and can be written as

$$\phi(z) = e^{|m|(z-z_0)-i\alpha}. \tag{3.5.15}$$

Here z_0 is the kink center while α is an arbitrary phase. In fact, these two parameters enter only in the combination $|m|z_0 + i\alpha$. We see that the notion of the kink center also gets complexified.

The physical meaning of the modulus α is obvious: there is a continuous family of solitons interpolating between the north and south poles of the target space sphere. This is due to U(1) symmetry. The soliton trajectory can follow any meridian (Fig. 3.12).

Equation (3.5.6) for the central charge implies that classically the kink mass is

$$M_0 = |m|\,(D(\infty) - D(0)) = \frac{2|m|}{g^2}. \tag{3.5.16}$$

(The subscript 0 emphasizes that this result is obtained at the classical level.) Quantum corrections will be considered shortly.

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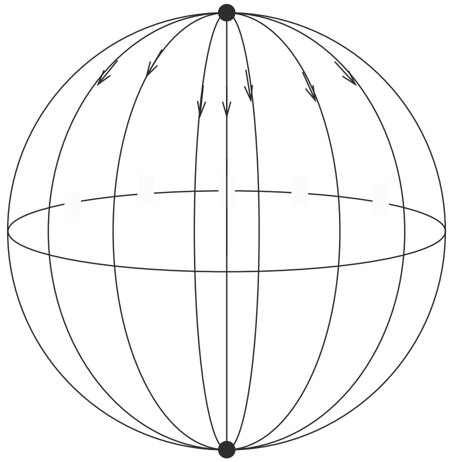

Figure 3.12. The soliton solution family. The collective coordinate α in Eq. (3.5.15) spans the interval $0 \leq \alpha \leq 2\pi$. For given α the soliton trajectory on the target space sphere follows a meridian, so that when α varies from 0 to 2π all meridians are covered.

3.5.2 Quantization of the bosonic moduli

To carry out conventional quasiclassical quantization we, as usual, assume the moduli z_0 and α in Eq. (3.5.15) to be (weakly) time-dependent, substitute (3.5.15) in the bosonic part of the Lagrangian (3.5.1), integrate over z and arrive at

$$\mathcal{L}_{\rm QM} = -M_0 + \frac{M_0}{2}\dot{z}_0^2 + \left\{\frac{1}{g^2|m|}\,\dot{\alpha}^2 - \frac{\theta}{2\pi}\,\dot{\alpha}\right\}. \tag{3.5.17}$$

The first term is the classical kink mass, the second describes free motion of the kink along the z axis. The term in the braces is most interesting. The variable α is compact. Its very existence is related to the exact U(1) symmetry of the model. The energy spectrum corresponding to α dynamics is quantized. It is not difficult to see that

$$E_{[\alpha]} = \frac{g^2|m|}{4}\; q^2_{\rm U(1)}, \tag{3.5.18}$$

where $q_{\rm U(1)}$ is the U(1) charge of the soliton,

$$q_{\rm U(1)} = k + \frac{\theta}{2\pi}\,, \quad k = \text{ an integer.} \tag{3.5.19}$$

Here we see the Witten phenomenon at work, analogously to that discussed in Section 3.3.8 for monopoles. The kink U(1) charge is no longer integer in the presence of the θ term, it is shifted by $\theta/(2\pi)$.

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3.5.3 The kink mass and holomorphy

Taking account of $E_{[\alpha]}$ – the energy of an "internal motion" – the kink mass can be written as

$$\begin{aligned} M &= \frac{2|m|}{g^2} + \frac{g^2|m|}{4}\left(k + \frac{\theta}{2\pi}\right)^2 \\ &= \frac{2|m|}{g^2}\left\{1 + \frac{g^4}{4}\left(k + \frac{\theta}{2\pi}\right)^2\right\}^{1/2} \\ &= 2|m|\left|\frac{1}{g^2} + i\,\frac{\theta + 2\pi k}{4\pi}\right|. \end{aligned} \tag{3.5.20}$$

Formally, the second equality here is approximate, valid to the leading order in the coupling constant. In fact, it is exact. The important circumstance to be stressed is that the kink mass depends on a special combination of the coupling constant and θ, namely,

$$\tau = \frac{1}{g^2} + i\,\frac{\theta}{4\pi} \tag{3.5.21}$$

In other words, it is the complexified coupling constant that enters.

Note that g^2 in Eq. (3.5.20) is the bare coupling constant. It is quite clear that the kink mass, being a physical parameter, should contain the renormalized constant $g^2(m)$, after taking account of radiative corrections.

