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DEPENDENT RISK MODELS WITH BIVARIATE PHASE-TYPE DISTRIBUTIONS

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Abstract

In this paper we consider an extension of the Sparre Andersen insurance risk model by relaxing one of its independence assumptions. The newly proposed dependence structure is introduced through the premise that the joint distribution of the interclaim time and the subsequent claim size is bivariate phase-type (see, e.g. Assaf *et al.* (1984) and Kulkarni (1989)). Relying on the existing connection between risk processes and fluid flows (see, e.g. Badescu *et al.* (2005), Badescu, Drekic and Landriault (2007), Ramaswami (2006), and Ahn, Badescu and Ramaswami (2007)), we construct an analytically tractable fluid flow that leads to the analysis of various ruin-related quantities in the aforementioned risk model. Using matrix-analytic methods, we obtain an explicit expression for the Gerber–Shiu discounted penalty function (see Gerber and Shiu (1998)) when the penalty function depends on the deficit at ruin only. Finally, we investigate how some ruin-related quantities involving the surplus immediately prior to ruin can also be analyzed via our fluid flow methodology.

Keywords: Surplus process; bivariate phase-type distribution; fluid queue; Gerber–Shiu function; deficit at ruin; surplus prior to ruin

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1. Introduction

In ruin theory, considerable attention has been given to various risk models which relax the stringent independence assumptions of the Sparre Andersen model. We recall that, in the framework of the Sparre Andersen model, the claim size and the interclaim time random variables (RVs) are all assumed to be independent. There exists a considerable number of insurable contexts for which such an assumption is severely violated (e.g. catastrophic insurance). An important class of risk models relaxing the stringent independence assumptions is the class of risk models with Markovian arrival processes (see Neuts (1979)), which allows for a complete dependence structure between claim sizes and interclaim times (see, e.g. Ahn *et al.* (2007) and Badescu *et al.* (2005), (2007) for a more detailed discussion).

Recently, another class of generalizations have received special attention, namely the class of risk models for which the increments of the surplus process between claims are independent and identically distributed. We refer to this class of risk models as *Sparre Andersen type*

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risk models. Risk models of this type have been proposed in Albrecher and Teugels (2006), Boudreault *et al.* (2006), and Cossette *et al.* (2008), among others. This class of risk models is of special interest given that some essential properties of the Sparre Andersen risk model are preserved (see Cheung *et al.* (2009) for more details).

In this paper, our goal is to enrich the latter class. The risk model of interest in this paper is described as follows. Let W_i be the ith interclaim time, and let X_i be the size of the ith claim ($i = 1, 2, \dots$). The claim number process $\{N(t),\ t \geq 0\}$ having interclaim times $\{W_i\}_{i=1}^{\infty}$ is defined as $N(t) = \max\{n \geq 0\colon \sum_{i=1}^{n} W_i \leq t\}$. Assuming that the insurer collects premium at a constant rate c per unit time, its surplus process $\underline{R} = \{R(t),\ t \geq 0\}$ follows the dynamics

$$R(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \qquad t \geq 0, \tag{1}$$

for an initial surplus of u. We assume that the random vectors (W_i, X_i) ($i = 1, 2, \dots$) are independent and identically distributed. Contrary to the Sparre Andersen risk model, we allow for the RVs W_i and X_i to be dependent. For reasons that will become apparent later, we model the generic pair $(W, X/c)$ by a bivariate phase-type distribution (see Section 2). As in the univariate case, we point out that the class of bivariate phase-type distributions has been proven to be dense in the set of distributions defined on $\mathbb{R}^+ \times \mathbb{R}^+$ (see Corollary 1 of Assaf *et al.* (1984)). The reader is referred to Assaf *et al.* (1984) for other properties of the bivariate phase-type distribution.

Of special interest for the surplus process $\underline{R}$ is the analysis of the time to ruin τ, defined as $\tau = \inf\{t \geq 0\colon R(t) < 0\}$ with $\tau = \infty$ if ruin does not occur (i.e. $R(t) \geq 0$ for all $t \geq 0$). To ensure that ruin does not occur almost surely, we assume that the positive security loading condition

$$\mathrm{E}[X] < c\,\mathrm{E}[W] \tag{2}$$

is fulfilled. In this paper, our main objective is the analysis of a subclass of Gerber–Shiu functions (see Gerber and Shiu (1998)), namely those for which the penalty function depends only on the deficit at ruin $|R(\tau)|$. To this end, we define

$$m(u) = \mathrm{E}[\mathrm{e}^{-s\tau} w(|R(\tau)|)\,\mathbf{1}(\tau < \infty) \mid R(0) = u], \tag{3}$$

where s ($s \geq 0$) is the force of interest, w is a penalty function, and $\mathbf{1}(A)$ is the indicator function of the event A. This special class of Gerber–Shiu functions has been considered in, for example, Willmot (2007) and Landriault and Willmot (2008). In this paper we propose to analyze $m(u)$ using the connection to a particular fluid flow model described in Section 3.

The paper is organized as follows. A review of the bivariate phase-type distribution can be found in Section 2. In Section 3, the construction of a fluid flow model is presented and its connection to the surplus process $\underline{R}$ is established. The main results of this paper are provided in Section 4. An explicit expression for the Laplace transform of the time to ruin τ is obtained followed by the identification of the distribution of the deficit at ruin. Finally, in Section 5 we examine the analysis of some ruin-related quantities involving the surplus immediately prior to ruin.

2. Bivariate phase-type distributions

We consider the class of bivariate phase-type distributions introduced in Assaf *et al.* (1984) and further analyzed in Kulkarni (1989). For purposes of completeness, this class of distributions is discussed in more detail here. Suppose that $\underline{Z} = \{Z(t),\ t \geq 0\}$ is a (time-homogeneous) continuous-time Markov chain (CTMC) with state space $E = \{1, 2, \dots, m\} \cup \{\Delta\}$, infinitesimal generator $\boldsymbol{A}$, and initial probability vector $\boldsymbol{\alpha}$. Let Γ_1 and Γ_2 be two nonempty subsets of E, and define $\Delta = \Gamma_1 \cap \Gamma_2$ (which can possibly be the empty set). We further assume that the CTMC $\underline{Z}$ is defined such that both subsets Γ_1 and Γ_2 are visited at least once almost surely.

Let Y_1 and Y_2 be the time of the first visit of $\underline{Z}$ to Γ_1 and Γ_2, respectively, i.e.

$$Y_i = \inf\{t \geq 0 \colon Z(t) \in \Gamma_i\}, \qquad i = 1, 2.$$

We assume without loss of generality that $\Delta = \Gamma_1 \cap \Gamma_2$ contains only one state (when Δ is not empty). The case in which $\Delta = \varnothing$ will be discussed later. We partition the state space E into the following four subsets: $E_0 = \Gamma_1^{\mathrm{c}} \cap \Gamma_2^{\mathrm{c}}$, $E_1 = \Gamma_1 \cap \Gamma_2^{\mathrm{c}}$, $E_2 = \Gamma_1^{\mathrm{c}} \cap \Gamma_2$, and $\Delta = \Gamma_1 \cap \Gamma_2$. The cardinality of the subsets E_0, E_1, and E_2 is m_0, m_1, and m_2, respectively (which satisfies $m_0 + m_1 + m_2 = m$). The infinitesimal generator $\boldsymbol{A}$ can be written as

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{T} & -\boldsymbol{T}\boldsymbol{e} \\ \boldsymbol{0} & 0 \end{bmatrix}, \tag{4}$$

where

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{T}_{00} & \boldsymbol{T}_{01} & \boldsymbol{T}_{02} \\ \boldsymbol{T}_{10} & \boldsymbol{T}_{11} & \boldsymbol{T}_{12} \\ \boldsymbol{T}_{20} & \boldsymbol{T}_{21} & \boldsymbol{T}_{22} \end{bmatrix}, \tag{5}$$

and $\boldsymbol{e}$ is a column vector of 1s while $\boldsymbol{0}$ is a row vector of 0s, both of appropriate dimension. Note that

- $\boldsymbol{T}_{ii}$ is an $m_i \times m_i$ matrix with negative diagonal elements and nonnegative off-diagonal elements containing the rate of transition in E_i;
- $\boldsymbol{T}_{ij}$ $(i \neq j)$ is an $m_i \times m_j$ matrix with nonnegative elements containing the rate of exit from any state in E_i to any state in E_j.

Owing to the form of (5), the column vector $-\boldsymbol{T}\boldsymbol{e}$ in (4) can be rewritten as

$$-\boldsymbol{T}\boldsymbol{e} = \begin{bmatrix} \boldsymbol{t}_0 \\ \boldsymbol{t}_1 \\ \boldsymbol{t}_2 \end{bmatrix},$$

where $\boldsymbol{t}_i = -\begin{bmatrix} \boldsymbol{T}_{i0} & \boldsymbol{T}_{i1} & \boldsymbol{T}_{i2} \end{bmatrix} \boldsymbol{e}$ for $i = 0, 1, 2$.

We will call the joint distribution of (Y_1, Y_2) a *bivariate phase-type distribution* with representation $(\boldsymbol{\alpha}, \boldsymbol{A})$. We point out that if $\Delta = \varnothing$ then $\boldsymbol{A} = \boldsymbol{T}$. It is also possible that either set E_1 or set E_2 is empty when Δ is not. This simply implies that some submatrices in representation (5) of $\boldsymbol{T}$ are empty.

The class of bivariate phase-type distributions contains various well-known bivariate distributions as special cases, notably Marshall and Olkin's (1967) bivariate exponential distribution and Freund's (1961) extension of the exponential distribution (see Assaf *et al.* (1984, Example 5.2) and Cai and Li (2007, Example 4.3) for their bivariate phase-type representations, respectively). We remark that while the former distribution allows for only positive correlation between the two RVs, the latter distribution can be used to model both positive and negative dependencies. Interested readers are referred to Assaf *et al.* (1984, Section 5) for other special cases of the bivariate phase-type distribution.

As pointed out in Kulkarni (1989), bivariate phase-type distributions may have singular components along $Y_1 = 0$, $Y_2 = 0$, or $Y_1 = Y_2$. In our context, given that the claim sizes and the interclaim times are assumed to be strictly positive RVs, the possible singular components along $Y_1 = 0$ or $Y_2 = 0$ are removed by assuming that the initial probability vector $\boldsymbol{\alpha}$ is of the form $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_0, \mathbf{0})$, where $\boldsymbol{\alpha}_0$ is a vector with m_0 nonnegative elements satisfying $\boldsymbol{\alpha}_0\boldsymbol{e} = 1$. In addition, it seems unreasonable to the authors to assume that the interclaim time and the claim size are equal with a positive probability. Such a possibility can be discarded by assuming that $\boldsymbol{t}_0$ is a column vector of 0s. However, in most cases, no significant simplification occurs when $\boldsymbol{t}_0$ is a null column vector. Thus, unless otherwise stated, we will treat the most general case in this paper.

