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WHEN IS THE ITERATE OF A FORMAL POWER SERIES ODD?

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Abstract

The formal power series (fps) $f(z) = \sum_{i=1}^{\infty} a_i z^i$ is *homozygous mod k* if $a_i \neq 0$ and $a_j \neq 0$ implies $i \equiv j \bmod k$. This generalizes even and odd fps. If f is homozygous mod k then all iterates of f ($f_n = f \circ f_{n-1}$) are also homozygous mod k, but the converse is false—there are many non-odd fps f for which $f(f(z)) = z$. It is shown that if f is not homozygous mod k but f_n is homozygous, then $f_{nr}(z) = z$ for some r. If all coefficients are real then, in fact, $f(f(z)) = z$.

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1. Introduction

A formal power series (fps) may be classified as even or odd if all exponents which appear are even or odd integers. Any iterate of an even or odd fps is itself an even or odd fps, but the converse is not true. For example, the fps

$$f(z) = c^{-1} \sum_{i=1}^{\infty} (-cz)^i = -z(1+cz)^{-1}$$

satisfies $f(f(z)) = z$. Roughly speaking, this will turn out to be the only type of exception; if an fps function which is not odd (or even) has an iterate which *is* odd (or even), then some further iterate is z.

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2. Notation and preliminaries

Throughout we shall consider formal power series with fixpoint 0 (that is, $f(0) = 0$). All theorems may be easily recast to allow for different fixpoints. Iteration without fixpoints requires conditions on convergence which remove us from the realm of the fps.

An fps f, $f(z) = \sum_{i=1}^{\infty} a_i z^i$, a_i complex, will be called *homozygous mod k* if $a_i \neq 0$, $a_j \neq 0$ implies $i \equiv j \bmod k$. If f is not homozygous mod k it will be called *hybrid mod k*. Every fps is homozygous mod 1, even or odd fps are homozygous mod 2. The composition of the two fps f and g is written $f \circ g$ and the iterates of f are defined inductively by $f_n = f \circ f_{n-1}$. If $a_1 \neq 0$ then the coefficients of an fps g satisfying $f(g(z)) = z$ may be determined recursively; $g(z) = a_1^{-1} z + \dots$ is written $f^{-1}(z)$. The fps f and g commute if $f \circ g = g \circ f$. Any two iterates of f commute.

The first lemma is trivial but useful. The proof is omitted.

LEMMA 1. *The following are equivalent.*

(i) *The fps $f(z)$ is homozygous mod k.*

(ii) *For some primitive* kth *root of unity ε and some integer r, $0 \leqslant r \leqslant k-1$, $f(\varepsilon z) = \varepsilon^r f(z)$.*

(iii) *One can write $f(z) = \sum_{j=0}^{\infty} a_{jk+r} z^{jk+r}$ $(1 \leqslant r \leqslant k)$.*

LEMMA 2. *If f is homozygous mod k then all iterates of f are homozygous mod k.*

PROOF. If $f(\varepsilon z) = \varepsilon^r f(z)$ then $f(\varepsilon^m z) = \varepsilon^{rm} f(z)$ by induction, hence

$$f_n(\varepsilon z) = \varepsilon^{r^n} f(z).$$

The remainder of this paper is devoted to finding a converse for Lemma 2. We shall need several results of Baker (1962). Baker's proofs are relatively short and will not be reproduced here.

THEOREM. (Baker (1962).)

(i) *If g and h commute, where $g(z) = \sum_{i=1}^{\infty} a_i z^i$ and $h(z) = z + \sum_{i=m+1}^{\infty} b_i z^i$, $b_{m+1} \neq 0$, $m \geqslant 1$, then $a_1^m = 1$.*

(ii) *If, in* (i), *$a_1 = 1$, then $a_2 = \dots = a_m = 0$ while a_{m+1} is arbitrary and to each λ, there is a unique fps g, denoted h_λ, with $a_{m+1} = \lambda b_{m+1}$.*

(iii) *For each a_1 with $a_1^m = 1$, there is exactly one fps $H(z) = \sum_{i=1}^{\infty} c_i z^i$ with $c_1 = a_1$, $H_m(z) = z$ and $H \circ h = h \circ H$. The series $g(z) = \sum_{i=1}^{\infty} a_i z^i$ with $a_1 z$ as leading term are precisely $H \circ h_\lambda$ for various λ.*

3. Spadework

Suppose $f(z) = \sum_{i=1}^{\infty} a_{n_i} z^{n_i}$ $(a_{n_i} \neq 0)$ is a hybrid fps mod k. Write $f \sim (n_1, n_j)_k$ where j is the smallest index for which $n_j \not\equiv n_1$ mod k.