Since the kink mass $M = |Z|$ radiative corrections must replace the bare $1/g^2$ by the renormalized $1/g^2(m)$ in Z. One-loop calculation is quite trivial. First, rotate the mass parameter m in such a way as to make it real, $m \to |m|$. Simultaneously, the θ angle is replaced by an effective θ,

$$\theta \to \theta_{\text{eff}} = \theta + 2\beta, \tag{3.5.22}$$

where the phase β is defined in Eq. (3.5.11). Next, decompose the field ϕ into a classical plus quantum part,

$$\phi \to \phi + \delta\phi.$$

Then the D part of the central charge

$$Z = mq - i\int dz\,\partial_z\, m\, D$$

becomes

$$D \to D + \frac{2}{g^2}\,\frac{1 - \phi^\dagger\phi}{\left(1 + \bar{\phi}\phi\right)^3}\,\delta\phi^\dagger\,\delta\phi. \tag{3.5.23}$$

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(The term in parentheses in (3.5.6) – the anomaly – gives a non-logarithmic contribution which we ignore for the time being.) Contracting $\delta\phi^\dagger\,\delta\phi$ into a loop and calculating this loop we arrive at

$$D \to \frac{\phi^\dagger\phi}{\chi}\left[\frac{2}{g^2} - \frac{2}{4\pi}\,\ln\frac{M_{\rm uv}^2}{|m|^2}\right], \tag{3.5.24}$$

which, in turn, yields

$$Z = 2im\left\{\tau - \frac{1}{4\pi}\,\ln\frac{M_{\rm uv}^2}{m^2} - i\,\frac{k}{2}\right\} \equiv 2im\left\{\tau_{\rm ren} - i\,\frac{k}{2}\right\}. \tag{3.5.25}$$

A salient feature of this formula is the holomorphic dependence of Z on m and τ. Such holomorphic dependence would be impossible if two and more loops contributed to D renormalization. Thus, D renormalization beyond one loop must cancel, and it does.[13] Note also that the bare coupling in Eq. (3.5.25) conspires with the logarithm in such a way as to replace the bare coupling by that renormalized at $|m|$, as was expected.

The analysis carried out above is quasiclassical. It tells us nothing about possible occurrence of nonperturbative terms in Z. In fact, all terms of the type

$$\left\{\frac{M_{\rm uv}^2}{m^2}\,\exp\left(-4\pi\tau\right)\right\}^{\ell}, \qquad \ell = \text{integer}$$

are fully compatible with holomorphy; they can and do emerge from instantons. An indirect calculation of nonperturbative terms was performed in Ref. [30].

The exact formula for this central charge obtained by Dorey is

$$Z = m\,q - im_D\,T, \tag{3.5.26}$$

where the subscript D in m_D appears for historical reasons, in parallel with the Seiberg–Witten solution (it stands for dual), and

$$m_D = \frac{m}{\pi}\left[\frac{1}{2}\,\ln\frac{m + \sqrt{m^2 + 4\Lambda^2}}{m - \sqrt{m^2 + 4\Lambda^2}} - \sqrt{1 + \frac{4\Lambda^2}{m^2}}\,\right]. \tag{3.5.27}$$

Furthermore, T is the topological charge of the kink under consideration, $T = \pm 1$. The limit $|m|/\Lambda \to \infty$ corresponds to the quasiclassical domain, while corrections of the type $(\Lambda/m)^{2k}$ are induced by instantons.

[13] Fermions are important for this cancellation.

It is instructive to consider the quasiclassical limit of Eq. (3.5.27) when the mass m is real and large, $m \gg \Lambda$. In this limit

$$\langle Z\rangle_{\rm kink} = -\frac{i\,m}{2\pi}\left[\ln\left(-\frac{m^2}{\Lambda^2}\right)-2\right]$$
$$= \frac{1}{2}\,m - im\left(\frac{2}{g_0^2}-\frac{1}{2\pi}\ln\frac{M_{\rm uv}^2}{m^2}\right)+\frac{i\,m}{\pi}, \qquad (3.5.28)$$

where g_0^2 is the bare coupling constant, and $M_{\rm uv}$ is the ultraviolet cut off. The first term in the second line reflects the fractional U(1) charge, $q = 1/2$, carried by the soliton at $\theta = 0$. The reason for the occurrence of half-integer charge will be explained in detail in Section 3.5.5. The second term coincides with the one-loop corrected average of $(-i\int dz\partial_z O_{\rm canon})$ in the central charge. The third term im/π represents the anomaly.

