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Homological Properties Relative to Injectively Resolving Subcategories

Zenghui Gao

Abstract. Let $\mathcal{E}$ be an injectively resolving subcategory of left R-modules. A left R-module M (resp. right R-module N) is called $\mathcal{E}$*-injective* (resp. $\mathcal{E}$*-flat*) if $\mathrm{Ext}^1_R(G, M) = 0$ (resp. $\mathrm{Tor}^R_1(N, G) = 0$) for any $G \in \mathcal{E}$. Let $\mathcal{E}$ be a covering subcategory. We prove that a left R-module M is $\mathcal{E}$-injective if and only if M is a direct sum of an injective left R-module and a reduced $\mathcal{E}$-injective left R-module. Suppose $\mathcal{F}$ is a preenveloping subcategory of right R-modules such that $\mathcal{E}^+ \subseteq \mathcal{F}$ and $\mathcal{F}^+ \subseteq \mathcal{E}$. It is shown that a finitely presented right R-module M is $\mathcal{E}$-flat if and only if M is a cokernel of an $\mathcal{F}$-preenvelope of a right R-module. In addition, we introduce and investigate the $\mathcal{E}$-injective and $\mathcal{E}$-flat dimensions of modules and rings. We also introduce $\mathcal{E}$-(semi)hereditary rings and $\mathcal{E}$-von Neumann regular rings and characterize them in terms of $\mathcal{E}$-injective and $\mathcal{E}$-flat modules.

1 Introduction

In classical homological algebra, homological dimensions are important and fundamental research objects, and every homological dimension of modules is defined relative to some certain subcategory of modules. When injective modules are generalized to FP-injective modules, divisible modules, cotorsion modules, and Gorenstein injective modules, respectively, FP-injective, divisible, cotorsion, and Gorenstein injective dimensions appear, and they share many nice properties of injective dimensions (see [2, 6, 11, 17, 19, 21, 23, 26, 27]). Then a natural question is: given a subcategory $\mathcal{X}$ of modules containing all injective modules, can we define $\mathcal{X}$-dimension of modules relative to the subcategory $\mathcal{X}$ in general? However, the answer is negative by the following example. Assume that R has infinite left self-injective dimension and suppose that $\mathcal{X} = \{M \in \mathrm{Mod}\,R : M \text{ is injective}\} \cup \{M \in \mathrm{Mod}\,R : M \cong {}_RR\}$. Then ${}_RR$ has, trivially, $\mathcal{X}$-dimension less than or equal to 1. But in each injective resolution $0 \to R \to E \to N \to 0$, N does not belong to $\mathcal{X}$. This is because N is not injective, as the left self-injective dimension of R is infinite, and N is not isomorphic to R, since, otherwise, the sequence would be split and ${}_RR$ would be injective. To fill the gap, we shall define a new homological dimension, named $\mathcal{E}$-dimension of modules, relative to an injectively resolving subcategory $\mathcal{E}$ of modules. Also, we will extend the ideas of Enochs and Jenda in [8, 9] and introduce $\mathcal{E}$-injectivity and $\mathcal{E}$-flatness relative to

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the subcategory $\mathcal{E}$, which unify some important properties possessed by some known modules.

In Section 2, we give the definition of $\mathcal{E}$-dimension of modules relative to an injectively resolving subcategory $\mathcal{E}$ of modules and a criterion for computing the $\mathcal{E}$-dimension of modules. Some notations and preliminary results are also given.

In Section 3, we introduce the notions of $\mathcal{E}$-injective and $\mathcal{E}$-flat modules and unify some important properties possessed by some known modules. For any ring R, it is shown that a left R-module M is $\mathcal{E}$-injective if and only if M is a kernel of an $\mathcal{E}$-precover $f: A \to B$ of a left R-module B with A injective. Let R be a ring such that $\mathcal{E}$ is a covering subcategory. Then a left R-module M is $\mathcal{E}$-injective if and only if M is a direct sum of an injective left R-module and a reduced $\mathcal{E}$-injective left R-module. We also prove that if $\mathcal{F}$ is a preenveloping subcategory of right R-modules such that $\mathcal{E}^+ \subseteq \mathcal{F}$ and $\mathcal{F}^+ \subseteq \mathcal{E}$, then a finitely presented right R-module M is $\mathcal{E}$-flat if and only if M is a cokernel of an $\mathcal{F}$-preenvelope $K \to F$ of a right R-module K with F flat.

Section 4 is devoted to $\mathcal{E}$-injective and $\mathcal{E}$-flat dimensions of modules and rings. We also characterize a generalization of (semi)hereditary rings, called $\mathcal{E}$-(semi)hereditary rings, in terms of the $\mathcal{E}$-injective and $\mathcal{E}$-flat modules. As applications, some known results can be obtained as corollaries.

Throughout this paper, R is an associative ring with identity, all modules are unitary, $\operatorname{Mod} R$ (resp. $\operatorname{Mod} R^{\mathrm{op}}$) is the category of left (resp. right) R-modules, and all subcategories of $\operatorname{Mod} R$ (resp. $\operatorname{Mod} R^{\mathrm{op}}$) are full and closed under isomorphisms. For an R-module M, the character module $\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+. For unexplained concepts and notations, we refer the reader to [9, 11, 25].

2 $\mathcal{E}$-dimension of Modules and Preliminaries

In this section, we give the definition of $\mathcal{E}$-dimension of modules relative to an injectively resolving subcategory $\mathcal{E}$ of modules and a criterion for computing the $\mathcal{E}$-dimension of modules. Some definitions and preliminary results are also given.

Definition 2.1 ([17]) Let $\mathcal{E}$ be a subcategory of $\operatorname{Mod} R$. Then $\mathcal{E}$ is called *injectively resolving* if it satisfies the following conditions:

(i) $\mathcal{E}$ contains all injective left R-modules;
(ii) $\mathcal{E}$ is closed under extensions;
(iii) $\mathcal{E}$ is closed under cokernels of monomorphisms.

In what follows, $\mathcal{E}$ always denotes an injectively resolving subcategory of left R-modules which is closed under finite direct products and direct summands.

Definition 2.2 For a module M in $\operatorname{Mod} R$, the $\mathcal{E}$-dimension of M, denoted by $\mathcal{E}\text{-dim}\, M$, is defined to be the smallest $n \geq 0$ such that there is an exact sequence

$$0 \longrightarrow M \longrightarrow E_0 \longrightarrow \cdots \longrightarrow E_n \longrightarrow 0$$

with all E_i in $\mathcal{E}$. Set $\mathcal{E}\text{-dim}\, M = \infty$ if no such integer exists.

