ITERATED TRANSFORMS

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1. Introduction. In his work on Laplace and Stieltjes transforms Widder [6, ch. 8] has investigated relationships of the type

$$f(x) = \int_0^\infty e^{-xt} g(t)dt, \tag{1}$$

$$g(x) = \int_0^\infty e^{-xt} h(t)dt, \tag{2}$$

$$f(x) = \int_0^\infty \frac{h(t)}{x+t}dt. \tag{3}$$

(1) and (2) are Laplace transforms and (3), which occurs in Stieltjes' [4] researches on continued fractions, is referred to by Widder as a Stieltjes transform. Widder also considers (3) in the more general form of a Stieltjes integral

$$f(x) = \int_0^\infty \frac{dk(t)}{x+t}.$$

These formulae bear a close resemblance to special cases of Chain transforms [1], whose theory I have developed in a previous paper. My object here is to investigate and generalize the relationships above by the methods used in Chain transform theory. For example, we prove that the factors e^{-xt} in (1) and (2) can be replaced by Laplace transforms of Fourier kernels and then show that this result can be generalized still further. In order to make use of the mean square theory of convergence we shall define a Laplace transform which is somewhat more general than the one in common use.

2. The Mellin transform. The Mellin transform [5, ch. 3], which is our main instrument of analysis, is given by

$$F(s) = \int_0^\infty f(u)u^{s-1}du, \tag{4}$$

$$f(u) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} F(s)u^{-s}ds. \tag{5}$$

Pairs of functions related in the manner of (4) and (5) are known as Mellin transforms of each other and will always be written in the form $f(u)$ and $F(s)$, $g(u)$ and $G(s)$, etc. Their main properties [5; §§ 3.17, 7.7, 7.8] are as follows:

2.1. If $f(u)$ belongs to $L^2(0, \infty)$, i.e.,

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$$\int_0^\infty |f(u)|^2 du$$

converges, then as a tends to infinity

2.11
$$\int_{1/a}^{a} f(u)u^{s-1}du$$

converges in mean square to $F(s)$, where $F(s)$ belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$,

2.12
$$\frac{1}{2\pi i}\int_{\frac{1}{2}-ia}^{\frac{1}{2}+ia} F(s)u^{-s}ds$$

converges in mean square to $f(u)$.

Conversely if $F(s)$ belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ then **2.12** holds, $f(u)$ belongs to $L^2(0, \infty)$ and is related to $F(s)$ by (4).

2.2 (The Parseval theorem). If $f(u)$ and $g(u)$ both belong to $L^2(0, \infty)$, or $F(s)$ and $G(s)$ both belong to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$, then

$$\int_0^\infty f(u)g(u)du = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} F(s)G(1-s)ds. \tag{6}$$

2.3 If $f(u)$ and $F(s)$ are Mellin transforms then so are

$$f(au) \quad \text{and} \quad F(s)a^{-s},$$
$$u^a f(u) \quad \text{and} \quad F(s+a),$$
$$f(u^a) \quad \text{and} \quad \frac{1}{a}F\left(\frac{s}{a}\right).$$

2.4 A pair of Mellin transforms is given by the equations $f(u) = 1$ $(0 < u < y)$, $f(u) = 0$ $(u > y)$, and $F(s) = y^s/s$.

To illustrate these results if y is real then, treated as a function of s, y^s/s belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. Hence if $M(s)$ also belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ we have from (6) and **2.4**,

$$\int_0^y m(u)du = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} M(s)\frac{y^{1-s}}{1-s}ds. \tag{7}$$

We shall, in future, write

$$\int_0^y m(u)du = m_1(y) \tag{8}$$

and all pairs of functions written in this way, e.g. $n(y)$, if it exists, and $n_1(y)$, will be related as in (8). Thus (7) becomes

$$m_1(y) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} M(s)\frac{y^{1-s}}{1-s}ds, \tag{9}$$

where $M(s)$ belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$.

