BULL. AUSTRAL. MATH. SOC. 58A99, 53C05
VOL. 10 (1974), 71-77.

Recurrent tensors and holonomy group

D.K. Datta

Let M be a connected C^∞ manifold of dimension n with a linear connection Γ. A method is being introduced here to study the action of the holonomy group and the restricted holonomy group of Γ on a recurrent tensor. The main result of this paper is that if the recurrence covector W of a recurrent tensor S on M is an exact form then the tensor S is invariant under the holonomy group of Γ and if W is a closed form then S is invariant under the restricted holonomy group of Γ. In the last section, this result is applied to some particular cases including the case of a riemannian manifold with recurrent curvature.

Introduction

Let M be a connected C^∞ manifold of dimension n and let a linear connection Γ be given on M. A tensor field S of type (r, s) on M is said to be recurrent if [4],

$$\nabla S = W \otimes S ,$$

where W is a differential 1-form on M and ∇ denotes covariant derivative with respect to Γ. The 1-form W is called the recurrence covector.

The purpose of this paper is to find the conditions under which a

---

Received 17 September 1973. This paper was written while the author was visiting the Australian National University as an Honorary Fellow. He is grateful to the Department of Mathematics for its hospitality.

recurrent tensor will be invariant by the action of the holonomy group or the restricted holonomy group of Γ .

1. Preliminaries

For a point p of M let $(x_1, \ldots, x_n)$ be a local coordinate system in a coordinate neighbourhood U of p . Let $S^{i_1 \ldots i_r}_{j_1 \ldots j_s}$ and W_i respectively be the components of S and W with respect to the coordinate system on U . Then the condition for S to be recurrent can be written as

$$\nabla_h S^{i_1 \ldots i_r}_{j_1 \ldots j_s} = W_h S^{i_1 \ldots i_r}_{j_1 \ldots j_s} .$$

For the 1-form W , we have the relation

$$dW = \mathrm{Alt}(\nabla W) ,$$

where dW denotes the exterior derivative of W and $\mathrm{Alt}(\nabla W)$ is given locally by $\mathrm{Alt}(\nabla W) = \nabla_j W_i - \nabla_i W_j$. This shows that W is a closed form if and only if ∇W is symmetric.

If W is an exact form we have the following

THEOREM 1.1. *Let S be a recurrent tensor field on a connected C^∞ manifold M such that the recurrence covector is an exact form. Then there exists a function σ on M such that σS is a parallel tensor field on S .*

Proof. Let $\nabla = W \otimes S$. If W is an exact form then there exists a function f on M such that $W = df$. Taking $\sigma = e^{-f}$, it follows that $\nabla\left(e^{-f} S\right) = 0$.

2. Main theorem

Let p be a point in M and let $\tau : t \to x(t)$, $(0 \le t \le 1)$, be a loop at p , that is, τ is a sectionally smooth closed curve with $\tau(0) = \tau(1) = p$. Since any sectionally smooth curve in a smooth manifold may be regarded as one each of whose smooth sections lies in a coordinate neighbourhood, it will be sufficient for our purpose to assume that τ is contained in a single coordinate neighbourhood U of p. Let S be a recurrent tensor with recurrence covector W. We shall denote by $S_{x(t)}$ the assignment of the tensor field S at $x(t) = \left(x_1(t), \ldots, x_n(t)\right)$. Then along τ, we get

$$\begin{aligned}\nabla_h S_{x(t)} \frac{dx_h}{dt} &= W_h\left(x(t)\right) S_{x(t)} \frac{dx_h}{dt} \\ &= f(t) S_{x(t)} ,\end{aligned}$$

where $f(t) = W_h\left(x(t)\right) \dfrac{dx_h}{dt}$.

To determine the action of the holonomy group of Γ on the recurrent tensor S, we construct a tensor field A of the same type as S and defined along τ such that

$$A_{x(0)} = S_{x(0)} = S_p$$

and

$$\nabla_h A_{x(t)} \frac{dx_h}{dt} = 0 .$$

The functions $A_{x(t)}$ defined along τ by

$$A_{x(t)} = \exp\left(-\int_0^t f(t)dt\right) S_{x(t)}$$

will give such a tensor field along τ since $A_{x(0)} = S_{x(0)}$ and

$$\begin{aligned}\nabla_h A_{x(t)} \frac{dx_h}{dt} &= \nabla_h \left[\exp\left(-\int_0^t f(t)dt\right) S_{x(t)}\right] \frac{dx_h}{dt} \\ &= \exp\left(-\int_0^t f(t)dt\right)\left(\nabla_h S_{x(t)}\right) \frac{dx_h}{dt} + \nabla_h\left(\exp\left(-\int_0^t f(t)dt\right)\right) S_{x(t)} \frac{dx_t}{dt} \\ &= \exp\left(-\int_0^t f(t)dt\right) f(t) S_{x(t)} - \exp\left(-\int_0^t f(t)dt\right) f(t) S_{x(t)} \\ &= 0 .\end{aligned}$$

Therefore, if we denote by τ the element in the holonomy group of Γ

https://doi.org/10.1017/S0004972700040648 Published online by Cambridge University Press

with reference point p corresponding to the curve $\tau : t \to x(t)$ then

$$\begin{aligned}\tau(S_p) &= \tau(A_p) \\ &= \exp\left(-\int_0^1 f(t)dt\right)S_{x(1)} \\ &= \rho S_p ,\end{aligned}$$

where $\rho = \exp\left(-\int_0^1 f(t)dt\right) = \exp\left(-\int_\tau f(t)dt\right)$.