LEMMA 3. *If $f \sim (c, d)_k$ and $c > 1$, then any iterate f_m is also hybrid mod k and $f_m \sim (c^m, c^m + d - c)_k$.*

PROOF. Suppose $f(z) = \sum_{i=1}^{\infty} a_{n_i} z^{n_i}$ and $f_s(z) = \sum_{k=1}^{\infty} b_{m_k} z^{m_k}$ where $a_{n_i}, b_{m_k} \neq 0$. Then $f_{s+1}(z) = \sum_{i=1}^{\infty} a_{n_i} (\sum_{k=1}^{\infty} b_{m_k} z^{m_k})^{n_i}$ and so z^r will appear in $f_{s+1}(z)$ only if r is a sum of n_i terms each of which is an m_k. Even so, there may be cancellations: $f(z) = \sum_{i=1}^{\infty} (-1)^i z^i$, $f(f(z)) = z$. The lemma will be proved by induction on m; the result is true for $m = 1$. Assume it for $m = s$; then, keeping the previous notation, $n_1 = c$, $m_1 = c^s$, $m_{j'} = c^s + d - c$ and $m_i \equiv m_1$ mod k for $1 \leqslant i < j'$. The smallest exponent in f_{s+1} will come from the smallest possible sum $\sum_{u=1}^{n_i} m_{k_u}$, which is clearly c^{s+1}. The coefficient of $z^{c^{s+1}}$ derived from $f \circ f_s$ is $a_c (b_{c^s})^c$ which does not vanish. If $n_i \equiv n_1$ and each $m_{k_u} \equiv m_1$, then the sum is congruent to c^{s+1}. The smallest exponent not congruent to c^{s+1} thus arises either when $n_i \not\equiv n_1$ and each $m_{k_u} \equiv m_1$ (giving $c^s d$) or when $n_i = n_1$, one m_{k_u} is m_j, and the rest are m_1 (giving $c^{s+1} + d - c$). The latter is smaller and the coefficient of $z^{c^{s+1}+d-c}$ derived in this way is $c a_c (b_{c^s})^{c-1} b_{c^s+d-c}$, which does not vanish. Accordingly, the induction hypothesis is verified for $m = s + 1$.

LEMMA 4. *Suppose $f(z) = \sum_{i=1}^{\infty} a_{n_i} z^{n_i}$, $a_{n_i} \neq 0$ and $f \sim (1, t)_k$. Then the coefficients of z and z^t in f_n are a_1^n and $a_t a_1^{n-1} \sum_{i=0}^{n-1} (a_1^{t-1})^i$ respectively.*

PROOF. The arguments of the last lemma demonstrate that any power of z less than t which appears in f_n is congruent to 1 mod k. This lemma is also proved by induction on n and is evidently true for $n = 1$. Assume it true for $n = r$. We have $f(z) = \sum_{i=1}^{\infty} a_{n_i} z^{n_i}$, $n_1 = 1$, $a_1 \neq 0$, $n_j = t$, $t \not\equiv 1$ mod k, $n_i \equiv 1$ mod k for $1 \leqslant i < j$ and $f_r(z) = \sum_{i=1}^{\infty} b_{m_i} z^{m_i}$, $m_1 = 1$, $b_1 = a_1^r$, $m_{j'} = t$, $m_i \equiv 1$ mod k for $1 \leqslant i < j'$. Also, $f_{r+1}(z) = f(f_r(z)) = a_1 f_r(z) + \sum_{i=2}^{\infty} a_{n_i} (f_r(z))^{n_i}$. If $d(s)$ is the coefficient of z^t in $f_s(z)$, then $d(1) = a_t$ and $d(r) = a_t a_1^{r-1} \sum_{i=0}^{r-1} (a_1^{t-1})^i$ by hypothesis. Considering how z^t can appear in $f_{r+1}(z)$, we have

$$d(r+1) = a_1 d(r) + a_t (a_1^r)^t = a_t a_1^r \sum_{i=0}^{r-1} (a_1^{t-1})^i + a_t a_1^{rt} = a_t a_1^r \sum_{i=0}^{r} (a_1^{t-1})^i$$

and the lemma is proved.

4. On the inheritance of the homozygous trait

The following theorem is a converse to Lemma 2.

THEOREM. *If f is hybrid mod k and f_n is homozygous mod k, then $f_{ns}(z) = z$ for some integer s. Further, if $f(z) = \sum_{i=1}^{\infty} a_i z^i$, then a_1 is a primitive nsth root of unity. In particular, if f is real, then $f(f(z)) = z$.*

PROOF. Suppose $f \sim (c, t)_k$. In the light of Lemma 3, we must have $c = 1$, and, by Lemma 4, the coefficient of z^t in f_n is $a_1^{n-1} a_t \sum_{i=0}^{n-1} (a_1^{t-1})^i$. As f_n is homozygous mod k and $t \equiv 1 \bmod k$, this coefficient must vanish. Hence, $(a_1^{t-1})^n = 1$ and $a_1^{t-1} \neq 1$. Let $h = f_{n(t-1)}$; h is homozygous with leading term $a_1^{(t-1)n} z = z$. Let ε be a primitive kth root of unity and define $g(z) = \varepsilon^{-1} f(\varepsilon z)$. As f is hybrid mod k, $f(\varepsilon z) \neq \varepsilon f(z)$, so $f \neq g$. The leading term of both f and g is $a_1 z$. Since f_n is homozygous, $f_n(\varepsilon z) = \varepsilon f_n(z)$ and so $g_n(z) = f_n(z)$. It follows that $f_{n(t-1)} = g_{n(t-1)} = h$.