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The function $M(s)$ plays a fundamental part in the theory of Fourier kernels. For all of these kernels, $M(s)$ is bounded on the line $s = \frac{1}{2} + i\tau$, where τ is real, but in general does not belong to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. However, since $M(s)$ is bounded, $M(s)/(1-s)$ belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. We can therefore deduce from **2.12** and (9) that $m_1(y)/y$ belongs to $L^2(0, \infty)$ and that its Mellin transform is $M(s)/(1-s)$. This may be true even if $m(y)$ does not exist, as the following example illustrates.

Let $M(s) = 1$, then

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} M(s)y^{-s}ds$$

oscillates finitely and does not converge. But from (9) we have $m_1(y) = 0$ if $0 < y < 1$ and $m_1(y) = 1$ if $y > 1$. Thus $m_1(y)$, defined by (9) instead of by (8), exists although $m(y)$ does not exist.

Partly for this reason and partly because $m_1(y)/y$ belongs to $L^2(0, \infty)$ it is much more convenient to formulate Fourier transform theory in terms of $m_1(u)$ rather than $m(u)$. For these reasons we shall also define our Laplace transform of §4 in terms of $m_1(u)$ rather than in terms of $m(u)$.

3. The general Fourier transform. If

$$M(s)N(1-s) = 1, \tag{10}$$

and $M(s)$ and $N(s)$ are both bounded on the line $s = \frac{1}{2} + i\tau$, where τ is real, and $A(x)$ belongs to $L^2(0, \infty)$, then we have the general Fourier transform

$$\begin{aligned} B(y) &= \frac{d}{dy}\int_0^\infty A(x)\frac{m_1(xy)}{x}dx, \\ A(y) &= \frac{d}{dy}\int_0^\infty B(x)\frac{n_1(xy)}{x}dx. \end{aligned} \tag{11}$$

In the course of the proof, it is shown that $B(x)$ also belongs to $L^2(0, \infty)$. The theory is given in Titchmarsh [**5**, p. 226]. The functions $m_1(x)/x$ and $n_1(x)/x$ are known as general Fourier kernels and are called symmetrical if $m_1(x) = n_1(x)$ and asymmetrical otherwise.

4. The general Laplace transform of Fourier kernels. From the asymptotic expansion of $\Gamma(s)$ we know that on the line $s = \frac{1}{2} + i\tau$, where τ is real, $\Gamma(s) = 0(e^{-\frac{1}{2}\pi|\tau|})$ and so belongs to $L^2(0, \infty)$. Also the Mellin transform of e^{-xu} is $\Gamma(s)\, x^{-s}$. Hence, if $M(s)$ belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ we have by (6),

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(s)M(1-s)x^{-s}ds = \int_0^\infty e^{-xu}m(u)du. \tag{12}$$

The right hand side is evidently the Laplace transform of $m(u)$.

Suppose that, as in Fourier transform theory, we know only that $M(s)$ is bounded. In this case $M(s)/(1-s)$ belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ and, from (9), is the Mellin transform of $m_1(u)/u$. Hence from (6) and **2.3** we have

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$$(13)\quad \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma(s)M(1-s)x^{-s}ds = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma(s+1)\frac{M(1-s)}{s}x^{-s}ds = \int_0^\infty xe^{-xu}m_1(u)du.$$

We shall call the right-hand side of (13) the general Laplace transform and write

$$(14)\qquad \mathbf{m}(x) = \int_0^\infty xe^{-xu}m_1(u)du.$$

All pairs of functions related in the manner defined by (14) will be written in the form $\mathbf{m}(x)$ and $m_1(x)$, $\mathbf{n}(x)$ and $n_1(u)$, etc.