What happens when one travels from the domain of large $|m|$ to that of small $|m|$? If $m = 0$ we know (e.g. from the mirror representation [120]) that there are two degenerate two-dimensional kink supermultiplets, corresponding to the Cecotti–Fendley–Intriligator–Vafa (CFIV) index = 2 [121]. They have quantum numbers $\{q,\ T\} = (0, 1)$ and $(1, 1)$, respectively. Away from the point $m = 0$ the masses of these states are no longer equal; there is one singular point with one of the two states becoming massless [34]. The region containing the point $m = 0$ is separated from the quasiclassical region of large m by the curve of marginal stability (CMS) on which an infinite number of other BPS states, visible quasiclassically, decay, see Fig. 3.13.[14] Thus, the infinite tower of the $\{q,\ T\}$ BPS states existing in the quasiclassical domain degenerates in just two stable BPS states in the vicinity of $m = 0$.

3.5.4 Fermions in quasiclassical consideration

Non-zero modes are irrelevant for our consideration since, being combined with the boson non-zero modes, they cancel for critical solitons, a usual story. Thus, for our purposes it is sufficient to focus on the (static) zero modes in the kink background (3.5.15). The coefficients in front of the fermion zero modes will become (time-dependent) fermion moduli, for which we are going to build corresponding quantum mechanics. There are two such moduli, $\bar\eta$ and η.

[14] CMS for CP($N-1$) with $N > 2$ is considered in [122].

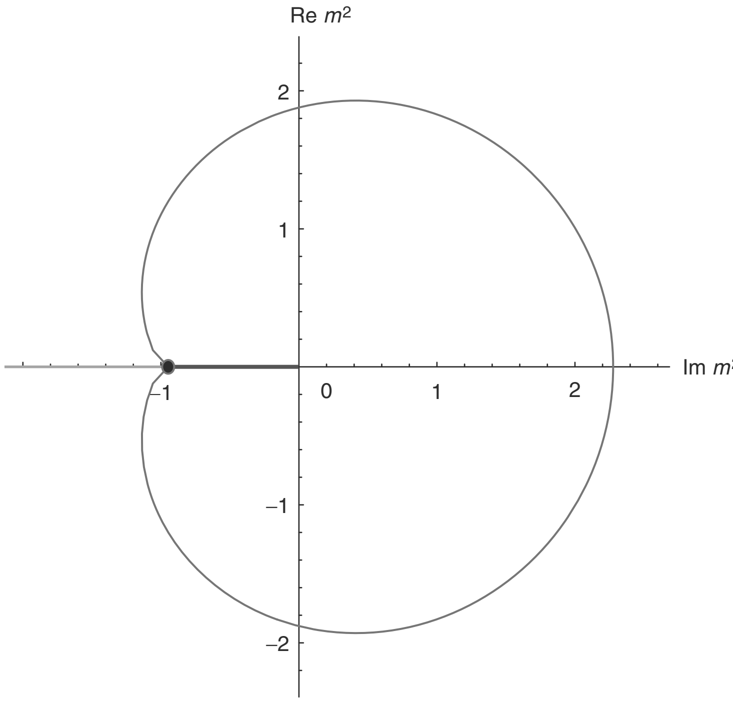

Figure 3.13. Curve of marginal stability in CP(1) with twisted mass. We set $4\Lambda^2 \to 1$. From Ref. [34].

The equations for the fermion zero modes are

$$\partial_z \Psi_L - \frac{2}{\chi}\left(\bar\phi \partial_z \phi\right)\Psi_L - i\,\frac{1-\bar\phi\phi}{\chi}\,|m|e^{i\beta}\Psi_R = 0,$$
$$\partial_z \Psi_R - \frac{2}{\chi}\left(\bar\phi \partial_z \phi\right)\Psi_R + i\,\frac{1-\bar\phi\phi}{\chi}\,|m|e^{-i\beta}\Psi_L = 0 \qquad (3.5.29)$$

(plus similar equations for $\bar\Psi$; since our operator is Hermitean we do not need to consider them separately).

It is not difficult to find a solution to these equations, either directly, or using supersymmetry. Indeed, if we know the bosonic solution (3.5.15), its fermionic superpartner – and the fermion zero modes are such superpartners – is obtained from the bosonic one by those two supertransformations which act on $\bar\phi$, ϕ nontrivially. In this way we conclude that the functional form of the fermion zero

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mode must coincide with the functional form of the boson solution (3.5.15). Concretely,

$$\begin{pmatrix} \Psi_R \\ \Psi_L \end{pmatrix} = \eta \left(\frac{g^2|m|}{2}\right)^{1/2} \begin{pmatrix} -ie^{-i\beta} \\ 1 \end{pmatrix} e^{|m|(z-z_0)} \tag{3.5.30}$$

and

$$\begin{pmatrix} \bar{\Psi}_R \\ \bar{\Psi}_L \end{pmatrix} = \bar{\eta} \left(\frac{g^2|m|}{2}\right)^{1/2} \begin{pmatrix} ie^{i\beta} \\ 1 \end{pmatrix} e^{|m|(z-z_0)}, \tag{3.5.31}$$

where the numerical factor is introduced to ensure proper normalization of the quantum-mechanical Lagrangian. Another solution which asymptotically, at large z, behaves as $e^{3|m|(z-z_0)}$ must be discarded as non-normalizable.