3. Connection with a particular fluid queue

To analyze the surplus process $\underline{R}$, we capitalize on the connection between surplus processes and fluid queues (see, e.g. Ramaswami (2006), Ahn *et al.* (2007), and Badescu *et al.* (2005), (2007)). To construct the fluid flow process $\underline{F}$ equivalent to the surplus process $\underline{R}$, we make the following assumptions:

- a period of increase in the surplus process $\underline{R}$ (i.e. collect premiums at a rate c) corresponds to a period of fluid increase (at a rate c) in the fluid queue $\underline{F}$;
- the sudden drop caused by a claim of size x in the surplus process $\underline{R}$ is replaced by a period of decrease (at a rate c) over a period of length x/c in the fluid queue $\underline{F}$ (see Figure 1).

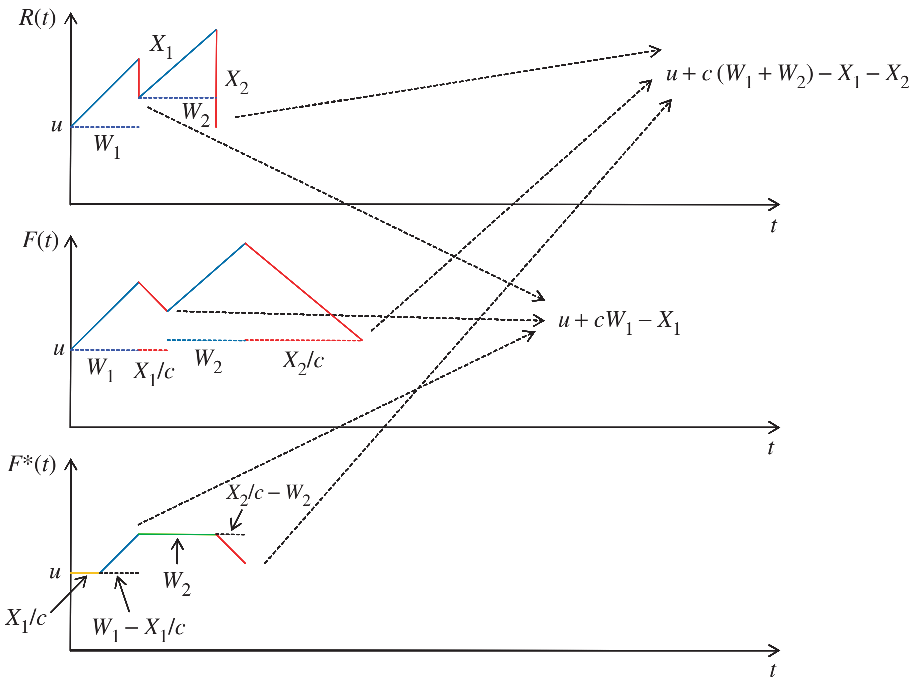

FIGURE 1: Translation of $R(t)$ into $F(t)$ and $F^*(t)$.

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From the above construction, the fluid process $\underline{F}$ has a rate of increase/decrease of c and alternates between increasing and decreasing fluid periods. The length of an increasing fluid period and its subsequent decreasing fluid period is distributed as the generic pair $(W, X/c)$. Also, all the increasing–decreasing cycles of $\underline{F}$ are mutually independent (given that the pairs (W_i, X_i) are assumed to be independent in the risk process $\underline{R}$). However, the fact that the length of an increasing fluid period and its subsequent decreasing fluid period are generally dependent compromises the use of the well-known matrix-analytic methods in fluid theory to analyze the fluid process $\underline{F}$.

In what follows, we define a *second* fluid flow process $\underline{F}^* = \{F^*(t),\ t \geq 0\}$ with three possible fluid patterns: increasing, decreasing, and constant. Defining the fluid process $\underline{F}^*$ together with its underlying CTMC $\underline{J}^* = \{J^*(t),\ t \geq 0\}$ such that the bivariate process $(\underline{F}^*, \underline{J}^*)$ is Markovian, represents the key step to analyze the risk process $\underline{R}$ via the existing matrix-analytic methods. To draw the connections between the surplus process $\underline{R}$ and the fluid process $\underline{F}^*$, we assume that the pairs $(W_i, X_i/c)$ $(i = 1, 2, \dots)$ are distributed as the pair (Y_1, Y_2) defined in Section 2. Note that every pair of $(W_i, X_i/c)$ is generated by a different sample path of the CTMC $\underline{Z}$. To avoid confusion, we will refer to the sample path of $\underline{Z}$ generating the pair $(W_i, X_i/c)$ as the *ith sample path of $\underline{Z}$*. The construction of the fluid process $\underline{F}^*$ with $F^*(0) = u$ is done as follows:

1. the fluid process $\underline{F}^*$ remains constant as long as the first sample path of $\underline{Z}$ has not yet visited any states in $\Gamma_1 \cup \Gamma_2$;
2. the fluid process $\underline{F}^*$ starts to decrease or increase at a rate c, when the first sample path of $\underline{Z}$ has visited at least one state in Γ_1 or, respectively, Γ_2, but not any states in Γ_2 or, respectively, Γ_1 (given that $W_1 < X_1/c$ or, respectively, $W > X_1/c$);
3. at the time that the first sample path of $\underline{Z}$ has visited both Γ_1 and Γ_2, the fluid process $\underline{F}^*$ stops its increasing or decreasing pattern;
4. from this new established fluid level, we repeat steps 1–3 by successively replacing the first sample path of $\underline{Z}$ by the 2nd, 3rd, ... sample path of $\underline{Z}$.

From the construction of the fluid process $\underline{F}^*$, it is immediate that the level of the surplus process $\underline{R}$ immediately after the payment of the ith claim corresponds to the fluid level of the process $\underline{F}^*$ at the end of the ith sample path of $\underline{Z}$ $(i = 1, 2, \dots)$. This is depicted in Figure 1. Thus, the ruin probability for the surplus process $\underline{R}$ coincides with the probability that the fluid process $\underline{F}^*$ eventually hits level 0 at least once. With regards to the time to ruin, it is not true that the first passage time to level 0 of $\underline{F}^*$ corresponds to the time to ruin τ in the surplus process $\underline{R}$. Indeed, when the fluid process $\underline{F}^*$ increases or remains constant, the surplus process $\underline{R}$ increases at a rate c, whereas time does not evolve in the surplus process $\underline{R}$ when the fluid level $\underline{F}^*$ decreases. Hence, the time to ruin τ in the surplus process $\underline{R}$ is equivalent to the total amount of time the process $\underline{F}^*$ takes to reach level 0, extracting periods of time for which the fluid decreases over that first passage time.

Let the time-homogeneous CTMC $\underline{J}^*$ have finite state space $S = S_0 \cup S_1 \cup S_2$ and infinitesimal generator

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{Q}_{00} & \boldsymbol{Q}_{01} & \boldsymbol{Q}_{02} \\ \boldsymbol{Q}_{10} & \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \boldsymbol{Q}_{20} & \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22} \end{bmatrix}, \tag{6}$$

where the submatrix $\boldsymbol{Q}_{ij}$ is an $|S_i| \times |S_j|$ matrix containing the (r, s)th elements of the infinitesimal generator $\boldsymbol{Q}$ for all $r \in S_i$ and all $s \in S_j$. The partition of the state space S into S_0, S_1, and S_2 is as follows:

- during sojourns of $\underline{J}^*$ in S_1 or S_2, the fluid process $\underline{F}^*$ increases or, respectively, decreases at a rate c;
- during sojourns of $\underline{J}^*$ in S_0, the fluid process $\underline{F}^*$ remains constant.

Note that, under our construction of the fluid process $\underline{F}^*$, S_1 or S_2 cannot be reached from S_2 or, respectively, S_1 without passing through S_0. Now it remains to formally define the $\boldsymbol{Q}_{ij}$ matrices in terms of the $\boldsymbol{T}_{ij}$ matrices. We first consider the most general case where the set $\Delta = \Gamma_1 \cap \Gamma_2$ is not empty. Cases for which Δ is empty can be obtained in an identical way. At time 0, we consider the first sample path of $\underline{Z}$ starting in some states in $E_0 = \Gamma_1^c \cap \Gamma_2^c$. The fluid level remains constant as long as $\underline{Z}$ is in E_0. Since the transition rates of $\underline{Z}$ within E_0 are governed by $\boldsymbol{T}_{00}$, it is clear that the cardinality of the set S_0, denoted by $|S_0|$, is $|S_0| = m_0$ and

$$\boldsymbol{Q}_{00} = \boldsymbol{T}_{00} + \boldsymbol{t}_0\boldsymbol{\alpha}_0. \tag{7}$$

When the process $\underline{Z}$ first leaves E_0, it enters either $E_1 = \Gamma_1 \cap \Gamma_2^c$ (governed by rate matrix $\boldsymbol{T}_{01}$) or $E_2 = \Gamma_1^c \cap \Gamma_2$ (governed by rate matrix $\boldsymbol{T}_{02}$) for the first time. Suppose that it enters E_1. From the time of entrance, as long as $\underline{Z}$ does not enter Γ_2 (or, equivalently, as long as $\underline{Z}$ moves only within $E_1 \cup E_0$), the fluid level decreases at a rate c. It follows that $|S_2| = |\Gamma_2^c| = m_1 + m_0$. We also recall that the transitions within $E_1 \cup E_0$ are governed by the $\boldsymbol{T}_{00}$, $\boldsymbol{T}_{01}$, $\boldsymbol{T}_{10}$, and $\boldsymbol{T}_{11}$ matrices. Combining the above observations, it is immediate that $\boldsymbol{Q}_{02}$ is an $m_0 \times (m_1 + m_0)$ matrix with representation

$$\boldsymbol{Q}_{02} = \begin{bmatrix} \boldsymbol{T}_{01} & \boldsymbol{0} \end{bmatrix}, \tag{8}$$

while $\boldsymbol{Q}_{22}$ is the following square matrix (of dimension $(m_1 + m_0)$):

$$\boldsymbol{Q}_{22} = \begin{bmatrix} \boldsymbol{T}_{11} & \boldsymbol{T}_{10} \\ \boldsymbol{T}_{01} & \boldsymbol{T}_{00} \end{bmatrix}. \tag{9}$$