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Proposition 2.3 *Let* $0 \to M \to E_0 \xrightarrow{f} E_1 \to K \to 0$ *be an exact sequence in* $\operatorname{Mod} R$ *with both* E_0 *and* E_1 *in* $\mathcal{E}$. *Then we have an exact sequence*

$$0 \longrightarrow M \longrightarrow I \longrightarrow E \longrightarrow K \longrightarrow 0$$

in $\operatorname{Mod} R$ *with* I *injective and* E *in* $\mathcal{E}$.

Proof Let $0 \to M \to E_0 \xrightarrow{f} E_1 \to K \to 0$ be an exact sequence in $\operatorname{Mod} R$ with both E_0 and E_1 in $\mathcal{E}$. Since $E_0 \in \mathcal{E}$, there exists an exact sequence $0 \to E_0 \to I \to E_2 \to 0$ with I injective and E_2 in $\mathcal{E}$. Then we have the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & E_0 & \longrightarrow & \operatorname{Im}(f) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & I & \longrightarrow & D & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & E_2 & = & E_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

We consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \operatorname{Im}(f) & \longrightarrow & E_1 & \longrightarrow & K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & E_2 & = & E_2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

In the sequence $0 \to E_1 \to E \to E_2 \to 0$, if both E_1 and E_2 belong to $\mathcal{E}$, then so does E, since $\mathcal{E}$ is closed under extensions. Assembling the sequences $0 \to M \to I \to D \to 0$ and $0 \to D \to E \to K \to 0$, we obtain the desired exact sequence. ∎

Theorem 2.4 *Let* $n \geq 1$ *and*

$$0 \longrightarrow M \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_{n-1} \longrightarrow A \longrightarrow 0$$

be an exact sequence in $\operatorname{Mod} R$ *with all* E_i *in* $\mathcal{E}$. *Then there exist exact sequences*

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{n-1} \longrightarrow B \longrightarrow 0$$

and $0 \to A \to B \to E \to 0$ *in* $\operatorname{Mod} R$ *with all* I_i *injective and* E *in* $\mathcal{E}$.

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Proof We assume that $n \geq 1$, and then proceed by induction on n. If $n = 1$, then $0 \to M \to E_0 \to A \to 0$ is exact with E_0 in $\mathcal{E}$. Thus, we have an exact sequence $0 \to E_0 \to I_0 \to E \to 0$ with I_0 injective. Because $\mathcal{E}$ is injectively resolving, we have E belongs to $\mathcal{E}$. The desired result follows from the pushout diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & E_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & I_0 & \longrightarrow & B & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & E & = & E & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0. & & \end{array}$$

Now let $n \geq 2$ and let $0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to A \to 0$ be an exact sequence in $\operatorname{Mod} R$ with all E_i in $\mathcal{E}$. Set $W = \operatorname{Coker}(E_0 \to E_1)$. Then we have the exact sequence $0 \to M \to E_0 \to E_1 \to W \to 0$. By Proposition 2.3, we get an exact sequence $0 \to M \to I_0 \to E_1' \to W \to 0$ with I_0 injective and E_1' in $\mathcal{E}$. Put $M' = \operatorname{Im}(I_0 \to E_1')$. Then we obtain the exactness of

$$0 \to M' \longrightarrow E_1' \longrightarrow E_2 \longrightarrow \cdots \longrightarrow E_{n-1} \longrightarrow A \longrightarrow 0.$$

Therefore, the assertion follows by the induction hypothesis. ■

Corollary 2.5 *The following statements are equivalent for any M in* $\operatorname{Mod} R$ *and* $n \geq 0$.

(i) $\mathcal{E}$*-dim* $M \leq n$.

(ii) *There exists an exact sequence* $0 \to M \to I_0 \to I_1 \to \cdots \to I_{n-1} \to E \to 0$ *in* $\operatorname{Mod} R$ *with all* I_i *injective and* E *in* $\mathcal{E}$.

Proof (ii) ⇒ (i) holds by definition.

(i) ⇒ (ii) follows from Theorem 2.4 and the fact that $\mathcal{E}$ is closed under extensions. ■

In the following, we collect some known notions and facts needed in the article.

Definition 2.6 ([7]) Let $\mathcal{F}$ be a subcategory of $\operatorname{Mod} R$. The morphism $f\colon F \to M$ in $\operatorname{Mod} R$ with $F \in \mathcal{F}$ is called an *$\mathcal{F}$-precover* of M if for any morphism $g\colon F_0 \to M$ in $\operatorname{Mod} R$ with $F_0 \in \mathcal{F}$, there exists a morphism $h\colon F_0 \to F$ such that the following diagram commutes:

$$\begin{array}{ccc} & & F_0 \\ & \overset{h}{\swarrow} & \downarrow g \\ F & \xrightarrow{f} & M. \end{array}$$

The morphism $f\colon F \to M$ is called *right minimal* if an endomorphism $h\colon F \to F$ is an automorphism whenever $f = fh$. An $\mathcal{F}$-precover $f\colon F \to M$ is called an *$\mathcal{F}$-cover* if f is right minimal. $\mathcal{F}$ is called a *covering subcategory* of $\operatorname{Mod} R$ if every module in $\operatorname{Mod} R$

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has an $\mathcal{F}$-cover. Dually, the notions of an *$\mathcal{F}$-preenvelope*, a *left minimal morphism*, an *$\mathcal{F}$-envelope*, and an *enveloping subcategory* are defined.

Recall that a module $M \in \operatorname{Mod} R$ is said to be *FP-injective* if $\operatorname{Ext}^1_R(F, M) = 0$ for any finitely presented left R-module F (see [26]). A module $M \in \operatorname{Mod} R$ is said to be *divisible* if $\operatorname{Ext}^1_R(R/aR, M) = 0$ for all $a \in R$ (see [23]). In [22, 23], Mao and Ding introduced the notions of FI-injective (D-injective) and FI-flat (D-flat) modules as follows.

Definition 2.7 ([22, 23]) A module $M \in \operatorname{Mod} R$ is called *FI-injective* (resp. *D-injective*) if $\operatorname{Ext}^1_R(G, M) = 0$ for any FP-injective (resp. divisible) left R-module G.

A module $N \in \operatorname{Mod} R^{\mathrm{op}}$ is said to be *FI-flat* (resp. *D-flat*) if $\operatorname{Tor}^R_1(N, G) = 0$ for any FP-injective (resp. divisible) left R-module G.

In 1995, Enochs and Jenda in [10] introduced the notion of Gorenstein injective modules. A left R-module M is called *Gorenstein injective* if there is an exact sequence of injective left R-modules

$$\cdots \longrightarrow I_1 \longrightarrow I_0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

with $M = \ker(I^0 \to I^1)$ such that $\operatorname{Hom}_R(E, -)$ leaves the sequence exact whenever E is an injective left R-module. The Gorenstein projective and Gorenstein flat modules are defined respectively (see [11]). In order to characterize the projective and flat dimension of Gorenstein injective modules, the notions of GI-injective and GI-flat modules were introduced respectively in [13, 14], that is, FP-injective (or divisible) modules are replaced by Gorenstein injective modules in Definition 2.7.