ρ will be different for different elements of the holonomy group of Γ with reference point p . The recurrent tensor S will be invariant under τ if $\rho = 1$.

From the theory of integration of forms, it follows that $\rho = \exp\left(-\int_\tau f(t)dt\right) = \exp\left(-\int_\tau W\right) = 1$, that is, $\int_\tau W = 0$ if

(i) W is a closed form and τ is homotopic to zero, that is, τ is an element of the restricted holonomy group of Γ , or if,

(ii) W is an exact form.

Since the holonomy group of Γ with different reference points of a connected C^∞ manifold are isomorphic, we have proved the following:

THEOREM 2.1. *Let M be a connected C^∞ manifold with a linear connection Γ and let S be a recurrent tensor on M . If the recurrence covector W of S is a closed form then S is invariant under the restricted holonomy group of Γ and if W is an exact form then S is invariant under the holonomy group of Γ .*

3. Applications

In this section we discuss applications of Theorem 2.1 to some interesting particular cases. For example, if S is a vector field on M then we get

THEOREM 3.1. *Let M be a connected C^∞-manifold with a linear connection Γ and let S be a vector field on M . Then the holonomy*

https://doi.org/10.1017/S0004972700040648 Published online by Cambridge University Press

group of Γ *has an invariant* 1-*plane*.

Proof. This follows from the relation $\tau(S_p) = \rho S_p$ as shown in §2. Clearly, the 1-plane generated by S is invariant under the holonomy group of Γ.

In [2] Sasaki and Goto have proved certain theorems when the restricted holonomy group of a complete riemannian connection fixes r vectors and these r vectors span an r-dimensional plane invariant under the holonomy group of the connection. In the following theorem we give an example of a set of vectors which will satisfy these conditions with respect to a linear connection on a connected C^∞ manifold.

THEOREM 3.2. *Let* M *be a connected* C^∞ *manifold with a linear connection* Γ *and let* $\{V_A \mid A = 1, \ldots, r\}$ *be a set of vector fields on* M *such that* $V_1, \ldots, V_r$ *are linearly independent at least at one point of* M *and* $\nabla V_A = W \otimes V_A$ $(A = 1, \ldots, r)$ *for some closed form* W. *Then*

(a) $V_1, \ldots, V_r$ *are invariant under the restricted holonomy group of* Γ,

(b) $V_1, \ldots, V_r$ *are linearly independent at every point of* M *and*

(c) span of $\{V_1, \ldots, V_r\}$ *is invariant under the holonomy group of* Γ.

Proof. Part *(a)* follows from Theorem 2.1 since each of $V_1, \ldots, V_r$ is a recurrent tensor with a closed form as recurrence covector.

Part *(b)* follows from Corollary 2.3 of Ludden [1] since $V_1, \ldots, V_r$ form a set of perfect tensors.

Part *(c)* follows from Theorem 3.1.

For a riemannian connection with $\nabla R = W \otimes R$, where R is the curvature tensor, it is known that ∇W is symmetric so that W is a closed form [3, 4]. Therefore, we get

THEOREM 3.3. *For a riemannian manifold with recurrent curvature, the curvature tensor is invariant under the restricted holonomy group of the* *riemannian connection.*

For a riemannian manifold with recurrent curvature, it can be shown that other tensors obtained from the curvature tensor are also recurrent with the same recurrence covector. According to Theorem 2.1 and the fact that ∇W is symmetric, it follows that all such tensors are also invariant under the restricted holonomy group of the riemannian connection.

Wong [5] has considered various types of recurrent tensors which satisfy either the condition that the recurrence covector is a closed form or the condition that the recurrence covector is an exact form. For example, if a recurrent tensor of type (r, r) is not almost parallel (a recurrent tensor S on M is almost parallel if there exists on M some linear connection with respect to which S is parallel) then the recurrence covector is a closed form. Therefore, we get

THEOREM 3.4. *Let S be a recurrent tensor of type (r, r) on a connected C^∞ manifold with a linear connection Γ. If S is not almost parallel then S is invariant under the restricted holonomy group of Γ.*

Also, if a recurrent tensor of type (r, r) on M is not completely nilpotent [5] then the recurrence covector is globally a gradient, that is, the recurrence covector is an exact form. Thus:

THEOREM 3.5. *Let S be a recurrent tensor of type (r, r) on a connected C^∞ manifold M with a linear connection Γ. If S is not completely nilpotent then S is invariant under the holonomy group of Γ.*

The main theorem of this paper may also be generalized to the case of a set of perfect tensors [1].

References

[1] Gerald D. Ludden, "Perfect tensors on a manifold", *J. Differential Geometry* 2 (1968), 41-53.

[2] Shigeo Sasaki and Morikuni Goto, "Some theorems on holonomy groups of Riemannian manifolds", *Trans. Amer. Math. Soc.* 80 (1955), 148-158.

[3] A.G. Walker, "On Ruse's spaces of recurrent curvature", *Proc. London Math. Soc.* (2) 52 (1951), 36-64.

[4] Yung-Chow Wong, "Recurrent tensors on a linearly connected differentiable manifold", *Trans. Amer. Math. Soc.* 99 (1961), 325-341.

[5] Yung-Chow Wong, "Existence of linear connections with respect to which given tensor fields are parallel or recurrent", *Nagoya Math. J.* 24 (1964), 67-108.

Department of Mathematics,
University of Rhode Island,
Kingston,
USA

and

Department of Mathematics,
Institute of Advanced Studies,
Australian National University,
Canberra, ACT.

https://doi.org/10.1017/S0004972700040648 Published online by Cambridge University Press