The definition (14) has two important advantages over the standard Laplace transform (12). First, it exists if $M(s)$ is bounded, whether $m(x)$ exists or not, and, secondly, it lends itself readily to the application of mean square arguments. (14) bears much the same relation to (12) as the general Fourier kernel bears to the ordinary Fourier kernel. To illustrate these remarks take $M(s) = 1$, then, from §2, $m(u)$ does not exist. But $m_1(u) = 0$ when $0 < u < 1$ and $m_1(u) = 1$ when $u > 1$. From (14) it follows that $\mathbf{m}(x) = e^{-x}$ so that any general theorems proved for general Laplace transforms will also be true for e^{-x}.

In most cases when $m(u)$ exists it can be shown, on integrating by parts, that the right-hand sides of (12) and (14) are equal. Integration by parts also shows that (14) can frequently be expressed as the Stieltjes integral

$$\int_0^\infty e^{-xu}dm_1(u),$$

which by many writers is considered to be a natural generalization of the Laplace transform. But for our purposes it is much more convenient to use the form (14) than the Stieltjes integral. This is because of the advantage gained by using $M(s)$ in the study of Laplace transforms of Fourier kernels (see (9)).

From (13) and (14) we have

$$(14a)\qquad \mathbf{m}(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma(s)M(1-s)x^{-s}ds,$$

where $\Gamma(s)\ M(1-s)$ belongs to $L^2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$. Hence, from **2.12**, $\mathbf{m}(x)$ belongs to $L^2(0, \infty)$ and its Mellin transform is $\Gamma(s)\ M(1-s)$.

If $M(s)$ and $N(s)$ satisfy (10) and are both bounded on the line $s = \frac{1}{2} + i\tau$ then $\mathbf{m}(x)$ and $\mathbf{n}(x)$ are the general Laplace transforms of a pair of conjugate Fourier kernels. We then have, from (6),

$$(15)\quad \int_0^\infty \mathbf{m}(ux)\mathbf{n}(u)du = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma(s)M(1-s)x^{-s}\Gamma(1-s)N(s)ds = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{\pi}{\sin \pi s}x^{-s}ds = \frac{1}{1+x}.$$

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That Laplace transforms of Fourier kernels satisfy an integral equation of the type (15) was conjectured by Ramanujan [**3**, ch. 11(F)] and proved recently by Goodspeed [**2**].

5. Iteration formulae. If $M(s)$ is the Mellin transform of a symmetrical Fourier kernel then (10) becomes

$$M(s)M(1-s) = 1. \tag{16}$$

THEOREM 1. *If* (i) $M(s)$ *satisfies* (16) *and is bounded on the line* $s = \frac{1}{2} + i\tau$, (ii) *the general Laplace transform* $\mathbf{m}(x)$ *is defined by* (14), (iii) $h(x)$ *belongs to* $L^2(0, \infty)$, *and*

$$g(u) = \int_0^\infty \mathbf{m}(ut)h(t)dt, \tag{17}$$

and

$$f(u) = \int_0^\infty \mathbf{m}(ut)g(t)dt, \tag{18}$$

then

$$f(u) = \int_0^\infty \frac{h(t)}{u+t}dt \tag{19}$$

and

$$\int_0^\infty f(ut)h(t)dt = \int_0^\infty g(ut)g(t)dt. \tag{20}$$

To prove (19) we note that $\mathbf{m}(ut)$, as a function of t, and $h(t)$ both belong to $L^2(0, \infty)$ and have Mellin transforms $\Gamma(s)\, M(1-s)\, u^{-s}$ and $H(s)$ respectively ((14a) and §2). Hence, from (6) and (17), we have

$$g(u) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(s)M(1-s)u^{-s}H(1-s)ds. \tag{21}$$