When the fluid level decreases at a rate c and the process $\underline{Z}$ is in some states in E_1, the process $\underline{J}^*$ would enter S_0 upon transition of $\underline{Z}$ into Γ_2, which could happen through (i) transition into E_2 (governed by $\boldsymbol{T}_{12}$) or (ii) transition into Δ (governed by $\boldsymbol{t}_1$). Once a transition into Γ_2 occurs, both RVs from the bivariate phase-type distribution have been generated and we immediately move on to the next sample path of $\underline{Z}$ via the initial probability vector $\boldsymbol{\alpha}$. Similarly, when the fluid level is decreasing at rate c, but the process $\underline{Z}$ is in some states in E_0, the process $\underline{J}^*$ would enter S_0 upon transition of $\underline{Z}$ into Γ_2 and this could happen through (i) transition into E_2 (governed by $\boldsymbol{T}_{02}$) or (ii) transition into Δ (governed by $\boldsymbol{t}_0$). From the above description, the matrix $\boldsymbol{Q}_{20}$ is of size $(m_1 + m_0) \times m_0$ and defined as

$$\boldsymbol{Q}_{20} = \begin{bmatrix} (\boldsymbol{T}_{12}\boldsymbol{e} + \boldsymbol{t}_1)\boldsymbol{\alpha}_0 \\ (\boldsymbol{T}_{02}\boldsymbol{e} + \boldsymbol{t}_0)\boldsymbol{\alpha}_0 \end{bmatrix}. \tag{10}$$

Furthermore, the process $\underline{J}^*$ cannot transit from any state in S_2 to a state in S_1 without going through S_0 and, therefore,

$$\boldsymbol{Q}_{21} = \boldsymbol{0}. \tag{11}$$

If the first transition out of E_0 is to any state in E_2, the use of identical arguments yields

$$\boldsymbol{Q}_{01} = \begin{bmatrix} \boldsymbol{T}_{02} & \boldsymbol{0} \end{bmatrix}, \tag{12}$$

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$$\boldsymbol{Q}_{11} = \begin{bmatrix} \boldsymbol{T}_{22} & \boldsymbol{T}_{20} \\ \boldsymbol{T}_{02} & \boldsymbol{T}_{00} \end{bmatrix}, \tag{13}$$

$$\boldsymbol{Q}_{10} = \begin{bmatrix} (\boldsymbol{T}_{21}\boldsymbol{e} + \boldsymbol{t}_2)\boldsymbol{\alpha}_0 \\ (\boldsymbol{T}_{01}\boldsymbol{e} + \boldsymbol{t}_0)\boldsymbol{\alpha}_0 \end{bmatrix}, \tag{14}$$

and

$$\boldsymbol{Q}_{12} = \boldsymbol{0}. \tag{15}$$

The characterization of the generator $\boldsymbol{Q}$ in (6) is now complete.

For the case in which $\Delta = \varnothing$, relations (7)–(15) remain valid by simply letting $\boldsymbol{t}_0 = \boldsymbol{0}$, $\boldsymbol{t}_1 = \boldsymbol{0}$, and $\boldsymbol{t}_2 = \boldsymbol{0}$ in (7), (10), and (14). Also, the case where some of the matrices $\boldsymbol{T}_{10}$, $\boldsymbol{T}_{12}$, $\boldsymbol{T}_{20}$, and $\boldsymbol{T}_{21}$ are 0s can be obtained in an analogous way, reducing the dimensions of some $\boldsymbol{Q}$ matrices in the process.

4. Analysis of a class of Gerber–Shiu functions

In this section we consider the analysis of the special Gerber–Shiu function defined in (3). This analysis is conducted via the connection of the surplus process $\underline{R}$ to the fluid flow process $\underline{F}^*$. We recall here some crucial observations made in Section 3:

- the surplus level immediately after the payment of the ith claim corresponds to the fluid level of the process $\underline{F}^*$ at the end of the ith sample path of $\underline{Z}$ $(i = 1, 2, \dots)$;
- with respect to time, time evolves in the surplus process $\underline{R}$ only when the CTMC $\underline{J}^*$ is either in the set of phases S_0 or S_1 (in contrast with the fluid process $\underline{F}^*$, where time evolves independently of the state of the CTMC $\underline{J}^*$).

Clearly, the analysis of the Gerber–Shiu function $m(u)$ in the risk process $\underline{R}$ requires a freeze in the clock time whenever the CTMC $\underline{J}^*$ is in the set of phases S_2. Then, consider the evolution of the fluid process $\underline{F}^*$ (with initial fluid level $F^*(0) = u$) and its underlying CTMC $\underline{J}^*$. For the time being, we assume that the fluid is in an ascending phase at time 0 (i.e. $J^*(0) \in S_1$). Thus, for the fluid level to become empty, the skip-free property of the fluid process $\underline{F}^*$ implies that the process must revisit level u at least once and ultimately make a transition from level u to level 0. We denote by θ the time it takes for the fluid process to return to its initial level u for the first time and by ξ the remaining time until the fluid level becomes empty for the first time.

In queueing theory, the (random) time θ is known as the *length of a busy period* for the fluid process $\underline{F}^*$. Given that θ is independent of the initial level u, let θ be defined as

$$\theta = \inf\{t \geq 0 \colon F^*(t) < 0\}$$

when $F^*(0) = 0$ with $\theta = \infty$ if $F^*(t) \geq 0$ for all $t \geq 0$. Also, let $\theta_i = \int_0^\theta \mathbf{1}(J^*(s) \in S_i)\,\mathrm{d}s$ $(i = 0, 1, 2)$ be the time spent in the set of phases S_i during the (infinite buffer) busy period $(0, \theta)$ (with $\theta_i = \infty$ if $\theta = \infty$). Given that time does not evolve in the surplus process $\underline{R}$ whenever the CTMC $\underline{J}^*$ is in the set of phases S_2, we are interested in the distribution of the time $\theta_0 + \theta_1$ in the busy period $(0, \theta)$ for the fluid process $\underline{F}^*$. Thus, let $\boldsymbol{\Gamma}(x)$ be an $|S_1| \times |S_2|$ matrix whose (k, l)th element is defined as

$$[\boldsymbol{\Gamma}(x)]_{kl} = \Pr(\theta < \infty,\ \theta_0 + \theta_1 \leq x,\ J^*(\theta) = l \mid J^*(0) = k)$$

for $k \in S_1$ and $l \in S_2$. The Laplace–Stieltjes transform (LST) of the time $\theta_0 + \theta_1$ is given by $\tilde{\boldsymbol{\Gamma}}(s) = \int_0^\infty \mathrm{e}^{-sx}\,\mathrm{d}\boldsymbol{\Gamma}(x)$. As we will see later, the determination of the LST $\tilde{\boldsymbol{\Gamma}}(s)$ plays a crucial role in this paper, being to some extent similar to the one defined for a certain fluid process in Badescu *et al.* (2005, Section 4).

Along the same lines of logic for the passage time ξ, we let $\xi_i = \int_0^{\xi} \mathbf{1}(J^*(\theta+s) \in S_i)\,\mathrm{d}s$ $(i = 0, 1, 2)$ be the time spent in the set of phases S_i during the interval $(\theta, \theta+\xi)$ with $\xi_i = \infty$ if $\xi = \infty$. Extracting the time intervals in which the fluid flow decreases over this passage time, we are interested in the identification of the distribution of $\xi_0 + \xi_1$. We define an $|S_2| \times |S_2|$ matrix $\boldsymbol{\Phi}_u(x)$ whose (k, l)th element is

$$[\boldsymbol{\Phi}_u(x)]_{kl} = \Pr(\xi < \infty,\ \xi_0 + \xi_1 \le x,\ J^*(\theta+\xi) = l \mid F^*(\theta) = u,\ J^*(\theta) = k),$$

where $k, l \in S_2$. The LST of the passage time $\xi_0 + \xi_1$ is given by $\tilde{\boldsymbol{\Phi}}_u(s) = \int_0^{\infty} \mathrm{e}^{-sx}\,\mathrm{d}\boldsymbol{\Phi}_u(x)$. We remark that the distribution of $\xi_0 + \xi_1$ has a mass point at 0 given that the CTMC $\underline{J}^*$ may remain in the S_2 set of phases during the interval $(\theta, \theta+\xi)$.

Conditioning on the length of the first ascending and subsequent constant period and on whether the fluid flow $\underline{F}^*$ returns to an ascending phase or visits for the first time a descending phase after the end of this constant period, we obtain

$$\begin{aligned}\boldsymbol{\Gamma}(x) = \int_0^x & \exp\{\boldsymbol{Q}_{11} y\}\boldsymbol{Q}_{10} \\ & \times \int_0^{x-y} \exp\{\boldsymbol{Q}_{00} z\}[\boldsymbol{Q}_{01}(\boldsymbol{\Gamma} * \boldsymbol{\Phi}_{cy})(x-y-z) + \boldsymbol{Q}_{02}\boldsymbol{\Phi}_{cy}(x-y-z)]\,\mathrm{d}z\,\mathrm{d}y, \end{aligned} \tag{16}$$

where $(\boldsymbol{\Gamma} * \boldsymbol{\Phi}_y)(x) = \int_0^x \boldsymbol{\Phi}_y(x-z)\,\mathrm{d}\boldsymbol{\Gamma}(z)$. Using (16) followed by some simple manipulations, the LST $\tilde{\boldsymbol{\Gamma}}(s)$ satisfies