In a recent article [16], we introduced the notions of weak injective and weak flat modules, and many results of a homological nature have been generalized from coherent rings to arbitrary rings.

Definition 2.8 ([16]) A module $M \in \operatorname{Mod} R$ (resp. $N \in \operatorname{Mod} R^{\mathrm{op}}$) is called *weak injective* (resp. *weak flat*) if $\operatorname{Ext}^1_R(F, M) = 0$ (resp. $\operatorname{Tor}^R_1(N, F)$) for any super finitely presented left R-module F, that is, for any left R-module F satisfying that there is an exact sequence: $\cdots \to P_n \to \cdots \to P_1 \to P_0 \to F \to 0$ in Mod$R$ with each P_i finitely generated projective. We use $\mathcal{WI}$ (resp. $\mathcal{WF}$) to denote the full subcategory of Mod$R$ (resp. $\operatorname{Mod} R^{\mathrm{op}}$) consisting of weak injective (resp. weak flat) modules.

Remark 2.9 For any ring R, it is known that all left R-modules have $\mathcal{WI}$-covers by [15, Theorem 3.1], and all right R-modules have $\mathcal{WF}$-preenvelopes by [16, Theorem 2.15]. Also, we have $\mathcal{WI}^+ \subseteq \mathcal{WF}$ and $\mathcal{WF}^+ \subseteq \mathcal{WI}$ by [16, Remark 2.2(2) and Theorem 2.10], respectively. By taking $\mathcal{E}$ (resp. $\mathcal{F}$) as the subcategory of $\operatorname{Mod} R$ (resp. $\operatorname{Mod} R^{\mathrm{op}}$) consisting of weak injective (resp. weak flat) modules in the article, one can deduce that the corresponding results also hold true.

3 $\mathcal{E}$-injective and $\mathcal{E}$-flat modules

In this section, we give a treatment of $\mathcal{E}$-injective and $\mathcal{E}$-flat modules, and we discuss some general properties of these modules.

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Definition 3.1 A module $M \in \operatorname{Mod} R$ is called $\mathcal{E}$-*injective* if $\operatorname{Ext}^1_R(G, M) = 0$ for any $G \in \mathcal{E}$. A module $N \in \operatorname{Mod} R^{\mathrm{op}}$ is called $\mathcal{E}$-*flat* if $\operatorname{Tor}^R_1(N, G) = 0$ for any $G \in \mathcal{E}$.

Remark 3.2 (i) If R is a ring such that $\mathcal{E}$ is a covering subcategory, then any kernel of an $\mathcal{E}$-cover is $\mathcal{E}$-injective by Wakamutsu's Lemma ([27, Lemma 2.1.1]). Also, one can easily verify that the class of $\mathcal{E}$-injective left R-modules (resp. $\mathcal{E}$-flat right R-modules) is closed under extensions, and the class of $\mathcal{E}$-flat right R-modules is closed under pure submodules.

(ii) Let $\mathcal{E}$ be the category of all left R-modules. Then $\mathcal{E}$-injective left R-modules and $\mathcal{E}$-flat right R-modules are just injective left R-modules and flat right R-modules, respectively.

(iii) Let $\mathcal{E}$ be the subcategory of injective left R-modules. Then $\mathcal{E}$-injective left R-modules and $\mathcal{E}$-flat right R-modules coincide with copure injective left R-modules and copure flat right R-modules in [9], respectively.

(iv) Let $\mathcal{E}$ be the subcategory of Gorenstein injective left R-modules. Then $\mathcal{E}$-injective left R-modules and $\mathcal{E}$-flat right R-modules coincide with GI-injective left R-modules [13] and GI-flat right R-modules [14], respectively.

(v) Let R be a left coherent ring and $\mathcal{E}$ the subcategory of FP-injective left R-modules. Then $\mathcal{E}$ is injectively resolving by [26, Lemma 3.1], and $\mathcal{E}$-injective left R-modules and $\mathcal{E}$-flat right R-modules coincide with FI-injective left R-modules and FI-flat right R-modules in [22], respectively.

(vi) Let R be a left strongly P-coherent ring and $\mathcal{E}$ the subcategory of divisible left R-modules. Then $\mathcal{E}$ is injectively resolving by [23, Lemmas 4.9 and 4.10]; and $\mathcal{E}$-injective left R-modules and $\mathcal{E}$-flat right R-modules are just D-injective left R-modules and D-flat right R-modules in [23], respectively.

(vii) A module $M \in \operatorname{Mod} R^{\mathrm{op}}$ is $\mathcal{E}$-flat if and only if M^+ is $\mathcal{E}$-injective by the standard isomorphism

$$\operatorname{Ext}^1_R(G, M^+) \cong \operatorname{Tor}^R_1(M, G)^+$$

for any $G \in \mathcal{E}$.

Proposition 3.3 (i) *A module $M \in \operatorname{Mod} R$ is injective if and only if M is $\mathcal{E}$-injective and $\mathcal{E}$-dim $M \le 1$.*

(ii) *A module $N \in \operatorname{Mod} R^{\mathrm{op}}$ is flat if and only if N is $\mathcal{E}$-flat and $\mathcal{E}$-dim $N^+ \le 1$.*

Proof (i) The "only if" part is trivial.

The "if" part: let M be an $\mathcal{E}$-injective left R-module and $\mathcal{E}$-dim $M \le 1$. There exists an exact sequence $0 \to M \to E \to N \to 0$ with E injective. Then $N \in \mathcal{E}$, since $\mathcal{E}$-dim $M \le 1$. Thus, we have $\operatorname{Ext}^1_R(N, M) = 0$, and the sequence $0 \to M \to E \to N \to 0$ is split. It follows that M is injective, as desired.

(ii) The "only if" part is trivial.

The "if" part: let N be an $\mathcal{E}$-flat right R-module and $\mathcal{E}$-dim $N^+ \le 1$. Then there exists an exact sequence $0 \to N^+ \to E \to L \to 0$, where E is injective and $L \in \mathcal{E}$. It follows that $\operatorname{Ext}^1_R(L, N^+) = 0$ since N^+ is $\mathcal{E}$-injective by Remark 3.2(vii). So the sequence $0 \to N^+ \to E \to L \to 0$ is split, and thus N^+ is injective. Hence, N is flat. ∎

Proposition 3.4 *The following are equivalent for a module M in* $\operatorname{Mod} R$.