Now on the line $s = \frac{1}{2} + i\tau$, $|\Gamma(s)\, M(1-s)|$ is bounded, say with upper bound K, and $H(s)$ belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ (from condition (iii) and **2.11**). Hence, on integrating along the line $s = \frac{1}{2} + i\tau$, we have

$$\int_{-\infty}^{\infty} |\Gamma(s)M(1-s)H(1-s)|^2 d\tau \leqslant K^2 \int_{-\infty}^{\infty} |H(1-s)|^2 d\tau$$

and so $\Gamma(s)\, M(1-s)\, H(1-s)$ belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. Hence, from **2.12** and (21), $g(u)$ belongs to $L^2(0, \infty)$ and its Mellin transform $G(s)$ is given by

$$G(s) = \Gamma(s)M(1-s)H(1-s). \tag{22}$$

Since $g(u)$ belongs to $L^2(0, \infty)$ we may similarly deduce from (18) that

$$F(s) = \Gamma(s)M(1-s)G(1-s). \tag{23}$$

On substituting $1 - s$ for s in (22) and eliminating $G(1-s)$ from (23) we have

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$$F(s) = \Gamma(s)M(1-s)\Gamma(1-s)M(s)H(s)$$
$$= \frac{\pi}{\sin \pi s} H(s)$$

from (16). Hence

$$f(u) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} H(s)u^{-s}\frac{\pi}{\sin \pi(1-s)}ds.$$

But $\pi/(\sin \pi s)$ belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ and its Mellin transform [5, p. 192 (7.7.8)] is $1/(1+u)$. From (6) we may then conclude that

$$f(u) = \int_0^\infty \frac{1}{1+t} h(ut)dt$$

(24)
$$= \int_0^\infty \frac{h(t)}{u+t}dt \qquad (u > 0)$$

by a slight change of variable. This completes the proof of (19).

For the proof of (20) we see from (22) and (23) that

(25)
$$F(s)H(1-s) = G(s)G(1-s).$$

Condition (iii) shows that $h(u)$ belongs to $L^2(0, \infty)$ and in the course of the proof of (24) it was shown that $f(u)$ and $g(u)$ also belong to $L^2(0, \infty)$. Hence, from **2.11**, $F(s)$, $G(s)$, and $H(s)$ all belong to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. On multiplying (25) by u^{-s} and integrating along the line $s = \frac{1}{2} + i\tau$ from $\frac{1}{2} - i\infty$ it follows from (6) that

$$\int_0^\infty f(ut)h(t)dt = \int_0^\infty g(ut)g(t)dt.$$

This completes the proof of Theorem 1. We have already noted that we can take $\mathbf{m}(x) = e^{-x}$ and then (17) and (18) reduce to known forms [6, p. 325].

Theorem 2 deals with the case of the asymmetrical kernels. If $M(s)$ and $N(s)$ are the Mellin transforms of a pair of conjugate Fourier kernels then the functional equation (10) is satisfied. This is dealt with in

THEOREM 2. *If* (i) $M(s)$ *and* $N(s)$ *satisfy* (10) *and are both bounded on the line* $s = \frac{1}{2} + i\tau$, (ii) *the general Laplace transforms* $\mathbf{m}(x)$ *and* $\mathbf{n}(x)$ *are defined by* (14), (iii) $h(x)$ *belongs to* $L^2(0, \infty)$, *and*

$$g(u) = \int_0^\infty \mathbf{m}(ut)h(t)dt,$$

and

$$f(u) = \int_0^\infty \mathbf{n}(ut)g(t)dt,$$

then

$$f(u) = \int_0^\infty \frac{h(t)}{u+t}dt.$$

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To prove this we use the same arguments as for Theorem 1. Equation (22) can be established as before, but instead of (23) we now prove

$$F(s) = \Gamma(s)N(1-s)G(1-s). \tag{26}$$

On eliminating $G(s)$ between (22) and (26), by the method of Theorem 1, and using (10) we establish once again

$$F(s) = \frac{\pi}{\sin \pi s} H(s).$$

From this we deduce (24), as in Theorem 1, and so complete the proof of Theorem 2. But if $M(s)$ is not equal to $N(s)$ then we cannot, from (22) and (26), establish a relation such as (25). Consequently, under the conditions of Theorem 2 the Parseval equation (20) does not in general exist.