$$\begin{aligned}\tilde{\boldsymbol{\Gamma}}(s) &= \int_0^{\infty} \mathrm{e}^{-sx} \int_0^x \exp\{\boldsymbol{Q}_{11} y\}\boldsymbol{Q}_{10} \\ &\quad \times \int_0^{x-y} \exp\{\boldsymbol{Q}_{00} z\}\frac{\partial}{\partial x}[\boldsymbol{Q}_{01}(\boldsymbol{\Gamma} * \boldsymbol{\Phi}_{cy})(x-y-z) + \boldsymbol{Q}_{02}\boldsymbol{\Phi}_{cy}(x-y-z)]\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\ &\quad + \int_0^{\infty} \mathrm{e}^{-sx} \int_0^x \exp\{\boldsymbol{Q}_{11} y\}\boldsymbol{Q}_{10}\exp\{\boldsymbol{Q}_{00}(x-y)\}\boldsymbol{Q}_{02}\boldsymbol{\Phi}_{cy}(0)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^{\infty} \exp\{\boldsymbol{Q}_{11} y\}\boldsymbol{Q}_{10} \int_y^{\infty} \mathrm{e}^{-sx} \\ &\quad \times \int_0^{x-y} \exp\{\boldsymbol{Q}_{00} z\}\frac{\partial}{\partial x}[\boldsymbol{Q}_{01}(\boldsymbol{\Gamma} * \boldsymbol{\Phi}_{cy})(x-y-z) + \boldsymbol{Q}_{02}\boldsymbol{\Phi}_{cy}(x-y-z)]\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_0^{\infty} \exp\{\boldsymbol{Q}_{11} y\}\boldsymbol{Q}_{10} \int_y^{\infty} \mathrm{e}^{-sx}\exp\{\boldsymbol{Q}_{00}(x-y)\}\boldsymbol{Q}_{02}\boldsymbol{\Phi}_{cy}(0)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^{\infty} \exp\{(\boldsymbol{Q}_{11} - s\boldsymbol{I})y\}\boldsymbol{Q}_{10} \int_0^{\infty} \mathrm{e}^{-sx} \\ &\quad \times \int_0^{x} \exp\{\boldsymbol{Q}_{00} z\}\frac{\partial}{\partial x}[\boldsymbol{Q}_{01}(\boldsymbol{\Gamma} * \boldsymbol{\Phi}_{cy})(x-z) + \boldsymbol{Q}_{02}\boldsymbol{\Phi}_{cy}(x-z)]\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_0^{\infty} \exp\{(\boldsymbol{Q}_{11} - s\boldsymbol{I})y\}\boldsymbol{Q}_{10}\bigg(\int_y^{\infty} \exp\{(\boldsymbol{Q}_{00} - s\boldsymbol{I})(x-y)\}\,\mathrm{d}x\bigg)\boldsymbol{Q}_{02}\boldsymbol{\Phi}_{cy}(0)\,\mathrm{d}y \\ &= \int_0^{\infty} \exp\{(\boldsymbol{Q}_{11} - s\boldsymbol{I})y\}\boldsymbol{Q}_{10}(s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}[\boldsymbol{Q}_{01}\tilde{\boldsymbol{\Gamma}}(s) + \boldsymbol{Q}_{02}]\tilde{\boldsymbol{\Phi}}_{cy}(s)\,\mathrm{d}y. \end{aligned}$$

It remains to identify a relation satisfied by the LST of the remaining time until ruin, namely $\tilde{\boldsymbol{\Phi}}_u(s)$. Now at level u in a decreasing phase, the fluid process $\underline{F}^*$ can become empty with or without a visit of the CTMC $\underline{J}^*$ in S_0 in the interim. Conditioning on the length of the descending period (from level u) and its subsequent constant period (if necessary), we obtain

$$\begin{aligned}\mathbf{\Phi}_u(x) &= \exp\left\{\frac{\boldsymbol{Q}_{22}}{c}u\right\} \\ &\quad + \int_0^{u/c} \exp\{\boldsymbol{Q}_{22}y\}\boldsymbol{Q}_{20} \\ &\qquad \times \int_0^x \exp\{\boldsymbol{Q}_{00}z\}[\boldsymbol{Q}_{01}(\mathbf{\Gamma} * \mathbf{\Phi}_{u-cy})(x-z) + \boldsymbol{Q}_{02}\mathbf{\Phi}_{u-cy}(x-z)]\,\mathrm{d}z\,\mathrm{d}y \\ &= \exp\left\{\frac{\boldsymbol{Q}_{22}}{c}u\right\} \\ &\quad + \frac{1}{c}\int_0^{u} \exp\left\{\frac{\boldsymbol{Q}_{22}}{c}y\right\}\boldsymbol{Q}_{20} \\ &\qquad \times \int_0^x \exp\{\boldsymbol{Q}_{00}z\}[\boldsymbol{Q}_{01}(\mathbf{\Gamma} * \mathbf{\Phi}_{u-y})(x-z) + \boldsymbol{Q}_{02}\mathbf{\Phi}_{u-y}(x-z)]\,\mathrm{d}z\,\mathrm{d}y.\end{aligned} \tag{17}$$

Taking the LST on both sides of (17), we find that

$$\begin{aligned}\tilde{\mathbf{\Phi}}_u(s) &= \exp\left\{\frac{\boldsymbol{Q}_{22}}{c}u\right\} \\ &\quad + \frac{1}{c}\int_0^{u} \exp\left\{\frac{\boldsymbol{Q}_{22}}{c}y\right\}\boldsymbol{Q}_{20}\int_0^\infty \mathrm{e}^{-sx} \\ &\qquad \times \int_0^x \exp\{\boldsymbol{Q}_{00}z\}\frac{\partial}{\partial x}[\boldsymbol{Q}_{01}(\mathbf{\Gamma} * \mathbf{\Phi}_{u-y})(x-z) + \boldsymbol{Q}_{02}\mathbf{\Phi}_{u-y}(x-z)]\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \frac{1}{c}\int_0^{u} \exp\left\{\frac{\boldsymbol{Q}_{22}}{c}y\right\}\boldsymbol{Q}_{20}\left(\int_0^\infty \exp\{(\boldsymbol{Q}_{00} - s\boldsymbol{I})x\}\,\mathrm{d}x\right)\boldsymbol{Q}_{02}\mathbf{\Phi}_{u-y}(0)\,\mathrm{d}y \\ &= \exp\left\{\frac{\boldsymbol{Q}_{22}}{c}u\right\} \\ &\quad + \frac{1}{c}\int_0^{u} \exp\left\{\frac{\boldsymbol{Q}_{22}}{c}y\right\}\boldsymbol{Q}_{20}(s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}[\boldsymbol{Q}_{01}\tilde{\mathbf{\Gamma}}(s) + \boldsymbol{Q}_{02}]\tilde{\mathbf{\Phi}}_{u-y}(s)\,\mathrm{d}y.\end{aligned} \tag{18}$$

Defining the Laplace transform

$$\tilde{\mathbf{\Phi}}(r,s) = \int_0^\infty \mathrm{e}^{-ru}\tilde{\mathbf{\Phi}}_u(s)\,\mathrm{d}u,$$

the use of (18) leads to

$$\tilde{\mathbf{\Phi}}(r,s) = \left(r\boldsymbol{I} - \frac{\boldsymbol{Q}_{22}}{c}\right)^{-1}\left(\boldsymbol{I} + \frac{\boldsymbol{Q}_{20}}{c}(s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}[\boldsymbol{Q}_{01}\tilde{\mathbf{\Gamma}}(s) + \boldsymbol{Q}_{02}]\tilde{\mathbf{\Phi}}(r,s)\right). \tag{19}$$

Simple manipulations of (19) yield

$$\begin{aligned}\tilde{\mathbf{\Phi}}(r,s) &= \left(\boldsymbol{I} - \left(r\boldsymbol{I} - \frac{\boldsymbol{Q}_{22}}{c}\right)^{-1}\frac{\boldsymbol{Q}_{20}}{c}(s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}[\boldsymbol{Q}_{01}\tilde{\mathbf{\Gamma}}(s) + \boldsymbol{Q}_{02}]\right)^{-1}\left(r\boldsymbol{I} - \frac{\boldsymbol{Q}_{22}}{c}\right)^{-1} \\ &= \left(r\boldsymbol{I} - \frac{\boldsymbol{Q}_{22}}{c} - \frac{\boldsymbol{Q}_{20}}{c}(s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}[\boldsymbol{Q}_{01}\tilde{\mathbf{\Gamma}}(s) + \boldsymbol{Q}_{02}]\right)^{-1}.\end{aligned} \tag{20}$$

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We remark that (20) is valid due to the fact that the spectral radius of the matrix

$$\frac{1}{c}\left(r\boldsymbol{I} - \frac{1}{c}\boldsymbol{Q}_{22}\right)^{-1}\boldsymbol{Q}_{20}(s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}[\boldsymbol{Q}_{01}\tilde{\boldsymbol{\Gamma}}(s) + \boldsymbol{Q}_{02}]$$

is strictly less than 1. To prove this, we use the fact that the matrix $\tilde{\boldsymbol{\Gamma}}(0)$ is strictly substochastic under the positive security loading condition (2). It is then easy to see that

$$\begin{aligned}
\boldsymbol{0} &\le \frac{1}{c}\left(r\boldsymbol{I} - \frac{1}{c}\boldsymbol{Q}_{22}\right)^{-1}\boldsymbol{Q}_{20}(s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}[\boldsymbol{Q}_{01}\tilde{\boldsymbol{\Gamma}}(s) + \boldsymbol{Q}_{02}]\boldsymbol{e} \\
&\le \frac{1}{c}\left(r\boldsymbol{I} - \frac{1}{c}\boldsymbol{Q}_{22}\right)^{-1}\boldsymbol{Q}_{20}(-\boldsymbol{Q}_{00})^{-1}[\boldsymbol{Q}_{01}\tilde{\boldsymbol{\Gamma}}(0) + \boldsymbol{Q}_{02}]\boldsymbol{e} \\
&< (-\boldsymbol{Q}_{22})^{-1}\boldsymbol{Q}_{20}(-\boldsymbol{Q}_{00})^{-1}[\boldsymbol{Q}_{01}\boldsymbol{e} + \boldsymbol{Q}_{02}\boldsymbol{e}] \\
&= (-\boldsymbol{Q}_{22})^{-1}\boldsymbol{Q}_{20}(-\boldsymbol{Q}_{00})^{-1}(-\boldsymbol{Q}_{00})\boldsymbol{e} \\
&= \boldsymbol{e}.
\end{aligned}$$

Thus, the use of Gershgorin's theorem (see Faddeev and Faddeeva (1963, Section 1.13)) proves the existence of the desired inverse.

Inverting (20) with respect to r leads to

$$\tilde{\boldsymbol{\Phi}}_u(s) = \exp\left\{\frac{1}{c}(\boldsymbol{Q}_{22} + \boldsymbol{Q}_{20}(s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}[\boldsymbol{Q}_{01}\tilde{\boldsymbol{\Gamma}}(s) + \boldsymbol{Q}_{02}])u\right\}.$$

In order to simplify the notation, we introduce the matrix $\boldsymbol{Q}^*(s)$ defined by

$$\boldsymbol{Q}^*(s) := \begin{bmatrix} \boldsymbol{Q}^*_{11}(s) & \boldsymbol{Q}^*_{12}(s) \\ \boldsymbol{Q}^*_{21}(s) & \boldsymbol{Q}^*_{22}(s) \end{bmatrix} = \begin{bmatrix} \boldsymbol{Q}_{10}(s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}\boldsymbol{Q}_{01} & \boldsymbol{Q}_{10}(s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}\boldsymbol{Q}_{02} \\ \boldsymbol{Q}_{20}(s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}\boldsymbol{Q}_{01} & \boldsymbol{Q}_{20}(s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}\boldsymbol{Q}_{02} \end{bmatrix}.$$

It is easy to observe that the four block matrices defining $\boldsymbol{Q}^*(s)$ give the Laplace transform of the time spent by the underlying CTMC $\underline{J}^*$ in S_0, given the states prior to entering and after leaving S_0. We are now ready to give the first result of this section.