(i) *M is $\mathcal{E}$-injective.*
(ii) *For each exact sequence $0 \to M \to E \to L \to 0$ with $E \in \mathcal{E}$, $E \to L$ is an $\mathcal{E}$-precover of L.*
(iii) *The map $E(M) \to E(M)/M$ is an $\mathcal{E}$-precover.*
(iv) *M is a kernel of an $\mathcal{E}$-precover $f: A \to B$ with A injective.*

Proof (i) ⇒ (ii) is by definition.

(ii) ⇒ (iii) Since there exists a short exact sequence

$$0 \longrightarrow M \longrightarrow E(M) \longrightarrow E(M)/M \longrightarrow 0,$$

and $E(M) \in \mathcal{E}$, then (iii) follows from (ii).

(iii) ⇒ (iv) follows immediately from (iii).

(iv) ⇒ (i) Let M be a kernel of an $\mathcal{E}$-precover $f: A \to B$ with A injective. Then there exists an exact sequence $0 \to M \to A \to A/M \to 0$. For any $G \in \mathcal{E}$, the sequence $\operatorname{Hom}_R(G, A) \to \operatorname{Hom}_R(G, A/M) \to \operatorname{Ext}^1_R(G, M) \to 0$ is exact. The sequence $\operatorname{Hom}_R(G, A) \to \operatorname{Hom}_R(G, A/M) \to 0$ is exact by (iv). Thus, $\operatorname{Ext}^1_R(G, M) = 0$, and so (i) follows. ∎

Recall from [11] that a module $M \in \operatorname{Mod} R$ is called *reduced* if M has no nonzero injective submodules.

Proposition 3.5 *Assume that R is a ring such that $\mathcal{E}$ is a covering subcategory. The following are equivalent for a module M in* Mod *R.*

(i) *M is a reduced $\mathcal{E}$-injective left R-module.*
(ii) *M is a kernel of an $\mathcal{E}$-cover $f: A \to B$ with A injective.*

Proof (i)⇒(ii) By Proposition 3.4, the natural map $\pi: E(M) \to E(M)/M$ is an $\mathcal{E}$-precover. Note that $E(M)/M$ has an $\mathcal{E}$-cover, and $E(M)$ has no nonzero direct summand K contained in M, since M is reduced. It follows from [27, Corollary 1.2.8] that $\pi: E(M) \to E(M)/M$ is an $\mathcal{E}$-cover, and so (ii) follows.

(ii)⇒(i) Let M be a kernel of an $\mathcal{E}$-cover $f: A \to B$ with A injective. Then M is $\mathcal{E}$-injective by Proposition 3.4. Now let K be an injective submodule of M. Suppose that $A = K \oplus L$, $p: A \to L$ is the projection, and $i: L \to A$ is the inclusion. It is easy to see that $f(ip) = f$, since $f(K) = 0$. Thus, ip is an isomorphism since f is a cover. Therefore, i is epic, and so $A = L, K = 0$. Consequently, M is reduced. ∎

Proposition 3.6 *Assume that R is a ring such that $\mathcal{E}$ is a covering subcategory. If M is $\mathcal{E}$-injective, then M has an $\mathcal{E}$-cover $f: E \to M$ with E injective. In particular,* ker *f is a reduced $\mathcal{E}$-injective left R-module.*

Proof Let $f: E \to M$ be an $\mathcal{E}$-cover of M. There is an exact sequence $0 \to E \xrightarrow{i} E_0 \to L \to 0$ with E_0 injective. Note that $L \in \mathcal{E}$, since $E \in \mathcal{E}$. So there exists $g: E_0 \to M$ such that $gi = f$, since $\operatorname{Ext}^1_R(L, M) = 0$. Thus, there is $h: E_0 \to E$ such that $fh = g$, since f is a cover. Therefore, $fhi = f$, and hence hi is an isomorphism. It follows that E is injective.

The second assertion follows directly by Proposition 3.5. ∎

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Theorem 3.7 *Let R be a ring such that $\mathcal{E}$ is a covering subcategory. Then a module $M \in \operatorname{Mod} R$ is $\mathcal{E}$-injective if and only if M is a direct sum of an injective left R-module and a reduced $\mathcal{E}$-injective left R-module.*

Proof The "if" part holds by definition.

The "only if" part: let M be an $\mathcal{E}$-injective left R-module. Then there exists an exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$. Then $E(M) \to E(M)/M$ is an $\mathcal{E}$-precover of $E(M)/M$ by Proposition 3.4. But $E(M)/M$ has an $\mathcal{E}$-cover $L \to E(M)/M$, so we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & E(M)/M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & E(M) & \longrightarrow & E(M)/M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \| & & \\
0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & E(M)/M & \longrightarrow & 0.
\end{array}$$

Because $L \to E(M)/M$ is an $\mathcal{E}$-cover, $\beta\delta$ is an isomorphism, and so $E(M) = \operatorname{Im}(\delta) \oplus \ker(\beta)$. Thus, L and $\ker(\beta)$ are injective, where $L \cong \operatorname{Im}(\delta)$. So K is a reduced $\mathcal{E}$-injective module by Proposition 3.5. By the five lemma, $\alpha\gamma$ is an isomorphism, and so $M = \operatorname{Im}(\gamma) \oplus \ker(\alpha)$, where $\operatorname{Im}(\gamma) \cong K$. By the snake lemma, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker(\alpha) & \longrightarrow & \ker(\beta) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & E(M) & \longrightarrow & E(M)/M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \| & & \\
0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & E(M)/M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

From the 3×3 lemma, it follows that $\ker(\alpha) \cong \ker(\beta)$. Thus, the desired result follows. ∎

Lemma 3.8 *Let $\mathcal{F}$ be a preenveloping subcategory of* $\operatorname{Mod} R^{\mathrm{op}}$ *such that $\mathcal{E}^+ \subseteq \mathcal{F}$ and $\mathcal{F}^+ \subseteq \mathcal{E}$.*

(i) *If M is a cokernel of an $\mathcal{F}$-preenvelope $f: K \to F$ of a right R-module K with F flat, then M is $\mathcal{E}$-flat.*

(ii) *If M is a finitely presented $\mathcal{E}$-flat right R-module, then M is a cokernel of an $\mathcal{F}$-preenvelope $K \to F$ of a right R-module K with F flat.*

Proof (i) Assume that M is a cokernel of an $\mathcal{F}$-preenvelope $f: K \to F$ of a right R-module K with F flat. Then we have an exact sequence $K \xrightarrow{f} F \to M \to 0$. Putting