6. Formal analysis. In this section I am concerned mainly with the methods by which iterated relationships such as (1), (2), and (3) can be obtained. The analysis is purely formal and difficulties, such as arise in changing the order of integration, etc., are for the moment ignored.

Consider the two equations

$$f(x) = \int_0^\infty p(xt)g(t)dt \tag{27}$$

and

$$g(x) = \int_0^\infty q(xt)h(t)dt. \tag{28}$$

On multiplying (27) by x^{s-1} we have formally

$$F(s) = \int_0^\infty \int_0^\infty p(xt)x^{s-1}g(t)dtdx$$

$$= \int_0^\infty \int_0^\infty p(u)u^{s-1}g(t)t^{-s}dtdu$$

$$= P(s)G(1-s). \tag{29}$$

Similarly from (28) we have

$$G(s) = Q(s)H(1-s). \tag{30}$$

On substituting $1 - s$ for s in (30) we can eliminate $G(1-s)$ from (29) and obtain

$$F(s) = P(s)Q(1-s)H(s).$$

Hence a relation exists between $f(x)$ and $h(x)$ which, in general, is independent of $g(x)$ but which depends largely upon the nature of the quantity $P(s)\,Q(1-s)$ and its Mellin transform. This observation is illustrated by the following examples:

If

$$f(x) = \int_0^\infty e^{-xt}(xt)^a g(t)dt \tag{31}$$

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and

$$(32) \qquad g(x) = \int_0^\infty e^{-xt}(xt)^a h(t)dt,$$

then

$$(33) \qquad f(x) = \Gamma(2a+1)\int_0^\infty \frac{h(t)(xt)^a}{(x+t)^{2a+1}}dt,$$

where $2a + 1 > 0$ and $x > 0$. By the method just outlined we have, from (31),

$$F(s) = \Gamma(s+a)G(1-s)$$

and from (32),

$$G(s) = \Gamma(s+a)H(1-s).$$

Hence

$$(34) \qquad F(s) = \Gamma(s+a)\Gamma(a+1-s)H(s).$$

But the Mellin transform [**5**, p. 195] of $\Gamma(a+1-s)\ \Gamma(a+s)$ is

$$\frac{\Gamma(2a+1)u^a}{(1+u)^{2a+1}}.$$

Hence from (34) and (6) we may deduce that

$$f(x) = \Gamma(2a+1)\int_0^\infty \frac{h(xt)t^a}{(1+t)^{2a+1}}dt.$$

This is finally reduced to (33) by a simple change of variable.

This analysis can be made rigorous by assuming that $h(x)$ belongs to $L^2(0, \infty)$. By the methods of Theorem 1 we can prove the following result:

THEOREM 3. *If $f(x)$ and $g(x)$ are related as in* (31) *and $g(x)$ and $h(x)$ as in* (32), *where $2a + 1 > 0$, and $h(u)$ belongs to $L^2(0, \infty)$ then $g(u)$ and $f(u)$ also belong to $L^2(0, \infty)$, $f(x)$ and $h(x)$ are related by* (33) *and also*

$$(35) \qquad \int_0^\infty f(ut)h(t)dt = \int_0^\infty g(ut)g(t)dt.$$

The equations (31), (32), and (33) reduce to (1), (2), and (3) in the special case $a = 0$.

It is not difficult to generalize Theorem 3 still further. For, write

$$\mu(t) = \int_0^\infty e^{-xt}(xt)^a m(x)dx$$

and

$$\nu(t) = \int_0^\infty e^{-xt}(xt)^a n(x)dx,$$

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where $m(x)$ and $n(x)$ are a pair of conjugate Fourier kernels, so that (10) is satisfied. Then we may, in general, replace (31) and (32) by

$$f(x) = \int_0^\infty \mu(xt)g(t)dt$$

and

$$g(x) = \int_0^\infty \nu(xt)h(t)dt,$$

and the relationship between $f(x)$ and $h(x)$ is still given by (33). If $m(x) = n(x)$ then we also have the Parseval equation (35), but not otherwise. This can be proved, on assuming suitable conditions, by the same arguments as are used in the proof of Theorem 1. But since this is a special case of Theorem 5, we shall omit the proof.