Theorem 1. *The Laplace transform of the time to ruin in the bivariate phase-type risk model defined in (1) is given by*

$$\tilde{\Psi}_u(s) := \mathrm{E}[\mathrm{e}^{-s\tau}\,\boldsymbol{1}(\tau < \infty) \mid R(0) = u] = \boldsymbol{\alpha}_0(s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}[\boldsymbol{Q}_{01}\tilde{\boldsymbol{\Gamma}}(s) + \boldsymbol{Q}_{02}]\tilde{\boldsymbol{\Phi}}_u(s)\boldsymbol{e}, \quad (21)$$

where

$$\tilde{\boldsymbol{\Gamma}}(s) = \frac{1}{c}\int_0^\infty \exp\left\{\frac{1}{c}(\boldsymbol{Q}_{11} - s\boldsymbol{I})y\right\}[\boldsymbol{Q}^*_{11}(s)\tilde{\boldsymbol{\Gamma}}(s) + \boldsymbol{Q}^*_{12}(s)]\tilde{\boldsymbol{\Phi}}_y(s)\,\mathrm{d}y \quad (22)$$

and

$$\tilde{\boldsymbol{\Phi}}_u(s) = \exp\left\{\frac{1}{c}(\boldsymbol{Q}_{22} + \boldsymbol{Q}^*_{21}(s)\tilde{\boldsymbol{\Gamma}}(s) + \boldsymbol{Q}^*_{22}(s))u\right\}. \quad (23)$$

Proof. Starting at time 0 with the initial probability vector $\boldsymbol{\alpha}_0$, the CTMC $\underline{J}^*$ sojourns in S_0 for a period of time whose Laplace transform is given by $(s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}$. At this instant, the CTMC $\underline{J}^*$ switches to either a phase in S_1 (with matrix exit rate $\boldsymbol{Q}_{01}$) or to a phase in S_2 (with matrix exit rate $\boldsymbol{Q}_{02}$). If the transition is made to S_1, the equivalent fluid flow $\underline{F}^*$ has to return to the same level u followed by a first passage from level u to 0 for ruin to occur. On the other hand, if the transition is made to S_2, the fluid flow has to make a first passage from u to 0. Combining these two cases and adding up all the possible phases (in S_2) of $\underline{J}^*$ when the fluid becomes empty for the first time results in (21).

It is easy to observe that $\tilde{\mathbf{\Gamma}}(s)$ plays a central role in the evaluation of the Laplace transform of the time to ruin τ. Substituting (23) into (22) followed by an integration by parts, we further obtain

$$[\boldsymbol{Q}_{11} - s\boldsymbol{I} + \boldsymbol{Q}^*_{11}(s)]\tilde{\mathbf{\Gamma}}(s) + \tilde{\mathbf{\Gamma}}(s)[\boldsymbol{Q}_{22} + \boldsymbol{Q}^*_{22}(s)] + \tilde{\mathbf{\Gamma}}(s)\boldsymbol{Q}^*_{21}(s)\tilde{\mathbf{\Gamma}}(s) + \boldsymbol{Q}^*_{12}(s) = 0. \quad (24)$$

Equation (24) for the LST $\tilde{\mathbf{\Gamma}}(s)$ is known as a *Riccati equation* (see, e.g. Abou-Kandil *et al.* (2003, Chapter 2)). Several numerical algorithms have been proposed in the literature to obtain solutions of a Riccati equation. We refer the interested reader to, e.g. Badescu *et al.* (2005), Bean *et al.* (2005), and Guo (2001). Furthermore, once $\tilde{\mathbf{\Gamma}}(s)$ is numerically determined, a numerical inversion of the LST $\tilde{\mathbf{\Gamma}}(s)$ leads to the determination of the finite-time ruin probability.

We point out that an alternative proof to the derivation of the Riccati equation for $\tilde{\mathbf{\Gamma}}(s)$ is provided in Appendix A. The advantage of the present approach over the more analytic-based one in Appendix A is that calculation of certain first passage probability matrices, whose evaluation is of special interest independently of the purpose of this paper, is obtained as a by-product.

Several natural consequences of the results obtained in Theorem 1 are stated as corollaries at the end of this section. All these results are obtained as a direct consequence of the Markov property exhibited by the bivariate fluid process (F^*, J^*).

Corollary 1. *The infinite-time ruin probability in the bivariate phase-type risk model $\underline{R}$ is given by*

$$\Pr(\tau < \infty \mid R(0) = u) = \boldsymbol{\alpha}_0(-\boldsymbol{Q}_{00})^{-1}[\boldsymbol{Q}_{01}\tilde{\mathbf{\Gamma}}(0) + \boldsymbol{Q}_{02}]\tilde{\mathbf{\Phi}}_u(0)\boldsymbol{e}. \quad (25)$$

Proof. The proof is immediate from (21) with $s = 0$.

We remark that (25) can also be retrieved from Theorem 3 of Ahn and Ramaswami (2005) with $s = 0$ by recalling that the ruin probability for the surplus process $\underline{R}$ and the probability that the fluid process $\underline{F}^*$ eventually hits level 0 at least once coincide.

Corollary 2. *The deficit at ruin in the bivariate phase-type risk model $\underline{R}$ is phase-type (PH) distributed with parameters* $\mathrm{PH}(\boldsymbol{\alpha}_0(-\boldsymbol{Q}_{00})^{-1}[\boldsymbol{Q}_{01}\tilde{\mathbf{\Gamma}}(0) + \boldsymbol{Q}_{02}]\tilde{\mathbf{\Phi}}_u(0), \boldsymbol{Q}_{22}/c)$.

Proof. Starting at level u, the fluid process must first reach level 0. The distribution of the phase in S_2 of the CTMC at that time is given by $\boldsymbol{\alpha}_0(-\boldsymbol{Q}_{00})^{-1}[\boldsymbol{Q}_{01}\tilde{\mathbf{\Gamma}}(0) + \boldsymbol{Q}_{02}]\tilde{\mathbf{\Phi}}_u(0)$. Finally, $\boldsymbol{Q}_{22}$ corresponds to the intensity matrix of a descending period. The correction factor of c arises because we are interested in the intensity matrix with respect to the level instead of time.

Corollary 3. *The Gerber–Shiu function $m(u)$ in the bivariate phase-type risk model $\underline{R}$ is given by*

$$m(u) = \boldsymbol{\alpha}_0(s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}[\boldsymbol{Q}_{01}\tilde{\mathbf{\Gamma}}(s) + \boldsymbol{Q}_{02}]\tilde{\mathbf{\Phi}}_u(s)\int_0^\infty \exp\left\{\frac{\boldsymbol{Q}_{22}}{c}y\right\}\left(\frac{\boldsymbol{q}_2}{c}\right)w(y)\,\mathrm{d}y,$$

where $\boldsymbol{q}_2 = -\boldsymbol{Q}_{22}\boldsymbol{e}$.

Proof. The proof follows directly from Theorem 1 and Corollary 2.

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5. The surplus prior to ruin

In this section we investigate how some ruin-related quantities involving the surplus immediately prior to ruin can be analyzed via the fluid flow approach discussed above. More precisely, we are interested in the identification of the so-called discounted density of the surplus prior to ruin $R(\tau^-)$ and the deficit at ruin $|R(\tau)|$ in the risk process $\underline{R}$ via its connection to the fluid process $\underline{F}^*$. We remind the reader that the discounted density of the surplus prior to ruin and the deficit at ruin has been studied by many authors in ruin theory (see, e.g. Gerber and Shiu (1997), Li and Garrido (2005), and Ren (2007)). In this section we further make the practical assumption that ties between the RVs W and X/c are not possible, i.e. $\boldsymbol{t}_0$ is a column vector of 0s.

In Section 3, the time to ruin τ and the deficit at ruin $|R(\tau)|$ in the surplus process $\underline{R}$ were connected to some particular quantities in the fluid process $\underline{F}^*$. However, a similar connection for the surplus prior to ruin $R(\tau^-)$ turns out to be more challenging to establish. Indeed, from the construction of the process $\underline{F}^*$, the initial upward segment of $\underline{R}$ (following a claim) is translated in the fluid process $\underline{F}^*$ to either a level segment (if $W < X/c$) or a combination of a level segment and the following upward segment (if $W > X/c$). Thus, it is clear that the construction of $\underline{F}^*$ does not allow us to directly associate the surplus prior to ruin $R(\tau^-)$ to any fluid level of $\underline{F}^*$. However, we already pointed out that the surplus level of $\underline{R}$ immediately after the payment of a claim, say the ith claim, corresponds to the fluid level of $\underline{F}^*$ at the end of the ith sample path of $\underline{Z}$ $(i = 1, 2, \dots)$. As a consequence, we propose to analyze $R(\tau^-)$ via the introduction of a new RV, say $R_{N(\tau)-1}$, defined as

$$R_{N(\tau)-1} = u + \sum_{i=1}^{N(\tau)-1} (cW_i - X_i). \tag{26}$$

From (26), $R_{N(\tau)-1}$ corresponds to the surplus level at the time of the penultimate claim before ruin (the claim just preceding the one causing ruin) where, by definition, $R_0 = u$ if ruin is caused by the first claim. By using sample path arguments we will obtain an expression for the discounted joint distribution of the triplet $(R_{N(\tau)-1}, R(\tau^-), |R(\tau)|)$. Given that the contributions to this discounted joint distribution have different functional forms based on whether ruin is caused by the first claim or any of its subsequent claims (due to the presence of $R_{N(\tau)-1}$), we introduce two $|S_0| \times 1$ column vectors, namely $\boldsymbol{h}_1(s, x_2, x_3 \mid u)$ and $\boldsymbol{h}_2(s, x_1, x_2, x_3 \mid u)$ whose ith elements $(i \in S_0)$ are respectively

$$\begin{aligned}&[\boldsymbol{h}_1(s, x_2, x_3 \mid u)]_i \,\mathrm{d}x_2 \,\mathrm{d}x_3 \\ &\quad = \mathrm{E}\left[\begin{matrix} \mathrm{e}^{-s\tau}\mathbf{1}(N(\tau) = 1)\mathbf{1}(R(\tau^-) \in (x_2, x_2 + \mathrm{d}x_2)) \\ \mathbf{1}(|R(\tau)| \in (x_3, x_3 + \mathrm{d}x_3)) \end{matrix} \;\middle|\; R(0) = u,\ J^*(0) = i\right]\end{aligned}$$

and

$$\begin{aligned}&[\boldsymbol{h}_2(s, x_1, x_2, x_3 \mid u)]_i \,\mathrm{d}x_1 \,\mathrm{d}x_2 \,\mathrm{d}x_3 \\ &\quad = \mathrm{E}\left[\begin{matrix} \mathrm{e}^{-s\tau}\mathbf{1}(N(\tau) > 1)\mathbf{1}(R_{N(\tau)-1} \in (x_1, x_1 + \mathrm{d}x_1)) \\ \mathbf{1}(R(\tau^-) \in (x_2, x_2 + \mathrm{d}x_2))\mathbf{1}(|R(\tau)| \in (x_3, x_3 + \mathrm{d}x_3)) \end{matrix} \;\middle|\; R(0) = u,\ J^*(0) = i\right].\end{aligned}$$