Suppose that $h(z) \neq z$, then $h(z) = z + \sum_{n=m+1}^{\infty} b_n z^n$, $b_{m+1} \neq 0$ and k divides m since h is homozygous. We now apply the several parts of Baker's theorem. Since both f and g commute with h and have the same leading coefficient, there exists $H(z) = \sum_{i=1}^{\infty} c_i z^i$ and $h_\lambda(z) = z + \lambda b_{m+1} z^{m+1} + \ldots$ such that $f = H \circ h_p$ and $g = H \circ h_q$. By construction, the coefficient of z^{m+1} in $H \circ h_\lambda$ is $\lambda c_1 b_{m+1} + c_{m+1}$, but the coefficient of z^{m+1} is the same in f and g because $m + 1 \equiv 1 \bmod k$. Thus $pc_1 b_{m+1} + c_{m+1} = qc_1 b_{m+1} + c_{m+1}$, and, since $c_1 b_{m+1} \neq 0$, $p = q$ so that $f = g$, a contradiction. Therefore, $h(z) = z$.

Finally, suppose $a_1^r = 1$ for $r < n(t-1)$; we may take $rr' = n(t-1)$, $r' > 1$. Let $k = f_r$, then $k_{r'}(z) = z$. If $k(z) \neq z$, then $k(z) = z + a_t z^t + \ldots a_t \neq 0$. By lemma 4, $k_{r'}(z) = z + r' a_t z^t + \ldots \neq z$, a contradiction, so $f_r(z) = z$ in any case. In particular, if f is real, then a_1 is real hence $a_1 = -1$ and $f(f(z)) = z$.

5. Examples

We may characterize the fps f in the last theorem still further. Suppose $f_r(z) = z$ and $f(z) = \varepsilon_r z + \ldots$, where ε_m, in general, is an mth root of unity. There is a canonical fps T such that $f(z) = T^{-1}(\varepsilon_r T(z))$. Indeed, let

$$T(z) = r^{-1} \sum_{j=0}^{r-1} \varepsilon_r^{-j} f_j(z);$$

$T(z) = z + \ldots$ is invertible. It is readily verified that $T(f(z)) = \varepsilon_r T(z)$; this is a solution to the Schröder equation. Further, the expressions of f and T in terms of each other show that they are either both homozygous mod k or both hybrid mod k. (Note that, if an fps h is invertible and homozygous, then so is h^{-1}.)

COROLLARY. *Using the previous notation, f is hybrid mod k and f_n is homozygous mod k (and so $f_{ns}(z) = z$) if and only if $f(z) = T^{-1}(\varepsilon_{ns} T(z))$, where $T = R \circ U$, R is hybrid mod k and homozygous mod s, and U is homozygous mod k, and both R and U are invertible.*

PROOF. If h is invertible and homozygous mod k and $h_1 = h \circ h_2$, then $h_2 = h^{-1} \circ h_1$ so that h_1 and h_2 are either both homozygous or both hybrid. Suppose first that T, R and U are as described and $f(z) = T^{-1}(\varepsilon_{ns} T(z))$. By the previous argument, T is hybrid mod k and so f is too. Let ε_s be the coefficient of z in $f_n(z)$; it is the sth root ε_{ns}^n. We have $f_n(z) = T^{-1}(\varepsilon_{ns}^n T(z)) = T^{-1}(\varepsilon_s T(z)) = U^{-1}(R^{-1}(\varepsilon_s R(U(z))))$. Since R is homozygous mod s, $\varepsilon_s R(U(z)) = R(\varepsilon_s^m(U(z)))$, so $f_n(z) = U^{-1}(\varepsilon_s^m U(z))$ and is homozygous mod k.

On the other hand, suppose f is hybrid mod k and f_n is homozygous mod k. Define T as before and let $U(z) = s^{-1} \sum_{j=0}^{s-1} \varepsilon_s^{-j} f_{nj}(z)$. Then U is homozygous mod k (since f_n is) and $f_n(z) = U^{-1}(\varepsilon_s U(z)) = T^{-1}(\varepsilon_s T(z))$. Therefore,

$$T(U^{-1}(\varepsilon_s U(z))) = \varepsilon_s T(z).$$

Let $R = T \circ U^{-1}$ and $z = U^{-1}(y)$, then $R(\varepsilon_s y) = \varepsilon_s R(y)$, so that R is homozygous mod s. Since T is hybrid and U is homozygous mod k, R is hybrid mod k.

Reference

I. N. Baker (1962), 'Permutable power series and regular iteration', *J. Austral. Math. Soc.* **2**, 265–294.

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