A second example is given by the following result: if

$$f(x) = a\int_0^\infty e^{-x^a t^a} g(t)dt$$

and

$$g(x) = a\int_0^\infty e^{-x^a t^a} h(t)dt,$$

where $a > 0$, then

$$f(x) = a\Gamma\left(\frac{1}{a}\right)\int_0^\infty \frac{h(t)}{(x^a + t^a)^{1/a}}dt,$$

and in addition (35) is true. This system reduces to (1), (2), and (3) when $a = 1$.

This set of transforms can be justified by the arguments used in Theorem 1 if we assume that $h(u)$ belongs to $L^2(0, \infty)$. The Mellin transforms required for the proof are

$$e^{-u^a} \text{ and } \frac{1}{a}\Gamma\left(\frac{s}{a}\right), \quad \frac{a\Gamma(1/a)}{(1+u^a)^{1/a}} \text{ and } \Gamma\left(\frac{s}{a}\right)\Gamma\left(\frac{1}{a}-\frac{s}{a}\right).$$

7. General iteration formulae. For the rest of this paper we shall find it convenient to use the following terminology. We write p for $p(x)$ or $p(u)$, Mel p for the Mellin transform of $p(u)$, P for $P(s)$, and $\bar{P}$ for $P(1-s)$. Thus

$$\text{Mel}\, p = P \text{ and } \overline{\text{Mel}\, p} = \bar{P}.$$

We shall also write

$$[p, q]\{u\} = [p, q] = \int_0^\infty p(ut)q(t)dt. \tag{36}$$

The form $[p,q]\{u\}$ will be used only when it is necessary to specify the variable u.

If $p(x)$ and $q(x)$ both belong to $L^2(0, \infty)$ then from (6) and **2.3** we have

$$[p, q] = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} Pu^{-s}\bar{Q}ds,$$

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where P and Q both belong to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. If, in addition, either P or Q is bounded on the line $s = \frac{1}{2} + i\tau$ then, as in the proof of Theorem 1, we can infer that $P\bar{Q}$ belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$, hence that $[p,q]$ belongs to $L^2(0, \infty)$ and, finally, that

$$\text{Mel}\,[p, q] = P\bar{Q} = \text{Mel}\,p\,\overline{\text{Mel}\,q}. \tag{37}$$

We can now prove

THEOREM 4. *If* (i) $p(x)$, $q(x)$, *and* $h(x)$ *all belong to* $L^2(0, \infty)$, (ii) P *and* Q *are bounded on the line* $s = \frac{1}{2} + i\tau$, *and*

$$g(u) = \int_0^\infty p(ut)h(t)dt \tag{38}$$

and

$$f(u) = \int_0^\infty q(ut)g(t)dt, \tag{39}$$

then

$$f(u) = \int_0^\infty [p, q]\{t\}h(ut)dt. \tag{40}$$

If, in addition, $p(u) = q(u)$, *then we also have*

$$\int_0^\infty f(ut)h(t)dt = \int_0^\infty g(ut)g(t)dt. \tag{41}$$

In the terminology just described conditions (38) and (39) become $g = [p,h]$ and $f = [q,g]$ and we are required to prove that $f = [h,[p,q]]$. Equation (41) can also be written in the form $[f,h] = [g,g]$.

To prove (40), apply the arguments preceding (37) to (38), using conditions (i) and (ii). We then deduce first that $G = P\bar{H}$ and secondly that $g(u)$ belongs to $L^2(0, \infty)$. We may further deduce from (39) that $F = Q\bar{G}$ and that $f(u)$ also belongs to $L^2(0, \infty)$.