To identify an expression for $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$, we will first define some new quantities in the fluid process $\underline{F}^*$ and its reflected version $(\underline{F}^*)^{\mathrm{r}}$. Note that the reflected fluid process $(\underline{F}^*)^{\mathrm{r}}$ is obtained by simply reversing the roles of S_1 and S_2 in $\underline{F}^*$, i.e. $(\underline{F}^*)^{\mathrm{r}}$ is increasing or decreasing at rate c

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whenever $\underline{J}^*$ is in S_2 or, respectively, S_1. For $y > x > 0$, let the $|S_1| \times |S_1|$ matrix ${}_0\tilde{\boldsymbol{g}}_{11}(x, y, s)$ be the LST (with argument s) of the total time spent by $\underline{J}^*$ in S_0 and S_1 during a first passage of $(\underline{F}^*, \underline{J}^*)$ from (x, S_1) to (y, S_1), avoiding level 0 enroute. In addition, analogous to $\tilde{\boldsymbol{\Gamma}}(s)$ defined in Section 4, we define the $|S_2| \times |S_1|$ matrix ${}^y\tilde{\boldsymbol{\Gamma}}^{\mathrm{r}}(s)$ as the LST (with argument s) of the total time spent by $\underline{J}^*$ in S_0 and S_1 during a first passage of $((\underline{F}^*)^{\mathrm{r}}, \underline{J}^*)$ from $(0, S_2)$ to $(0, S_1)$, avoiding level y enroute.

We remark that these two quantities have been analyzed in a (similar) context where the time spent by $\underline{J}^*$ in S_0, S_1, and S_2 is accounted for (see, e.g. Ahn *et al.* (2007, Theorem 1) and Ramaswami (2006, Theorem 4), respectively, and the references therein). Here those quantities are defined by extracting the time spent by the process $\underline{J}^*$ in S_2 over the desired first passage times. Inspired from the existing methodology, these newly defined LSTs are defined in Appendix B.

Theorem 2. *For $u \geq 0$, we have*

$$\boldsymbol{h}_1(s, x_2, x_3 \mid u) = c^{-2} \exp\left\{(\boldsymbol{Q}_{00} - s\boldsymbol{I})\left(\frac{x_2 - u}{c}\right)\right\} \boldsymbol{Q}_{02} \exp\left\{\boldsymbol{Q}_{22}\left(\frac{u + x_3}{c}\right)\right\} \boldsymbol{Q}_{20}\boldsymbol{e}$$

for $x_2 > u$ and $x_3 > 0$, and

$$\boldsymbol{h}_2(s, x_1, x_2, x_3 \mid u) = (s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}[\boldsymbol{Q}_{01}\boldsymbol{k}_1(s, x_1, x_2, x_3 \mid u) + \boldsymbol{Q}_{02}\boldsymbol{k}_2(s, x_1, x_2, x_3 \mid u)]$$

for $x_1 > 0$, $x_2 > x_1$, and $x_3 > 0$, where

$$\begin{aligned}\boldsymbol{k}_1(s, x_1, x_2, x_3 \mid u) = c^{-1}{}_0\tilde{\boldsymbol{g}}_{11}(u, x_1, s)[\boldsymbol{I} - \tilde{\boldsymbol{\Gamma}}(s)^{x_1}\tilde{\boldsymbol{\Gamma}}^{\mathrm{r}}(s)]^{-1} \\ \times (\boldsymbol{Q}_{10} + \tilde{\boldsymbol{\Gamma}}(s)\boldsymbol{Q}_{20})\boldsymbol{h}_1(s, x_2, x_3 \mid x_1)\end{aligned}$$

and

$$\boldsymbol{k}_2(s, x_1, x_2, x_3 \mid u) = {}^u\tilde{\boldsymbol{\Gamma}}^{\mathrm{r}}(s)\boldsymbol{k}_1(s, x_1, x_2, x_3 \mid u)$$

for $x_1 > u$, while

$$\boldsymbol{k}_1(s, x_1, x_2, x_3 \mid u) = \tilde{\boldsymbol{\Gamma}}(s)\boldsymbol{k}_2(s, x_1, x_2, x_3 \mid u)$$

and

$$\begin{aligned}\boldsymbol{k}_2(s, x_1, x_2, x_3 \mid u) = c^{-1}\,\tilde{\boldsymbol{\Phi}}_{u-x_1}(s)[\boldsymbol{I} - {}^{x_1}\tilde{\boldsymbol{\Gamma}}^{\mathrm{r}}(s)\tilde{\boldsymbol{\Gamma}}(s)]^{-1} \\ \times (\boldsymbol{Q}_{20} + {}^{x_1}\tilde{\boldsymbol{\Gamma}}^{\mathrm{r}}(s)\boldsymbol{Q}_{10})\boldsymbol{h}_1(s, x_2, x_3 \mid x_1)\end{aligned}$$

for $x_1 < u$.

Proof. We first consider the expression for $\boldsymbol{h}_1(s, x_2, x_3 \mid u)$. Given that the first claim causes ruin, the surplus prior to ruin, x_2, will be greater than the initial surplus u. For the surplus prior to ruin to be x_2, the following statements must hold.

- The surplus process $\underline{R}$ has to first reach level x_2 from level u without a claim. This is translated into a level segment of duration $(x_2 - u)/c$ in the associated fluid flow process $\underline{F}^*$. Accounting for the time spent by $\underline{J}^*$ in S_0 during this level segment, this yields a contribution of $\exp\{(\boldsymbol{Q}_{00} - s\boldsymbol{I})(x_2 - u)/c\}$ to $\boldsymbol{h}_1$.

- Then, the surplus process experiences its first claim within $c^{-1}\,\mathrm{d}x_2$ after reaching level x_2. In order for ruin to occur upon the first claim, this is translated into a transition from a level segment period to a decreasing period in the fluid process $\underline{F}^*$ within $c^{-1}\,\mathrm{d}x_2$ (with a contribution of $c^{-1}\boldsymbol{Q}_{02}$ to $\boldsymbol{h}_1$).

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In addition, the duration of this decreasing period in the fluid process $\underline{F}^*$ will be $(u + x_3)/c$ to enable a deficit at ruin of x_3. Thus, the following statements must hold.

- The fluid process $\underline{F}^*$ must decrease over a period of length $(u + x_3)/c$. Given that the time spent by $\underline{J}^*$ in S_2 is factored out, this yields a contribution of $\exp\{\boldsymbol{Q}_{22}(u + x_3)/c\}$ to $\boldsymbol{h}_1$.
- The fluid process $\underline{F}^*$ should stop its descending behavior within $c^{-1}\,\mathrm{d}x_3$ after reaching x_3, which results in a contribution of $c^{-1}\boldsymbol{Q}_{20}$ to $\boldsymbol{h}_1$.

Finally, the term $\boldsymbol{e}$ allows us to consider all the phases in S_0 that the process ends up when the descending period ends.

Now, let us look at the expression for $\boldsymbol{h}_2(s, x_1, x_2, x_3 \mid u)$. Note that $\boldsymbol{h}_2(s, x_1, x_2, x_3 \mid u)$ is nonzero only if $x_2 > x_1$ (given that $R(\tau^-) > R_{N(\tau)-1}$ almost surely). First, the term $(s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}$ corresponds to the Laplace transform of the time that the CTMC $\underline{J}^*$ first leaves S_0 given that $J^*(0) \in S_0$. Upon this first exit from S_0, $\underline{J}^*$ enters either S_1 (governed by $\boldsymbol{Q}_{01}$) if $W_1 > X_1/c$ or S_2 (governed by $\boldsymbol{Q}_{02}$) if $W_1 < X_1/c$. Provided that the first exit was made to a phase in S_i, we denote the LST of the remaining time until ruin (with a surplus level at the time of the penultimate claim before ruin of x_1, a surplus prior to ruin of x_2, and a deficit of x_3) by $\boldsymbol{k}_i(s, x_1, x_2, x_3 \mid u)$ $(i = 1, 2)$.

Now let us consider the case in which $x_1 > u$ for $\boldsymbol{k}_i(s, x_1, x_2, x_3 \mid u)$ $(i = 1, 2)$. First, we look at the expression for $\boldsymbol{k}_1(s, x_1, x_2, x_3 \mid u)$. For $R_{N(\tau)-1}$ to be x_1 $(x_1 > u)$, the surplus process $\underline{R}$ has to first reach level x_1 from level u before ruin. Equivalently, the fluid level process $\underline{F}^*$, starting with level u in S_1, has to first attain level x_1 in S_1, avoiding level 0 enroute. The LST of the total time spent in S_0 and S_1 during this first passage time is ${}_0\tilde{\boldsymbol{g}}_{11}(u, x_1, s)$. Being at level x_1 in S_1 for the first time, it is possible to revisit level x_1 in S_1 an arbitrary number of times prior to ruin. The LST of the time spent by $\underline{J}^*$ in S_0 and S_1 before the last visit of $\underline{F}^*$ to level x_1 in S_1 is given by $[\boldsymbol{I} - \tilde{\boldsymbol{\Gamma}}(s)^{x_1}\tilde{\boldsymbol{\Gamma}}^{\mathrm{r}}(s)]^{-1}$. Now, having the fluid process $\underline{F}^*$ at level x_1 in S_1 for the last time, $R_{N(\tau)-1}$ can be x_1 via two scenarios:

- the fluid process $\underline{F}^*$ should stop its ascending pattern within $c^{-1}\,\mathrm{d}x_1$ after reaching x_1 for the last time in S_1, which results in a contribution of $c^{-1}\boldsymbol{Q}_{10}$ to $\boldsymbol{k}_1$; or
- the fluid process $\underline{F}^*$ continues its ascending pattern, returns to level x_1 this time in S_2, and then stops its descending pattern within $c^{-1}\,\mathrm{d}x_1$ after reaching x_1 in S_2 (with a total contribution $\tilde{\boldsymbol{\Gamma}}(s)\,c^{-1}\boldsymbol{Q}_{20}$ to $\boldsymbol{k}_1$).

