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POINTWISE APPROXIMATION BY BERNSTEIN POLYNOMIALS

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Abstract

We improve the degree of pointwise approximation of continuous functions $f(x)$ by Bernstein operators, when x is close to the endpoints of $[0, 1]$. We apply the new estimate to establish upper and lower pointwise estimates for the test function $g(x) = x\log(x) + (1-x)\log(1-x)$. At the end we prove a general statement for pointwise approximation by Bernstein operators.

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1. Introduction

In 1994 Ditzian showed in [4] that for the Bernstein polynomials

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\cdot\binom{n}{k}x^k(1-x)^{n-k}, \quad x \in [0,1],$$

the pointwise approximation

$$|B_n(f,x) - f(x)| \le C\omega_2^{\varphi^\lambda}(f, n^{-1/2}\varphi(x)^{1-\lambda}), \quad x \in [0,1], \tag{1.1}$$

holds true for $\lambda \in [0,1]$, $\varphi(x) := \sqrt{x(1-x)}$ and $f \in C[0,1]$, where the Ditzian–Totik modulus of second order is given by

$$\omega_2^{\varphi^\lambda}(f,t) := \sup_{0<h\le t}\ \sup_{x\pm h\varphi^\lambda(x)\in[0,1]} |f(x-h\varphi^\lambda(x)) - 2f(x) + f(x+h\varphi^\lambda(x))|. \tag{1.2}$$

We recall that this modulus is equivalent to the K-functional

$$K_{\varphi^\lambda}(f,t^2) = \inf(\|f-h\|_{C[0,1]} + t^2\|\varphi^{2\lambda}h''\|_{C[0,1]}). \tag{1.3}$$

The infimum is taken on functions satisfying $h \in AC$, $h' \in AC_{\rm loc}$ where AC is the set of all absolutely continuous functions on $[0,1]$ and $AC_{\rm loc}$ is the set of absolutely continuous functions on compact subsets of $(0,1)$. (See [5].)

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In 1998 Felten proved in [6] the more general inequality

$$|B_n(f,x)-f(x)|\le C\omega_2^{\phi}\Big(f, n^{-1/2}\frac{\varphi(x)}{\phi(x)}\Big),\quad x\in[0,1],$$

where $\phi:[0,1]\to\mathbb{R}$ is an admissible step-weight function of the Ditzian–Totik modulus and ϕ^2 is a concave function. The aim of this note is to improve the estimate (1.1) for $\lambda=1$, when x is close to the endpoints of [0, 1].

Let us define

$$\delta(n,x):=\min\Big\{n^{-1/2},\Big(\frac{x(1-x)}{n}\Big)^{1/4}\Big\}.$$

The following theorem is our main result.

Theorem 1.1. *The pointwise estimate*

$$|B_n(f,x)-f(x)|\le C\omega_2^{\varphi}(f,\delta(n,x)),\quad x\in[0,1],\tag{1.4}$$

holds true for all $f\in C[0,1]$, $n\in\mathbb{N}$.

In Section 2 we give the proof of Theorem 1.1. In Section 3 we establish upper and lower bounds for approximation of the function $g(x)$, defined in (2.1), by Bernstein operators.

2. Proof of Theorem 1.1

Let us define $g:[0,1]\to\mathbb{R}$ as

$$g(x)=x\log(x)+(1-x)\log(1-x),\quad x\in(0,1),\tag{2.1}$$

and $g(0)=g(1):=0$. The problem of evaluating the remainder term

$$R_n(g,x)=B_n(g,x)-g(x),\quad x\in[0,1],$$

was formulated by the author in [14] during the fifth Romanian–German Seminar on Approximation Theory, held in Sibiu, Romania, in 2002. More precisely, we proposed to find (best) bounds of the type

$$k_1\cdot\frac{x^{\alpha_1}(1-x)^{\alpha_2}}{n^{\beta}}\le R_n(g,x)\le K_2\cdot\frac{x^{a_1}(1-x)^{a_2}}{n^{b}},\quad x\in[0,1],$$