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$L = \operatorname{Im} f$, then $0 \to L \to F \to M \to 0$ is exact. We claim that $L \to F$ is an $\mathcal{F}$-preenvelope of L. In fact, for any $F_0 \in \mathcal{F}$, we consider the following commutative diagram:

$$\begin{array}{ccccccc} & & \operatorname{Hom}_R(F, F_0) & = & \operatorname{Hom}_R(F, F_0) & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \operatorname{Hom}_R(f, F_0)} & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Hom}_R(L, F_0) & \longrightarrow & \operatorname{Hom}_R(K, F_0) & \longrightarrow & \operatorname{Hom}_R(\ker f, F_0). & & \end{array}$$

Because $f: K \to F$ is an $\mathcal{F}$-preenvelope, $\operatorname{Hom}_R(f, F_0)$ is an epimorphism. By the snake lemma, we obtain that $\operatorname{Hom}_R(F, F_0) \to \operatorname{Hom}_R(L, F_0) \to 0$ is exact. Therefore, $L \to F$ is an $\mathcal{F}$-preenvelope of L. For each $E \in \mathcal{E}$, we have $E^+ \in \mathcal{F}$ by assumption. So we obtain the exactness of $\operatorname{Hom}_R(F, E^+) \to \operatorname{Hom}_R(L, E^+) \to 0$, which induces the exact sequence $(F \otimes_R E)^+ \to (L \otimes_R E)^+ \to 0$. Thus, the sequence $0 \to L \otimes_R E \to F \otimes_R E$ is exact. Note that F is flat; it follows that $0 \to \operatorname{Tor}_1^R(M, E) \to L \otimes_R E \to F \otimes_R E$ is exact, and hence $\operatorname{Tor}_1^R(M, E) = 0$. Thus, M is $\mathcal{E}$-flat.

(ii) Let M be a finitely presented right R-module. Then there exists an exact sequence $0 \to N \to P \to M \to 0$ with P finitely generated projective and N finitely generated. Next we will prove that $N \to P$ is an $\mathcal{F}$-preenvelope. For any $F \in \mathcal{F}$, we have $F^+ \in \mathcal{E}$ by assumption, and so $\operatorname{Tor}_1^R(M, F^+) = 0$ since M is $\mathcal{E}$-flat. Thus, we get the following exact commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & N \otimes_R F^+ & \longrightarrow & P \otimes_R F^+ \\ & & {\scriptstyle \theta_N}\downarrow & & \downarrow{\scriptstyle \theta_P} \\ & & \operatorname{Hom}_R(N, F)^+ & \xrightarrow{\theta} & \operatorname{Hom}_R(P, F)^+. \end{array}$$

Note that if N is finitely generated and P is finitely presented, we obtain that θ_N is epic and θ_P is isomorphic by [3, Lemma 2]. It follows that θ is a monomorphism. Hence the sequence $\operatorname{Hom}_R(P, F) \to \operatorname{Hom}_R(N, F) \to 0$ is exact, and thus $N \to P$ is an $\mathcal{F}$-preenvelope. ■

By Lemma 3.8, we immediately get the following theorem.

Theorem 3.9 *Assume that $\mathcal{F}$ is a preenveloping subcategory of* Mod R^{op} *such that $\mathcal{E}^+ \subseteq \mathcal{F}$ and $\mathcal{F}^+ \subseteq \mathcal{E}$. Then a finitely presented right R-module M is $\mathcal{E}$-flat if and only if M is a cokernel of an $\mathcal{F}$-preenvelope $f: K \to F$ of a right R-module K with F flat.*

Remark 3.10 (i) Let R be a left Noetherian ring. Since the subcategory of all injective left R-modules is covering by [7, Theorem 2.1], it follows that a left R-module M is copure injective if and only if it is a direct sum of an injective left R-module and a reduced copure injective left R-module.

(ii) Let R be a left coherent ring. Then the subcategory of all flat right R-modules is preenveloping by [7, Proposition 5.1], and so a finitely presented right R-module M is copure flat if and only if it is a cokernel of a flat preenvelope $f: K \to F$ of a right R-module K.

Eklof and Trlifaj proved in [5, Theorem 12] that if $\mathcal{B}$ is a class of left R-modules, then every right R-module has a $\ker \operatorname{Tor}_1^R(-,\mathcal{B})$-cover, where

$$\ker \operatorname{Tor}_1^R(-,\mathcal{B}) = \{A \mid \operatorname{Tor}_1^R(A,B) = 0 \text{ for any } B \in \mathcal{B}\}.$$

By taking $\mathcal{B}$ as the subcategory of $\mathcal{E}$, we can deduce that every right R-module has an $\mathcal{E}$-flat cover. For a module $M \in \operatorname{Mod} R^{\mathrm{op}}$, the $\mathcal{E}$-flat cover and the injective envelope of M are denoted by $\mathcal{E}F_0(M)$ and $E^0(M)$, respectively.

Proposition 3.11 *The following statements are equivalent:*

(i) *$E^0(M)$ is $\mathcal{E}$-flat for any $\mathcal{E}$-flat right R-module M;*
(ii) *$\mathcal{E}F_0(I)$ is injective for any injective right R-module I.*

Proof (i)⇒(ii) Suppose that I is an injective right R-module and $\alpha\colon \mathcal{E}F_0(I) \to I$ is the $\mathcal{E}$-flat cover of I, and $\beta\colon \mathcal{E}F_0(I) \to E^0(\mathcal{E}F_0(I))$ is the injective envelope. Then there exists $\theta\colon E^0(\mathcal{E}F_0(I)) \to I$ such that $\alpha = \theta\beta$. On the other hand, since $E^0(\mathcal{E}F_0(I))$ is $\mathcal{E}$-flat by (i), there exists $\lambda\colon E^0(\mathcal{E}F_0(I)) \to \mathcal{E}F_0(I)$ such that $\alpha\lambda = \theta$. Thus, $\alpha = \alpha\lambda\beta$, and hence $\lambda\beta$ is an isomorphism since α is a cover. Therefore, $\mathcal{E}F_0(I)$ is a direct summand of $E^0(\mathcal{E}F_0(I))$, and hence it is injective.

(ii)⇒(i) Suppose that M is an $\mathcal{E}$-flat right R-module and $\phi\colon M \to E^0(M)$ is the injective envelope, and $\phi\colon \mathcal{E}F_0(E^0(M)) \to E^0(M)$ is the $\mathcal{E}$-flat cover of $E^0(M)$. Then there exists $\mu\colon M \to \mathcal{E}F_0(E^0(M))$ such that $\phi = \phi\mu$. On the other hand, because $\mathcal{E}F_0(E^0(M))$ is injective by (ii), there exists $\gamma\colon E^0(M) \to \mathcal{E}F_0(E^0(M))$ such that $\mu = \gamma\phi$. Thus, $\phi = \phi\mu = \phi\gamma\phi$, and so $\phi\gamma$ is an isomorphism, since ϕ is an envelope. It follows that $E^0(M)$ is $\mathcal{E}$-flat. ■

It is well known that every module over any ring R has a flat cover ([1]). The following result was proved in [20, Theorem 2.2] when R is a commutative Noetherian ring.