Note that in both cases, the fluid process is now at level x_1 in S_0. Given that ruin has to occur at the time of the next claim with a surplus prior to ruin of x_2 and a deficit at ruin of x_3, this yields a final contribution of $\boldsymbol{h}_1(s, x_2, x_3 \mid x_1)$.

For $\boldsymbol{k}_2(s, x_1, x_2, x_3 \mid u)$, we note that the fluid process $\underline{F}^*$ (now at level u in S_2) must return to level u in S_1 avoiding level 0 enroute for $R_{N(\tau)-1}$ to be x_1. The LST of the total time spent in S_0 and S_1 during this first passage time is ${}^{u}\tilde{\boldsymbol{\Gamma}}^{\mathrm{r}}(s)$. Being back at level u in S_1, the remaining contribution is easily found to be $\boldsymbol{k}_1(s, x_1, x_2, x_3 \mid u)$.

The formulae provided for $\boldsymbol{k}_1(s, x_1, x_2, x_3 \mid u)$ and $\boldsymbol{k}_2(s, x_1, x_2, x_3 \mid u)$ when $x_1 < u$ can be found using a similar line of logic.

From Theorem 2, it is immediate that the discounted joint distribution of $R(\tau^-)$ and $|R(\tau)|$, denoted by $\tilde{\boldsymbol{f}}(s, x_2, x_3 \mid u)$ and having as its ith element

$$[\tilde{\boldsymbol{f}}(s, x_2, x_3 \mid u)]_i \, \mathrm{d}x_2 \, \mathrm{d}x_3$$
$$= \mathrm{E}[\mathrm{e}^{-s\tau} \mathbf{1}(R(\tau^-) \in (x_2, x_2 + \mathrm{d}x_2)) \mathbf{1}(|R(\tau)| \in (x_3, x_3 + \mathrm{d}x_3)) \mid R(0) = u, \, J^*(0) = i],$$

is given by

$$\tilde{\boldsymbol{f}}(s, x_2, x_3 \mid u) = \mathbf{1}(x_2 > u)\boldsymbol{h}_1(s, x_2, x_3 \mid u) + \int_0^{x_2} \boldsymbol{h}_2(s, x_1, x_2, x_3 \mid u) \, \mathrm{d}x_1.$$

We point out that the discounted density of the surplus prior to ruin and the deficit at ruin represents a key component of the defective renewal equation satisfied by the general Gerber–Shiu discounted penalty function in a large class of commonly analyzed risk models in ruin theory (see Cheung *et al.* (2009)).

Appendix A

In this appendix we provide an alternative analytic proof for the derivation of the Riccati equation (24). For the fluid process $\underline{F}^*$ with initial fluid $F^*(0) = u$, let ν be the time taken by the fluid process to become empty for the first time with $\nu = \infty$ if $F^*(t) > 0$ for all $t \geq 0$. Also, let $\nu_i = \int_0^\nu \mathbf{1}(J^*(s) \in S_i) \mathrm{d}s$ $(i = 0, 1, 2)$ be the time spent in the set of phases S_i during this first passage time (with $\nu_i = \infty$ if $\nu = \infty$). Note that the (k, l)th element of $\boldsymbol{\Phi}_u(x)$ previously defined in Section 4 can also be expressed as

$$[\boldsymbol{\Phi}_u(x)]_{kl} = \Pr(\nu < \infty, \; \nu_0 + \nu_1 \leq x, \; J^*(\nu) = l \mid F^*(0) = u, \; J^*(0) = k)$$

for $k \in S_2$ and $l \in S_2$. We further introduce an $|S_1| \times |S_2|$ matrix $\boldsymbol{\Lambda}_u(x)$ and an $|S_0| \times |S_2|$ matrix $\boldsymbol{\Upsilon}_u(x)$ whose (k, l)th elements are respectively defined as

$$[\boldsymbol{\Lambda}_u(x)]_{kl} = \Pr(\nu < \infty, \; \nu_0 + \nu_1 \leq x, \; J^*(\nu) = l \mid F^*(0) = u, \; J^*(0) = k)$$

for $k \in S_1$ and $l \in S_2$, and

$$[\boldsymbol{\Upsilon}_u(x)]_{kl} = \Pr(\nu < \infty, \; \nu_0 + \nu_1 \leq x, \; J^*(\nu) = l \mid F^*(0) = u, \; J^*(0) = k)$$

for $k \in S_0$ and $l \in S_2$. We also denote their respective LSTs by $\tilde{\boldsymbol{\Lambda}}_u(s) = \int_0^\infty \mathrm{e}^{-sx} \, \mathrm{d}\boldsymbol{\Lambda}_u(x)$ and $\tilde{\boldsymbol{\Upsilon}}_u(s) = \int_0^\infty \mathrm{e}^{-sx} \, \mathrm{d}\boldsymbol{\Upsilon}_u(x)$.

Our objective consists in establishing some relationships between the LSTs $\tilde{\boldsymbol{\Phi}}_u(s)$, $\tilde{\boldsymbol{\Lambda}}_u(s)$, and $\tilde{\boldsymbol{\Upsilon}}_u(s)$. To this end, we condition on the first transition of the process $\underline{J}^*$ into another set of phases and readily obtain

$$\tilde{\boldsymbol{\Upsilon}}_u(s) = (s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}[\boldsymbol{Q}_{01}\tilde{\boldsymbol{\Lambda}}_u(s) + \boldsymbol{Q}_{02}\tilde{\boldsymbol{\Phi}}_u(s)], \tag{27}$$

$$\begin{aligned} \tilde{\boldsymbol{\Lambda}}_u(s) &= \int_0^\infty \exp\{(\boldsymbol{Q}_{11} - s\boldsymbol{I})y\}\boldsymbol{Q}_{10}\tilde{\boldsymbol{\Upsilon}}_{u+cy}(s) \, \mathrm{d}y \\ &= \frac{1}{c}\int_u^\infty \exp\biggl\{\biggl(\frac{\boldsymbol{Q}_{11} - s\boldsymbol{I}}{c}\biggr)(y - u)\biggr\}\boldsymbol{Q}_{10}\tilde{\boldsymbol{\Upsilon}}_y(s) \, \mathrm{d}y, \end{aligned} \tag{28}$$

and

$$\begin{aligned} \tilde{\boldsymbol{\Phi}}_u(s) &= \exp\biggl\{\frac{\boldsymbol{Q}_{22}}{c}u\biggr\} + \int_0^{u/c} \exp\{\boldsymbol{Q}_{22}y\}\boldsymbol{Q}_{20}\tilde{\boldsymbol{\Upsilon}}_{u-cy}(s) \, \mathrm{d}y \\ &= \exp\biggl\{\frac{\boldsymbol{Q}_{22}}{c}u\biggr\} + \frac{1}{c}\int_0^u \exp\biggl\{\frac{\boldsymbol{Q}_{22}}{c}(u - y)\biggr\}\boldsymbol{Q}_{20}\tilde{\boldsymbol{\Upsilon}}_y(s) \, \mathrm{d}y. \end{aligned} \tag{29}$$

Differentiating (28) with respect to u and then making use of (27), we find that

$$\begin{aligned}\frac{\partial}{\partial u}\tilde{\boldsymbol{\Lambda}}_u(s) &= -\frac{1}{c}[\boldsymbol{Q}_{10}\tilde{\boldsymbol{\Upsilon}}_u(s) + (\boldsymbol{Q}_{11} - s\boldsymbol{I})\tilde{\boldsymbol{\Lambda}}_u(s)] \\ &= -\frac{1}{c}[(\boldsymbol{Q}_{10}(s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}\boldsymbol{Q}_{01} + (\boldsymbol{Q}_{11} - s\boldsymbol{I}))\tilde{\boldsymbol{\Lambda}}_u(s) \\ &\qquad + \boldsymbol{Q}_{10}(s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}\boldsymbol{Q}_{02}\tilde{\boldsymbol{\Phi}}_u(s)] \\ &= -\frac{1}{c}[(\boldsymbol{Q}_{11} - s\boldsymbol{I} + \boldsymbol{Q}^*_{11}(s))\tilde{\boldsymbol{\Lambda}}_u(s) + \boldsymbol{Q}^*_{12}(s)\tilde{\boldsymbol{\Phi}}_u(s)].\end{aligned} \tag{30}$$

Note that, under the positive security loading condition (2),

$$\lim_{u\to\infty}\tilde{\boldsymbol{\Lambda}}_u(s) = \boldsymbol{0}. \tag{31}$$

Similarly, the differentiation of (29) with respect to u followed by the use of (27) yields

$$\begin{aligned}\frac{\partial}{\partial u}\tilde{\boldsymbol{\Phi}}_u(s) &= \frac{1}{c}[\boldsymbol{Q}_{22}\tilde{\boldsymbol{\Phi}}_u(s) + \boldsymbol{Q}_{20}\tilde{\boldsymbol{\Upsilon}}_u(s)] \\ &= \frac{1}{c}[\boldsymbol{Q}_{22}\tilde{\boldsymbol{\Phi}}_u(s) + \boldsymbol{Q}_{20}((s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}[\boldsymbol{Q}_{01}\tilde{\boldsymbol{\Lambda}}_u(s) + \boldsymbol{Q}_{02}\tilde{\boldsymbol{\Phi}}_u(s)])] \\ &= \frac{1}{c}[(\boldsymbol{Q}_{22} + \boldsymbol{Q}^*_{22}(s))\tilde{\boldsymbol{\Phi}}_u(s) + \boldsymbol{Q}^*_{21}(s)\tilde{\boldsymbol{\Lambda}}_u(s)]\end{aligned} \tag{32}$$

with boundary condition

$$\tilde{\boldsymbol{\Phi}}_0(s) = \boldsymbol{I}. \tag{33}$$

From (30), (31), (32), and (33), we observe that $\tilde{\boldsymbol{\Lambda}}_u(s)$ and $\tilde{\boldsymbol{\Phi}}_u(s)$ satisfy a Feynman–Kac equation (see, e.g. Asmussen *et al.* (2002, Theorem 2)). Given that

$$\tilde{\boldsymbol{\Lambda}}_u(s) = \tilde{\boldsymbol{\Gamma}}(s)\tilde{\boldsymbol{\Phi}}_u(s), \tag{34}$$

we arrive at

$$\frac{\partial}{\partial u}\tilde{\boldsymbol{\Phi}}_u(s) = \frac{1}{c}[\boldsymbol{Q}_{22} + \boldsymbol{Q}^*_{22}(s) + \boldsymbol{Q}^*_{21}(s)\tilde{\boldsymbol{\Gamma}}(s)]\tilde{\boldsymbol{\Phi}}_u(s). \tag{35}$$

Equation (35), together with the boundary condition (33), yields (23). Finally, using (23), (34) becomes

$$\tilde{\boldsymbol{\Lambda}}_u(s) = \tilde{\boldsymbol{\Gamma}}(s)\exp\left\{\frac{1}{c}(\boldsymbol{Q}_{22} + \boldsymbol{Q}^*_{21}(s)\tilde{\boldsymbol{\Gamma}}(s) + \boldsymbol{Q}^*_{22}(s))u\right\}. \tag{36}$$

The substitution of (23) and (36) into (30) followed by some algebraic manipulations leads to the Riccati equation obtained in (24) for $\tilde{\boldsymbol{\Gamma}}(s)$.