We now have

$$F = Q\bar{G} = Q\overline{P\bar{H}} = Q\bar{P}H.$$

From the conditions of integrable square, some of which have been assumed and some proved, we may apply (6) and (37) to this result and rewrite it in the form $\text{Mel}\,f = \text{Mel}\,[h,r]$, where $\text{Mel}\,r = P\bar{Q} = \text{Mel}\,[p,q]$. Hence $f = [h,[p,q]]$, which is equivalent to (40).

If $p = q$ then $P = Q$ and from $G = P\bar{H}$ and $F = Q\bar{G}$ we deduce that $F\bar{H} = G\bar{G}$. From (37) this may be written in the form $\text{Mel}\,[f,h] = \text{Mel}\,[g,g]$. Hence $[f,h] = [g,g]$, which is equivalent to (41).

Theorem 4 contains the following results as special cases. When

$$p(x) = q(x) = e^{-x}$$

it reduces to equations (1), (2), and (3). When

$$p(x) = q(x) = e^{-x}x^a$$

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it reduces to Theorem 3. When

$$p(x) = q(x) = ae^{-x^a}$$

it reduces to the system of equations at the end of §6.

8. The Fourier kernel transform. In Theorem 1 it was shown that the factors e^{-xt} in (1) and (2) could be replaced by the Laplace transforms of Fourier kernels. In this section we shall show that Theorem 4 is capable of an analogous generalization.

For this purpose we shall introduce two operators T_1 and T_2. Let $M = M(s)$ and $N = N(s)$ be two functions which are bounded on the line $s = \frac{1}{2} + i\tau$ and which satisfy the functional equation

$$MN̄ = 1. \tag{42}$$

Then by using (9) we can find functions $m_1(x)$ and $n_1(x)$ which form the basis of general Fourier transforms of the type (11) [**5**, p. 226]. We define the operators as follows:

$$T_1 p\{x\} = T_1 p = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} Px^{-s}\bar{M}ds \tag{43}$$

and

$$T_2 q\{x\} = T_2 q = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} Qx^{-s}\bar{N}ds. \tag{44}$$

The forms $T_1p\{x\}$ and $T_2q\{x\}$ will be used only when it is necessary to specify the variable x.

As an illustration, when $M = N = 1$ and $P = Q = \Gamma(s)$ then $T_1p = T_2q = e^{-x}$.

The Parseval equation (6) is often true even when the conditions of **2.2** are not fulfilled. Assuming that it is true for (43) and (44) we should then have

$$\begin{aligned} T_1 p &= \int_0^\infty p(xt)m(t)dt, \\ T_2 q &= \int_0^\infty q(xt)n(t)dt. \end{aligned} \tag{45}$$

If, for example, $p(x) = q(x) = e^{-x}$ then T_1p and T_2q reduce to the Laplace transforms of $m(x)$ and $n(x)$ respectively.

Again, since M and N are bounded on $s = \frac{1}{2} + i\tau$, M/s and N/s both belong to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. Hence if sP belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ we may write $sPx^{-s}\bar{M}/s$ for the right-hand integrand of (43) and use (6). We can then deduce in general that

$$T_1 p = \int_0^\infty - xp'(xt)m_1(t)dt, \tag{46}$$

where $m_1(t)$ is defined by (9) and the prime denotes differentiation. Similarly, if sQ belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ then we have, in general,

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$$(47) \qquad T_2 q = \int_0^\infty - xq'(xt)n_1(t)dt.$$

These bear the same relation to (45) as the general Laplace transform (14) bears to the ordinary Laplace transform (12). When $m(x)$ and $n(x)$ exist it is usually possible to prove that (46) and (47) reduce to (45) (on integrating by parts).

The advantage of defining T_1 and T_2 by (43) and (44) instead of by (45) or by (46) and (47) lies in the great generality of (43) and (44). Thus if $M = N = 1$, $m(t)$ and $n(t)$ do not exist; but if $p(x)$ and $q(x)$ belong to $L^2(0, \infty)$ then, from **2.12** and (43), (44), we have $T_1p = p$ and $T_2q = q$.

