SOME INTEGRALS INVOLVING E-FUNCTIONS

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§ 1. *Introductory.* This paper describes a simple method of evaluating integrals of a type some cases of which have been discussed by Dr. Ragab.

The following formulae are required in the proofs

If $p \leqq q+1$, $z \neq 0$,

$$E(p\,;\alpha_r:q\,;\rho_s:z)=\prod_{r=1}^{p}\Gamma(\alpha_r)\{\prod_{s=1}^{q}\Gamma(\rho_s)\}^{-1}F(p\,;\alpha_r:q\,;\rho_s:\,-1/z). \quad \text{......(1)}$$

If $p=q+1$, $|z|>1$ in this formula.

If $R(\alpha_{p+1})>0$,

$$\int_0^\infty e^{-\mu}\mu^{\alpha_{p+1}-1}E(p\,;\alpha_r:q\,;\rho_s:z/\mu)\,d\mu=E(p+1\,;\alpha_r:q\,;\rho_s:z). \quad \text{......(2)}$$

If $R(\alpha+\gamma)>0$, $R(\beta+\gamma)>0$,

$$\int_0^\infty e^{-\lambda}\lambda^{\gamma-1}E(\alpha,\beta::\lambda)\,d\lambda=\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)/\Gamma(\alpha+\beta+\gamma) \quad \text{......(3)}$$

$$\frac{1}{2\pi i}\int e^{\zeta}\zeta^{-z}\,d\zeta=\frac{1}{\Gamma(z)}, \quad \text{......(4)}$$

where the contour starts from $-\infty$ on the ξ-axis, passes round the origin in the positive direction, and ends at $-\infty$ on the ξ-axis, the initial value of amp ζ being $-\pi$.

If $R(m\pm n)>0$,

$$\int_0^\infty K_n(\lambda)\lambda^{m-1}\,d\lambda=2^{m-2}\Gamma\left(\frac{m+n}{2}\right)\Gamma\left(\frac{m-n}{2}\right). \quad \text{......(5)}$$

If m is a positive integer,

$$\Gamma(mz)=m^{mz-\frac{1}{2}}(2\pi)^{\frac{1}{2}-\frac{1}{2}m}\Gamma(z)\Gamma\left(z+\frac{1}{m}\right)\dots\Gamma\left(z+\frac{m-1}{m}\right). \quad \text{......(6)}$$

§ 2. *Method of Proof.* The first formula to be proved is as follows. If $R(\alpha)>0$, $R(\rho-\alpha)>0$, and m is a positive integer,

$$\int_0^1\lambda^{\alpha-1}(1-\lambda)^{\rho-\alpha-1}E(p\,;\alpha_r:q\,;\rho_s:z/\lambda^m)\,d\lambda$$
$$=\Gamma(\rho-\alpha)m^{\alpha-\rho}E(p+m\,;\alpha_r:q+m\,;\rho_s:z), \quad \text{......(7)}$$

where $\alpha_{p+\nu+1}=(\alpha+\nu)/m$, $\rho_{q+\nu+1}=(\rho+\nu)/m$, $\nu=0,1,2,\dots,m-1$.

To prove this, assume that $q\geqq p$, so that (1) can be applied for all non-zero values of z. The L.H.S. of (7) can then be written

$$\Pi\Gamma(\alpha_r)[\Pi\Gamma(\rho_s)]^{-1}\int_0^1\lambda^{\alpha-1}(1-\lambda)^{\rho-\alpha-1}F(p\,;\alpha_r:q\,;\rho_s:\,-\lambda^m/z)\,d\lambda.$$

On expanding in powers of $1/z$ and integrating term by term this becomes

$$B(\alpha,\rho-\alpha)\Pi\Gamma(\alpha_r)[\Pi\Gamma(\rho_s)]^{-1}F(p+m\,;\alpha_r:q+m\,;\rho_s:\,-1/z).$$

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Now from (6), with α/m in place of z,

$$\Gamma(\alpha)=m^{\alpha-\frac{1}{2}}(2\pi)^{\frac{1}{2}-\frac{1}{2}m}\,\Gamma\left(\frac{\alpha}{m}\right)\Gamma\left(\frac{\alpha+1}{m}\right)\ldots\Gamma\left(\frac{\alpha+m-1}{m}\right),$$

with a similar formula for $\Gamma(\rho)$. On applying (1) it is seen that the integral has the value given in formula (7).