Appendix B

Let $\{J^{**}(t),\ t \ge 0\}$ and $\{F^{**}(t),\ t \ge 0\}$ be the time-modified versions of the CTMC $\{J^*(t),\ t \ge 0\}$ and the fluid flow $\{F^*(t),\ t \ge 0\}$ by incising out the intervals of time in which the fluid flow $\{F^*(t),\ t \ge 0\}$ decreases and gluing the remaining pieces together. Let $\boldsymbol{V}$ be a matrix for which its (i, j)th element is defined as

$$[\boldsymbol{V}]_{i,j}(t, x) = \Pr(F^{**}(t) \le x,\ J^{**}(t) = j \mid J^{**}(0) = i,\ F^{**}(0) = 0),$$

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and its associated density is defined as

$$[\boldsymbol{v}]_{i,j}(t,x) = \frac{\partial}{\partial x}[\boldsymbol{V}]_{i,j}(t,x) \quad \text{for } i \in S_1 \text{ and } j \in S_1 \cup S_2.$$

Following a similar line of logic as in the proof of Lemma 3.3.2 of Ahn and Ramaswami (2006) (with $\boldsymbol{Q}_{21} = \boldsymbol{0}$), we find that

$$\frac{\partial}{\partial t}\boldsymbol{v}_{11}(t,x) = \boldsymbol{v}_{11}(t,x)\,\boldsymbol{Q}_{11} + \boldsymbol{v}_{10}(t,x)\,\boldsymbol{Q}_{01} - c\frac{\partial}{\partial x}\boldsymbol{v}_{11}(t,x) \tag{37}$$

and

$$\frac{\partial}{\partial t}\boldsymbol{v}_{10}(t,x) = \boldsymbol{v}_{11}(t,x)\,\boldsymbol{Q}_{10} + \boldsymbol{v}_{10}(t,x)\,\boldsymbol{Q}_{00} + \boldsymbol{h}_{12}(t,x)\,\boldsymbol{Q}_{20}, \tag{38}$$

where

$$\boldsymbol{h}_{12}(t,x) = \int_0^\infty \boldsymbol{v}_{10}(t,x+cy)\,\boldsymbol{Q}_{02}\exp\{\boldsymbol{Q}_{22}y\}\,\mathrm{d}y.$$

Note that, by a probabilistic argument, it is clear that we reach level $x + cy$ in S_0 by being in the interim at level x in S_1. By conditioning $\boldsymbol{h}_{12}(t,x)$ on the time of the last visit to x in S_1 before time t, we find that

$$\boldsymbol{h}_{12}(t,x) = \int_0^t \boldsymbol{v}_{11}(t-a,x)\,\mathrm{d}\boldsymbol{\Gamma}(a). \tag{39}$$

Taking Laplace transforms (with respect to time) of (37) and (38) yields

$$s\widetilde{\boldsymbol{v}}_{11}(s,x) = \widetilde{\boldsymbol{v}}_{11}(s,x)\,\boldsymbol{Q}_{11} + \widetilde{\boldsymbol{v}}_{10}(s,x)\,\boldsymbol{Q}_{01} - c\frac{\partial}{\partial x}\widetilde{\boldsymbol{v}}_{11}(s,x)$$

and

$$s\widetilde{\boldsymbol{v}}_{10}(s,x) = \widetilde{\boldsymbol{v}}_{11}(s,x)\,\boldsymbol{Q}_{10} + \widetilde{\boldsymbol{v}}_{10}(s,x)\,\boldsymbol{Q}_{00} + \int_0^\infty \widetilde{\boldsymbol{v}}_{10}(s,x+cy)\,\boldsymbol{Q}_{02}\exp\{\boldsymbol{Q}_{22}y\}\,\boldsymbol{Q}_{20}\,\mathrm{d}y, \tag{40}$$

respectively. Given that

$$\int_0^\infty \widetilde{\boldsymbol{v}}_{10}(s,x+cy)\,\boldsymbol{Q}_{02}\exp\{\boldsymbol{Q}_{22}y\}\,\mathrm{d}y = \widetilde{\boldsymbol{v}}_{11}(s,x)\widetilde{\boldsymbol{\Gamma}}(s),$$

(40) becomes

$$s\widetilde{\boldsymbol{v}}_{10}(s,x) = \widetilde{\boldsymbol{v}}_{11}(s,x)\,\boldsymbol{Q}_{10} + \widetilde{\boldsymbol{v}}_{10}(s,x)\,\boldsymbol{Q}_{00} + \widetilde{\boldsymbol{v}}_{11}(s,x)\widetilde{\boldsymbol{\Gamma}}(s)\,\boldsymbol{Q}_{20},$$

or, equivalently,

$$\widetilde{\boldsymbol{v}}_{10}(s,x) = \widetilde{\boldsymbol{v}}_{11}(s,x)(\boldsymbol{Q}_{10} + \widetilde{\boldsymbol{\Gamma}}(s)\,\boldsymbol{Q}_{20})(s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}. \tag{41}$$

Substituting (41) into (B) yields

$$s\widetilde{\boldsymbol{v}}_{11}(s,x) = \widetilde{\boldsymbol{v}}_{11}(s,x)\,\boldsymbol{Q}_{11} + \widetilde{\boldsymbol{v}}_{11}(s,x)(\boldsymbol{Q}_{10} + \widetilde{\boldsymbol{\Gamma}}(s)\,\boldsymbol{Q}_{20})(s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}\,\boldsymbol{Q}_{01} - c\frac{\partial}{\partial x}\widetilde{\boldsymbol{v}}_{11}(s,x),$$

or, equivalently,

$$c^{-1}\widetilde{\boldsymbol{v}}_{11}(s,x)(\boldsymbol{Q}_{11} - s\boldsymbol{I} + (\boldsymbol{Q}_{10} + \widetilde{\boldsymbol{\Gamma}}(s)\,\boldsymbol{Q}_{20})(s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}\,\boldsymbol{Q}_{01}) = \frac{\partial}{\partial x}\widetilde{\boldsymbol{v}}_{11}(s,x). \tag{42}$$

The solution to (42) is

$$\widetilde{\boldsymbol{v}}_{11}(s,x) = c^{-1}\mathrm{e}^{\boldsymbol{K}(s)x}, \tag{43}$$

where

$$\boldsymbol{K}(s) = c^{-1}(\boldsymbol{Q}_{11} - s\boldsymbol{I} + (\boldsymbol{Q}_{10} + \widetilde{\boldsymbol{\Gamma}}(s)\boldsymbol{Q}_{20})(s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}\boldsymbol{Q}_{01}).$$

Note that the occurrence of the c^{-1} term in (43) is precisely the effect of the infinitesimal changes in time to infinitesimal changes in fluid level (for a detailed discussion see Section 3.5 of Ahn *et al.* (2007)).

Given that $(F^{**}(0), J^{**}(0)) = (x, i)$ for $i \in S_1$, the LST (with respect to time) of the expected number of visits to (x, j), $j \in S_1$, avoiding level 0 in the interval $(0, t)$ is the (i, j)th element of the matrix

$$\boldsymbol{\Xi}(s,x) = c^{-1}\widetilde{\boldsymbol{\Gamma}}(s)\int_0^x \widetilde{\boldsymbol{\Phi}}_y(s)\boldsymbol{Q}_{20}(s\boldsymbol{I} - \boldsymbol{Q}_{00})^{-1}\boldsymbol{Q}_{01}\widetilde{\boldsymbol{v}}_{11}(s,y)\,\mathrm{d}y.$$

For computation of the integral term, we refer the interested reader to Lemma 2 of Ramaswami (2006). Clearly,

$$\widetilde{\boldsymbol{v}}_{11}(s,y) = c^{-1}{}_0\tilde{\boldsymbol{g}}_{11}(0,y,s) + c^{-1}{}_0\tilde{\boldsymbol{g}}_{11}(0,y,s)\boldsymbol{\Xi}(s,x),$$

which leads to

$${}_0\tilde{\boldsymbol{g}}_{11}(0,y,s) = c\widetilde{\boldsymbol{v}}_{11}(s,y)(\boldsymbol{I} + \boldsymbol{\Xi}(s,x))^{-1}.$$

Using a sample path argument,

$${}^{b}\widetilde{\boldsymbol{\Gamma}}(s) = \widetilde{\boldsymbol{\Gamma}}(s) - {}_0\tilde{\boldsymbol{g}}_{11}(0,b,s)\widetilde{\boldsymbol{\Gamma}}(s)\widetilde{\boldsymbol{\Phi}}_b(s). \tag{44}$$

For the reflected fluid flow, ${}^{b}\widetilde{\boldsymbol{\Gamma}}^{\mathrm{r}}(s)$ is calculated from (44) by first changing the roles of the states in S_1 and S_2. Finally, an explicit expression for ${}_0\tilde{\boldsymbol{g}}_{11}(x,y,s)$ is found to be

$${}_0\tilde{\boldsymbol{g}}_{11}(x,y,s) = (\boldsymbol{I} - {}^{y-x}\widetilde{\boldsymbol{\Gamma}}(s)\,{}^{x}\widetilde{\boldsymbol{\Gamma}}^{\mathrm{r}}(s))^{-1}{}_0\tilde{\boldsymbol{g}}_{11}(0,y-x,s).$$

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