where k_1, K_2 are positive numbers, independent of x and n. Some days after the conference, Lupaş showed that the above holds with $\alpha_1=\alpha_2=\beta=1$, $k_1=\frac{1}{2}$ and $a_1=a_2=b=\frac{1}{2}$, $K_2=\sqrt{2}$ (see [8, 9]), that is,

$$\frac{x(1-x)}{2n}\le R_n(g,x)\le\sqrt{2}\cdot\sqrt{\frac{x(1-x)}{n}}.\tag{2.2}$$

The function g was applied in the following direct estimate, proved by Parvanov and Popov in [12].

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If $L : C[0, 1] \to C[0, 1]$ is a linear positive operator, preserving linear functions, then

$$|L(f, x) - f(x)| \le 2\|f - h\|_{C[0,1]} + |L(g, x) - g(x)| \cdot \|\varphi^2 h''\|_{C[0,1]}$$

holds for arbitrary $h \in AC$, $h' \in AC_{\mathrm{loc}}$, $\|\varphi^2 h''\|_{C[0,1]} < \infty$. Instead of L we write B_n and apply the right-hand side of (2.2). Hence

$$|B_n(f, x) - f(x)| \le 2\|f - h\|_{C[0,1]} + \sqrt{2}\Big(\frac{x(1-x)}{n}\Big)^{1/2} \cdot \|\varphi^2 h''\|_{C[0,1]}.$$

Therefore

$$|B_n(f, x) - f(x)| \le 2K_\varphi\Big(f, \Big(\frac{x(1-x)}{n}\Big)^{1/2}\Big).$$

From the equivalence between $K_\varphi(f, t^2)$ and $\omega_2^\varphi(f, t)$, it follows that

$$|B_n(f, x) - f(x)| \le C\omega_2^\varphi\Big(f, \Big(\frac{x(1-x)}{n}\Big)^{1/4}\Big). \tag{2.3}$$

The estimates (2.3) and (1.1) with $\lambda = 1$ complete the proof. □

3. Upper and lower pointwise bounds

The following is a straightforward corollary of Theorem 1.1.

Corollary 3.1. *The pointwise estimate*

$$|B_n(g, x) - g(x)| \le C\omega_2^\varphi\Big(g, \sqrt[4]{\frac{x(1-x)}{n}}\Big), \quad x \in [0, 1], \tag{3.1}$$

holds true for all $n \in \mathbb{N}$.

Remark 3.2. If x is close to the endpoints of $[0, 1]$, then the estimate (3.1) is better than that in (2.1) for $\lambda = 1$, established by Ditzian in [4].

Remark 3.3. Other direct pointwise estimates in terms of K_φ are proved in [6]. We point out that neither from [6] nor from [4] is it possible to deliver (3.1) as a straightforward corollary.

We continue with lower pointwise bounds. In [1, Theorem 11], using the function $g(x)$ as a 'universal' tool, the authors proved that

$$c(g)\omega_2\Big(g, \sqrt{\frac{x(1-x)}{n}}\Big) \le |B_n(g, x) - g(x)|$$

does not hold. So the question arises: what kind of modulus is appropriate to serve as a lower pointwise bound for $|B_n(g, x) - g(x)|$? The answer is given in the next theorem.