Corollary 3.12 *The following statements are equivalent:*

(i) *$E^0(F)$ is flat for any flat right R-module F;*
(ii) *$F_0(I)$ (the flat cover of I) is injective for any injective right R-module I.*

4 The $\mathcal{E}$-injective and $\mathcal{E}$-flat Dimensions of Modules and Rings

In this section, we introduce and investigate the $\mathcal{E}$-injective and $\mathcal{E}$-flat dimensions of modules and rings. Then we characterize $\mathcal{E}$-(semi)hereditary rings, which is a generalization of (semi)hereditary rings, in terms of $\mathcal{E}$-injective and $\mathcal{E}$-flat modules.

Definition 4.1 For a module M in $\operatorname{Mod} R$, the $\mathcal{E}$-injective dimension $\mathcal{E}\text{-}\operatorname{id}_R(M)$ of M is defined as $\max\{n \mid \operatorname{Ext}_R^n(G,M) \neq 0 \text{ for some } G \in \mathcal{E}\}$. The left global $\mathcal{E}$-injective dimension $l.\mathcal{E}\mathcal{I}\text{-}\dim(R)$ of R is defined as

$$l.\mathcal{E}\mathcal{I}\text{-}\dim(R) = \sup\{\mathcal{E}\text{-}\operatorname{id}_R(M) \mid M \in \operatorname{Mod} R\}.$$

The $\mathcal{E}$-flat dimension, $\mathcal{E}\text{-}\operatorname{fd}_R(N)$, of a module N in $\operatorname{Mod} R^{\mathrm{op}}$ is defined as

$$\max\{n \mid \operatorname{Tor}_n^R(N,G) \neq 0 \text{ for some } G \in \mathcal{E}\}.$$

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The right global $\mathcal{E}$-flat dimension $r.\mathcal{EF}\text{-}\dim(R)$ of R is defined as $l.\mathcal{EF}\text{-}\dim(R) = \sup\{\mathcal{E}\text{-}\operatorname{fd}_R(N) \mid N \in \operatorname{Mod} R^{\mathrm{op}}\}$.

Similarly, we have $r.\mathcal{EI}\text{-}\dim(R)$ and $l.\mathcal{EF}\text{-}\dim(R)$, respectively (when R is commutative, we drop the unneeded letters r and l).

Remark 4.2 It is clear that $\operatorname{cid}_R(M) \le \mathcal{E}\text{-}\operatorname{id}_R(M) \le \operatorname{id}_R(M)$ for every $M \in \operatorname{Mod} R$. If a module $M \in \operatorname{Mod} R$ with $\operatorname{id}_R(M) < \infty$, then $\operatorname{cid}_R(M) = \mathcal{E}\text{-}\operatorname{id}_R(M) = \operatorname{id}_R(M)$ by [9, Corollary 3.2]. Also, we have $\operatorname{cfd}_R(N) \le \mathcal{E}\text{-}\operatorname{fd}_R(N) \le \operatorname{fd}_R(N)$ for every N in $\operatorname{Mod} R^{\mathrm{op}}$.

We shall say that a module M in $\operatorname{Mod} R$ (resp. N in $\operatorname{Mod} R^{\mathrm{op}}$) is *strongly $\mathcal{E}$-injective* (resp. *strongly $\mathcal{E}$-flat*) if $\operatorname{Ext}^i_R(G, M) = 0$ (resp. $\operatorname{Tor}^R_i(G, N) = 0$) for all $i \ge 1$ and all $G \in \mathcal{E}$. We set $\mathcal{E}\text{-}\operatorname{id}_R(M) = 0$ ($\mathcal{E}\text{-}\operatorname{fd}_R(M) = 0$) if M is strongly $\mathcal{E}$-injective (strongly $\mathcal{E}$-flat).

The proofs of the next two propositions are standard homological algebra.

Proposition 4.3 *The following are equivalent for a module M in* Mod R*:*

(i) $\mathcal{E}\text{-}\operatorname{id}_R(M) \le n$;
(ii) $\operatorname{Ext}^{n+j}_R(G, M) = 0$ *for all* $G \in \mathcal{E}$ *and all* $j \ge 1$;
(iii) *for every exact sequence* $0 \to M \to E^0 \to \cdots \to E^{n-1} \to C^n \to 0$ *where* $E^0, \dots, E^{n-1}$ *are injective, then* C^n *is strongly $\mathcal{E}$-injective.*

Proposition 4.4 *The following are equivalent for a module M in* Mod R^{op}*:*

(i) $\mathcal{E}\text{-}\operatorname{fd}_R(M) \le n$;
(ii) $\operatorname{Tor}^R_{n+j}(M, G) = 0$ *for all* $G \in \mathcal{E}$ *and all* $j \ge 1$;
(iii) *for every exact sequence* $0 \to K_n \to F_{n-1} \cdots \to F_0 \to M \to 0$ *where* $F_0, \cdots, F_{n-1}$ *are flat, then* K_n *is strongly $\mathcal{E}$-flat.*

Theorem 4.5 *The following quantities are identical:*

(i) $l.\mathcal{EI}\text{-}\dim(R)$;
(ii) $\sup\{\operatorname{pd}_R(M) \mid M \in \mathcal{E}\}$;
(iii) $\sup\{\operatorname{pd}_R(M) \mid M \in \operatorname{Mod} R$ *with* $\mathcal{E}\text{-}\dim M < \infty\}$.

Proof (i) $\le$ (ii) We may suppose that $\sup\{\operatorname{pd}_R(M) \mid M \in \mathcal{E}\} = m < \infty$. Let M be a left R-module. Then $\operatorname{Ext}^{m+1}_R(N, M) = 0$ for any $N \in \mathcal{E}$, since $\operatorname{pd}_R(N) \le m$, and hence $\mathcal{E}\text{-}\operatorname{id}_R(M) \le m$ by Proposition 4.3. Thus $l.\mathcal{EI}\text{-}\dim(R) \le m$.

(ii) $\le$ (i) Assume that $l.\mathcal{EI}\text{-}\dim(R) = n < \infty$. For any left R-module M, we have $\mathcal{E}\text{-}\operatorname{id}_R(M) \le n$. Let $N \in \mathcal{E}$. Then $\operatorname{Ext}^{n+1}_R(N, M) = 0$ by Proposition 4.3, which implies that $\operatorname{pd}_R(N) \le n$, as desired.

(ii) $\le$ (iii) is obvious.