Finally, to remove the restrictions on p, replace z in (7) by z/μ and apply formula (2) as often as is required.

On applying the same procedure it is found that

$$\frac{1}{2\pi i}\int e^{\zeta}\,\zeta^{-\rho}\,E(p\,;\alpha_r:q\,;\rho_s:z\zeta^m)\,d\zeta$$
$$=(2\pi)^{\frac{1}{2}m-\frac{1}{2}}\,m^{\frac{1}{2}-\rho}\,E(p\,;\alpha_r:q+m\,;\rho_s:zm^m), \qquad \ldots\ldots(8)$$

where the contour is that of formula (4).

Similarly, if $R(\alpha)>0$, and assuming initially that $q\geqq p+m$,

$$\int_0^\infty e^{-\lambda}\,\lambda^{\alpha-1}\,E(p\,;\alpha_r:q\,;\rho_s\,;z/\lambda^m)\,d\lambda$$
$$=(2\pi)^{\frac{1}{2}-\frac{1}{2}m}\,m^{\alpha-\frac{1}{2}}\,E(p+m\,;\alpha_r:q\,;\rho_s:z/m^m). \qquad \ldots\ldots(9)$$

Using formula (5), and assuming initially that $q\geqq p+2m$, it can be shown that, if $R(k\pm n)>0$,

$$\int_0^\infty K_n(\lambda)\,\lambda^{k-1}\,E(p\,;\alpha_r:q\,;\rho_s:z/\lambda^{2m})\,d\lambda$$
$$=(2\pi)^{1-m}\,2^{k-2}\,m^{k-1}\,E\{p+2m\,;\alpha_r:q\,;\rho_s:z/(2m)^{2m}\}, \qquad \ldots\ldots(10)$$

where $\alpha_{p+\nu+1}=(k+n+2\nu)/(2m)$, $\alpha_{p+m+\nu+1}=(k-n+2\nu)/(2m)$, $\nu=0,\,1,\,2,\,\ldots,\,m-1$.

Again, on applying (3) and assuming initially that $q\geqq p+m$, it is found that, if $R(k+\gamma)>0$, $R(k+\delta)>0$,

$$\int_0^\infty e^{-\lambda}\,\lambda^{k-1}\,E(\gamma,\,\delta::\lambda)\,E(p\,;\alpha_r:q\,;\rho_s:z/\lambda^m)\,d\lambda$$
$$=(2\pi)^{\frac{1}{2}-\frac{1}{2}m}\,m^{k-\frac{1}{2}}\,\Gamma(\gamma)\,\Gamma(\delta)\,E(p+2m\,;\alpha_r:q+m\,;\rho_s:z/m^m), \qquad \ldots\ldots(11)$$

where $\alpha_{p+\nu+1}=(\gamma+k+\nu)/m$, $\alpha_{p+m+\nu+1}=(\delta+k+\nu)/m$, $\rho_{q+\nu+1}=(\gamma+\delta+k+\nu)/m$, $\nu=0,\,1,\,2,\,\ldots,\,m-1$.

Finally,

$$\int e^{\zeta}\,\zeta^{k-1}\,E(p\,;\alpha_r:q\,;\rho_s:z/\zeta^m)\,d\zeta$$
$$=(2\pi)^{\frac{1}{2}-\frac{1}{2}m}m^{\,k-\frac{1}{2}}\begin{Bmatrix} e^{i\pi k}\,E(p+m\,;\alpha_r:q\,;\rho_s:ze^{-im\pi}/m^m) \\ -e^{-i\pi k}\,E(p+m\,;\alpha_r:q\,;\rho_s:ze^{im\pi}/m^m \end{Bmatrix}, \qquad \ldots\ldots(12)$$

where $\alpha_{p+\nu+1}=(k+\nu)/m$, $\nu=0,\,1,\,2,\,m-1$, and the contour is that of formula (4).

To prove this, assume that $R(k)>0$ and $q\geqq p+m$; then deform the contour into a small circle about the origin with radius tending to zero and the negative real axis described twice.

Formulae (9), (10), (11) were given by Ragab in somewhat different forms in Part III of these *Proceedings*, pages 131, 119 and 134 respectively.

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