Now, to perform quasiclassical quantization we follow the standard route: the moduli are assumed to be time-dependent, and we derive quantum mechanics of moduli starting from the original Lagrangian (3.5.1). Substituting the kink solution and the fermion zero modes for Ψ one gets

$$\mathcal{L}'_{\rm QM} = i\,\bar{\eta}\dot{\eta}. \tag{3.5.32}$$

In the Hamiltonian approach the only remnants of the fermion moduli are the anticommutation relations

$$\{\bar{\eta}\eta\} = 1, \quad \{\bar{\eta}\bar{\eta}\} = 0, \quad \{\eta\eta\} = 0, \tag{3.5.33}$$

which tell us that the wave function is two-component (i.e. the kink supermultiplet is two-dimensional). One can implement Eq. (3.5.33) by choosing e.g. $\bar{\eta} = \sigma^+$, $\eta = \sigma^-$.

The fact that there are two critical kink states in the supermultiplet is consistent with the multiplet shortening in $\mathcal{N} = 2$. Indeed, in two dimensions the full $\mathcal{N} = 2$ supermultiplet must consist of four states: two bosonic and two fermionic. 1/2 BPS multiplets are shortened – they contain twice less states than the full supermultiplets, one bosonic and one fermionic. This is to be contrasted with the single-state kink supermultiplet in the minimal supersymmetric model of Section 3.1.1. The notion of the fermion parity remains well-defined in the kink sector of the CP(1) model.

3.5.5 Combining bosonic and fermionic moduli

Quantum dynamics of the kink at hand is summarized by the Hamiltonian

$$H_{\rm QM} = \frac{M_0}{2}\,\dot{\bar{\zeta}}\,\dot{\zeta} \tag{3.5.34}$$

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acting in the space of two-component wave functions. The variable ζ here is a complexified kink center,

$$\zeta = z_0 + \frac{i}{|m|}\,\alpha. \tag{3.5.35}$$

For simplicity, we set the vacuum angle $\theta = 0$ for the time being (it will be reinstated later).

The original field theory we deal with has four conserved supercharges. Two of them, $\mathcal{Q}$ and $\bar{\mathcal{Q}}$, see Eq. (3.5.11), act trivially in the critical kink sector. In moduli quantum mechanics they take the form

$$\mathcal{Q} = \sqrt{M_0}\,\dot{\zeta}\,\eta, \quad \bar{\mathcal{Q}} = \sqrt{M_0}\,\dot{\bar{\zeta}}\,\bar{\eta}\,; \tag{3.5.36}$$

they do indeed vanish provided that the kink is at rest. Superalgebra describing kink quantum mechanics is $\{\bar{\mathcal{Q}}\,\mathcal{Q}\} = 2H_{\mathrm{QM}}$. This is nothing but Witten's $\mathcal{N}=1$ supersymmetric quantum mechanics [123] (two supercharges). The realization we deal with is peculiar and distinct from that of Witten. Indeed, the standard Witten quantum mechanics includes one (real) bosonic degree of freedom and two fermionic, while we have two bosonic degrees of freedom, x_0 and α. Nevertheless, superalgebra remains the same due to the fact that the bosonic coordinate is complexified.

Finally, to conclude this section, let us calculate the U(1) charge of the kink states. We start from Eq. (3.5.8), substitute the fermion zero modes and get[15]

$$\Delta q_{\mathrm{U}(1)} = \frac{1}{2}[\bar{\eta}\eta] \tag{3.5.37}$$

(this is to be added to the bosonic part, Eq. (3.5.19)). Given that $\bar{\eta} = \sigma^+$ and $\eta = \sigma^-$ we arrive at $\Delta q_{\mathrm{U}(1)} = \frac{1}{2}\sigma_3$. This means that the U(1) charges of two kink states in the supermultiplet split from the value given in Eq. (3.5.19): one has the U(1) charge

$$k + \frac{1}{2} + \frac{\theta}{2\pi},$$

[15] To set the scale properly, so that the U(1) charge of the vacuum state vanishes, one must antisymmetrize the fermion current, $\bar{\Psi}\gamma^\mu\Psi \to (1/2)\left(\bar{\Psi}\gamma^\mu\Psi - \bar{\Psi}^c\gamma^\mu\Psi^c\right)$ where the superscript c denotes C conjugation.

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and another

$$k - \frac{1}{2} + \frac{\theta}{2\pi}.$$

In this way we explain the occurrence of $1/2$ seen from the quasiclassical expansion of the exact formula (3.5.28).