(iii) $\le$ (i) We suppose that $l.\mathcal{EI}\text{-}\dim(R) = n < \infty$. Let $N \in \operatorname{Mod} R$ with $\mathcal{E}\text{-}\dim N < \infty$, we may assume that $\mathcal{E}\text{-}\dim N = m < \infty$. Then there exists an exact sequence

$$0 \longrightarrow N \to E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^m \longrightarrow 0$$

in $\operatorname{Mod} R$ with E^i in $\mathcal{E}$. From the equality between (i) and (ii), it follows that $\operatorname{pd}_R(E^i) \le n$. Consequently $\operatorname{pd}_R(N) \le n$, and so (iii) $\le$ (i) holds. ∎

Similarly, we have the following result.

Theorem 4.6 *The following quantities are identical:*

(i) $r.\mathcal{EF}\text{-}\dim(R)$;
(ii) $\sup\{\mathrm{fd}_R(M) \mid M \in \mathcal{E}\}$;
(iii) $\sup\{\mathrm{fd}_R(M) \mid M \in \mathrm{Mod}\, R$ *with* $\mathcal{E}\text{-}\dim M < \infty\}$.

Definition 4.7 A ring R is called *left $\mathcal{E}$-hereditary* if $\mathrm{pd}_R(B) \le 1$ for any $B \in \mathcal{E}$; and R is called *left $\mathcal{E}$-semihereditary* if $\mathrm{fd}_R(B) \le 1$ for any $B \in \mathcal{E}$.

A ring R is called $\mathcal{E}$*-semisimple* if every module in $\mathcal{E}$ is projective; and R is called $\mathcal{E}$*-von Neumann regular* if every module in $\mathcal{E}$ is flat.

Mahdou and Tamekkante [24] introduced the notion of Gorenstein (semi) hereditary rings. Recall a ring R is called *Gorenstein hereditary* if every submodule of a projective module is Gorenstein projective; R is called *Gorenstein semihereditary* if it is coherent and every submodule of a flat module is Gorenstein flat. We say a ring R is *n-IF* if $\mathrm{fd}_R(M) \le n$ for every injective left R-module M.

Remark 4.8 (i) Let $\mathcal{E}$ be the class of all left R-modules. Then left $\mathcal{E}$-(semi)hereditary rings are just the well-known left (semi)hereditary rings. A ring R is $\mathcal{E}$-semisimple if and only if it is semisimple; and R is $\mathcal{E}$-von Neumann regular if and only if it is a von Neumann regular ring.

(ii) Let $\mathcal{E}$ be the class of (FP-)injective left R-modules. Then a (coherent) ring R is left $\mathcal{E}$-semihereditary if and only if R is a left 1-IF ring by [4, Theorem 3.5] (if and only if R is left Gorenstein semihereditary by [24, Proposition 3.3]); a commutative ring R is $\mathcal{E}$-hereditary if and only if $\mathrm{pd}_R(I) \le 1$ for all injective R-modules I if and only if R is Gorenstein hereditary by [24, Theorem 2.3] and [18, Corollary 1.3]. Moreover, a ring R is $\mathcal{E}$-semisimple if and only if R is quasi-Frobenius, and R is $\mathcal{E}$-von Neumann regular if and only if R is a left IF ring.

(iii) Let $\mathcal{E}$ be the class of Gorentein injective left R-modules. Then R is left $\mathcal{E}$-hereditary (resp. $\mathcal{E}$-semihereditary) if and only if $\mathrm{l.GI\text{-}dim}(R) \le 1$ (resp. $\mathrm{r.GIFD}(R) \le 1$); a ring R is $\mathcal{E}$-semisimple if and only if R is semisimple by [13, Theorem 2.13]; and a commutative ring R is $\mathcal{E}$-von Neumann regular if and only if R is von Neumann regular by [14, Theorem 3.16].

We now are in position to characterize $\mathcal{E}$-(semi)hereditary rings.

Theorem 4.9 *The following statements are equivalent:*

(i) *R is an $\mathcal{E}$-hereditary ring;*
(ii) $l.\mathcal{EI}\text{-}\dim(R) \le 1$;
(iii) *every quotient module of any injective left R-module is strongly $\mathcal{E}$-injective;*
(iv) *every quotient module of any strongly $\mathcal{E}$-injective left R-module is strongly $\mathcal{E}$-injective.*

Proof (i) $\Leftrightarrow$ (ii) is by definition.

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(i) ⇒ (iii) Let E be an injective left R-module and K a submodule of E. For any $G \in \mathcal{E}$ and all $i \geq 1$, the exactness of $0 \to K \to E \to E/K \to 0$ induces the exact sequence

$$0 = \operatorname{Ext}_R^i(G, E) \longrightarrow \operatorname{Ext}_R^i(G, E/K) \longrightarrow \operatorname{Ext}_R^{i+1}(G, K).$$

Note that $\operatorname{Ext}_R^{i+1}(G, K) = 0$ by (i). Then we have $\operatorname{Ext}_R^i(G, E/K) = 0$, which implies E/K is strongly $\mathcal{E}$-injective, and hence (iii) follows.

(iii) ⇒ (ii) Let M be a left R-module. There exists an exact sequence $0 \to M \to E \to E/M \to 0$ with E injective. Then E/M is strongly $\mathcal{E}$-injective by (iii), and it follows that $\mathcal{E}\text{-}\operatorname{id}_R(M) \leq 1$ by Proposition 4.3. Thus, (ii) holds.

(iv) ⇒ (iii) follows from the fact that every injective module is strongly $\mathcal{E}$-injective.

(iii) ⇒ (iv) Let M be a strongly $\mathcal{E}$-injective left R-module and K a submodule of M. There exists an exact sequence $0 \to K \to E(K) \to L \to 0$. We consider the pushout diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & M/K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & E(K) & \longrightarrow & H & \longrightarrow & M/K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & L & = & L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Then L is strongly $\mathcal{E}$-injective by (iii). Since M is strongly $\mathcal{E}$-injective, one easily checks that H is strongly $\mathcal{E}$-injective. For any $G \in \mathcal{E}$ and all $i \geq 1$, we get the exactness of

$$0 = \operatorname{Ext}_R^i(G, H) \longrightarrow \operatorname{Ext}_R^i(G, M/K) \longrightarrow \operatorname{Ext}_R^{i+1}(G, E(K)) = 0.$$

Thus, $\operatorname{Ext}_R^i(G, M/K) = 0$ for all $i \geq 1$, and so M/K is strongly $\mathcal{E}$-injective. ■

For a module $M \in \operatorname{Mod} R$, we have $\mathcal{E}\text{-}\operatorname{fd}_R(M) = \mathcal{E}\text{-}\operatorname{id}_R(M^+)$ by the standard isomorphism: $\operatorname{Tor}_j^R(G, M)^+ \cong \operatorname{Ext}_R^j(G, M^+)$ for all $G \in \mathcal{E}$ and all $j \geq 1$.