Another advantage lies in the fact that important deductions can be made from (43) and (44) with the help of reasonably simple assumptions. The most useful one from our point of view is as follows:

If $p(x)$ and $q(x)$ both belong to $L^2(0, \infty)$ and M and N are bounded on the line $s = \frac{1}{2} + i\tau$ then T_1p and T_2q also belong to $L^2(0, \infty)$ and

$$(48) \qquad \begin{aligned} \text{Mel } T_1 p &= P\bar{M}, \\ \text{Mel } T_2 q &= Q\bar{N}. \end{aligned}$$

For $p(x)$ belongs to $L^2(0, \infty)$ and so, from **2.11**, P belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. Hence, since M is bounded, $P\bar{M}$ also belongs to $L^2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. The results stated then follow from **2.12**.

We now prove our final theorem.

THEOREM 5. *If* (i) $p(x)$, $q(x)$, *and* $h(x)$ *all belong to* $L^2(0, \infty)$, (ii) P *and* Q *are bounded on the line* $s = \frac{1}{2} + i\tau$, (iii) M *and* N *are bounded on the line* $s = \frac{1}{2} + i\tau$ *and satisfy the equation* $M\bar{N} = 1$, *and*

$$(49) \qquad g(u) = \int_0^\infty T_1 p\{ut\}h(t)dt$$

and

$$(50) \qquad f(u) = \int_0^\infty T_2 q\{ut\}g(t)dt,$$

then

$$(51) \qquad f(u) = \int_0^\infty [p, q]\{t\}h(ut)dt.$$

If, in addition, $p(u) = q(u)$ *and* $M = N$, *then we also have*

$$(52) \qquad \int_0^\infty f(ut)h(t)dt = \int_0^\infty g(ut)g(t)dt.$$

We first prove (51) by means of Theorem 4. Since $p(x)$ belongs to $L^2(0, \infty)$ it follows from (48) that T_1p belongs to $L^2(0, \infty)$ and that Mel $T_1p = P\bar{M}$. Also from conditions (ii) and (iii) it is evident that $P\bar{M}$ is bounded on the line $s = \frac{1}{2} + i\tau$. Similar remarks apply to T_2q and to Mel $T_2q = Q\bar{N}$. Hence conditions (i) and (ii) of Theorem 4 are satisfied. On applying the results of that theorem to (49) and (50) we find that

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$$f(u) = \int_0^\infty [T_1 p, T_2 q]\{t\} h(ut)dt. \tag{53}$$

The proof is then completed if we can show that $[T_1p, T_2q] = [p, q]$.

Since $p(x)$, $q(x)$, T_1p, and T_2q all belong to $L^2(0, \infty)$ it follows from (37) that

$$\begin{aligned} \text{Mel } [T_1 p, T_2 q] &= \text{Mel } T_1 p \,.\, \overline{\text{Mel } T_2 q} & \\ &= P\bar{M} \,.\, \bar{Q}N & \text{from (48)} \\ &= P\bar{Q} & \text{from condition (iii) above} \\ &= \text{Mel } [p, q] & \end{aligned}$$

from (37) again. Hence

$$[T_1 p, T_2 q] = [p, q]$$

and the proof of (51) is completed.

To prove (52), from (6) and (49) we have

$$G = \text{Mel } T_1p \,.\, \bar{H} = P\bar{M}\bar{H}$$

and from (6) and (50) we have

$$F = \text{Mel } T_2q \,.\, \bar{G} = Q\bar{N}\bar{G}.$$

But we now have two extra conditions: $p(x) = q(x)$, from which we derive $P = Q$, and $M = N$. Hence $F\bar{H} = G\bar{G}$. From (37) this may be written in the form Mel $[f, h]$ = Mel $[g, g]$ and so $[f, h] = [g, g]$. Finally, from (36) this is equivalent to (52).

Theorem 5 contains most of the other theorems as special cases. When $p(x) = q(x) = e^{-x}$, it reduces to Theorem 1, and when $M = N = 1$ it reduces to Theorem 4.

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