Theorem 4.10 *The following statements are equivalent:*

(i) *R is an $\mathcal{E}$-semihereditary ring;*
(ii) $r.\mathcal{E}\mathcal{F}\text{-}\dim(R) \leq 1$;
(iii) $\mathcal{E}\text{-}\operatorname{id}_R(M) \leq 1$ *for all cotorsion left R-modules M;*
(iv) $\mathcal{E}\text{-}\operatorname{id}_R(M) \leq 1$ *for all pure injective left R-modules M;*
(v) $\mathcal{E}\text{-}fd_R(M) \leq 1$ *for all finitely presented right R-modules M;*
(vi) *every submodule of a projective right R-module is $\mathcal{E}$-flat;*
(vii) *every submodule of a flat right R-module is $\mathcal{E}$-flat;*
(viii) *every submodule of an $\mathcal{E}$-flat right R-module is $\mathcal{E}$-flat;*
(ix) $\operatorname{id}_R \operatorname{Hom}_R(G, I) \leq 1$ *for all $G \in \mathcal{E}$ and all injective left R-modules I.*

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Proof (i) ⇔ (ii) is by definition.

(ii) ⇒ (iii) Let M be a cotorsion left R-module. For any $N \in \mathcal{E}$, we have $\mathrm{fd}_R(N) \leq 1$ by the definition of $\mathcal{E}$-semihereditary rings. Thus, N has a flat resolution: $0 \to F_1 \to F_0 \to N \to 0$. It is easy to check that $\mathrm{Ext}_R^{j+1}(N, M) = 0$ for all $j \geq 1$, and hence $\mathcal{E}\text{-}\mathrm{id}_R(M) \leq 1$.

(iii) ⇒ (iv) follows from the fact that every pure injective module is cotorsion.

(iv) ⇒ (ii) Let M be a right R-module. Then M^+ is pure injective by [11, Proposition 5.3.7], and so $\mathcal{E}\text{-}\mathrm{id}_R(M^+) \leq 1$ by (iv). Hence $\mathcal{E}\text{-}\mathrm{fd}_R(M) \leq 1$.

(ii) ⇒ (v) is trivial.

(v) ⇒ (i) Let $G \in \mathcal{E}$ and M a finitely presented right R-module. Then $\mathrm{Tor}_j^R(M, G) = 0$ for any $j \geq 2$ by (v) and Proposition 3.5. Hence, $\mathrm{fd}_R(G) \leq 1$, and thus R is $\mathcal{E}$-semihereditary, as desired.

(i) ⇒ (viii) Let N be a submodule of an $\mathcal{E}$-flat right R-module M. There exists an exact sequence $0 \to N \to M \to M/N \to 0$. For any $G \in \mathcal{E}$, we have the exactness of

$$\mathrm{Tor}_2^R(M/N, G) \longrightarrow \mathrm{Tor}_1^R(N, G) \longrightarrow \mathrm{Tor}_1^R(M, G).$$

The first term is zero, since $\mathrm{fd}_R(G) \leq 1$ by (i), and the last term is zero, since M is $\mathcal{E}$-flat. Consequently, $\mathrm{Tor}_1^R(N, G) = 0$, and hence (viii) holds.

(viii) ⇒ (vii) ⇒ (vi) are trivial.

(vi) ⇒ (i) Let $G \in \mathcal{E}$ and M any right R-module. Then there exists an exact sequence $0 \to K \to P \to M \to 0$ with P projective. Thus we obtain the exactness of

$$0 \longrightarrow \mathrm{Tor}_2^R(M, G) \longrightarrow \mathrm{Tor}_1^R(K, G).$$

The last term is zero, since K is $\mathcal{E}$-flat by (vi). Therefore, $\mathrm{Tor}_2^R(M, G) = 0$, which implies $\mathrm{fd}_R(G) \leq 1$, and so (i) follows.

(i) ⇒ (ix) Let $G \in \mathcal{E}$ and let I be an injective left R-module. Then $\mathrm{fd}_R(G) \leq 1$ by (i), and there exists a flat resolution of G: $0 \to F_1 \to F_0 \to G \to 0$, which induces the exactness of the sequence

$$0 \longrightarrow \mathrm{Hom}_R(G, I) \longrightarrow \mathrm{Hom}_R(F_0, I) \longrightarrow \mathrm{Hom}_R(F_1, I) \longrightarrow 0.$$

Since each F_i $(i = 0, 1)$ is flat and I is injective, it follows that $\mathrm{Hom}_R(F_i, I)$ is injective by [25, Theorem 3.44]. Consequently, $\mathrm{id}_R \mathrm{Hom}_R(G, I) \leq 1$, and so (ix) holds.

(ix) ⇒ (i) Let $G \in \mathcal{E}$ and let $\cdots \to F_1 \to F_0 \to G \to 0$ be a flat resolution of G. Set $K = \mathrm{Im}(F_1 \to F_0)$. Then we get a short exact sequence $0 \to K \to F_0 \to G \to 0$. For any injective R-module I, the sequence

$$0 \longrightarrow \mathrm{Hom}_R(G, I) \longrightarrow \mathrm{Hom}_R(F_0, I) \longrightarrow \mathrm{Hom}_R(K, I) \longrightarrow 0$$

is exact. Since $\mathrm{id}_R \mathrm{Hom}_R(G, I) \leq 1$ and $\mathrm{Hom}_R(F_0, I)$ is injective, $\mathrm{Hom}_R(K, I)$ is injective for any injective R-module I. Hence, K is flat by [12, Proposition 11.35], and so $\mathrm{fd}_R(G) \leq 1$. Thus (i) follows. ∎

Specializing Theorem 4.10, we obtain the following characterizations of $\mathcal{E}$-von Neumann regular rings.

Corollary 4.11 *The following statements are equivalent:*

(i) *R is $\mathcal{E}$-von Neumann regular;*

(ii) *every right R-module is $\mathcal{E}$-flat;*

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(iii) *every cotorsion left R-module is $\mathcal{E}$-injective;*
(iv) *every pure injective left R-module is $\mathcal{E}$-injective;*
(v) *every finitely presented right R-module is $\mathcal{E}$-flat;*
(vi) $\mathrm{Hom}_R(G, I)$ *is injective for any* $G \in \mathcal{E}$ *and any injective left R-module I.*

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College of Applied Mathematics, Chengdu University of Information Technology, Chengdu, Sichuan 610225, P.R. China
e-mail: gaozenghui@cuit.edu.cn