Theorem 3.4. *The following inequality holds true:*

$$c \cdot \omega_2^\varphi\Big(g, \sqrt{\frac{x(1-x)}{n}}\Big) \le |B_n(g, x) - g(x)|. \tag{3.2}$$

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Proof. Using the equivalence between $K_\varphi(g, t^2)$ and $\omega_2^\varphi(g, t)$, we compute

$$\begin{aligned} c\omega_2^\varphi\Big(g, \sqrt{\frac{x(1-x)}{n}}\Big) &\le K_\varphi\Big(g, \frac{x(1-x)}{n}\Big) \\ &:= \inf_h\Big\{\|g-h\|_{C[0,1]} + \frac{x(1-x)}{n}\cdot\|\varphi^2 h''\|_{C[0,1]}\Big\} \\ &\le \|g-g\|_{C[0,1]} + \frac{x(1-x)}{n}\cdot\|\varphi^2 g''\|_{C[0,1]} \\ &= \frac{x(1-x)}{n} \le 2|B_n(g,x)-g(x)|, \end{aligned}$$

where the last inequality follows from (2.2). The proof is complete. □

Remark 3.5. It was pointed out in [1] that for $f(x) = x^3$, $x \in [0, 1]$, an estimate similar to (3.2) is not possible.

Remark 3.6. Theorems 3.4 and 3.7 imply for the function $g(x)$ in (2.1) the two-sided pointwise inequality

$$c\omega_2^\varphi\Big(g, \sqrt{\frac{x(1-x)}{n}}\Big) \le |B_n(g,x)-g(x)| \le C\omega_2^\varphi\Big(g, \sqrt[4]{\frac{x(1-x)}{n}}\Big). \tag{3.3}$$

Very recently, motivated by the result of Lupaş and considerations set out in [1, 2, 12] we proved in [15] that the values of $\alpha_1 = \alpha_2 = 1$ and $a_1 = a_2 = \frac{1}{2}$ in (1.4) are optimal, that is, we proved the following result.

Theorem A. *It is not possible to find $a_1 > \frac{1}{2}$, or $a_2 > \frac{1}{2}$, or $\alpha_1 < 1$, or $\alpha_2 < 1$, such that*

$$k_1 \cdot \frac{x^{\alpha_1}(1-x)^{\alpha_2}}{n} \le R_n(g,x) \le K_2 \cdot \frac{x^{a_1}(1-x)^{a_2}}{\sqrt{n}}$$

holds true for all $x \in [0, 1]$ with some positive numbers k_1, K_2, independent of x and n.

Our next statement is the following theorem.

Theorem 3.7. *In both sides of (3.3) it is not possible to put one and the same modulus: neither $\omega_2^\varphi(g, \sqrt{x(1-x)/n})$ nor $\omega_2^\varphi(g, \sqrt[4]{x(1-x)/n})$.*

Proof. First we suppose that $\omega_2^\varphi(g, \sqrt[4]{x(1-x)/n})$ could be placed in the left-hand side of (3.3). Setting $x = \frac{1}{2}$ in (1.2), we obtain

$$\Delta_{h\varphi}^2 g\Big(\frac{1}{2}\Big) = h^2\cdot\varphi^2\Big(\frac{1}{2}\Big)\cdot g''(\xi) \ge h^2\cdot\Big(\frac{1}{2}\Big)^2\cdot\frac{1}{\frac{1}{2}(1-\frac{1}{2})} = h^2.$$

Hence by

$$t := \sqrt[4]{\frac{x(1-x)}{n}}, \qquad x \in [0, 1] \text{ fixed},$$

we have

$$\omega_\varphi^2(g, t) \ge t^2 = \sqrt{\frac{x(1-x)}{n}}.$$

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From our supposition and the last inequality we get

$$c\sqrt{\frac{x(1-x)}{n}} \le |B_n(g,x) - g(x)|,$$

which contradicts the statement of Theorem A (left-hand side of the inequality, as $x \to 0$). Also if we suppose that $\omega_2^{\varphi}(g, \sqrt{x(1-x)/n})$ could be placed in the right-hand side of (3.3) due to the fact that (see [3, Theorem 6.1])

$$\omega_2^{\varphi}(g,t) \le Ct^2\|\varphi^2 g''\| = Ct^2 \cdot 1,$$

the last inequality would imply that

$$|B_n(g,x) - g(x)| \le C\frac{x(1-x)}{n},$$

which again contradicts Theorem A (right-hand side of the inequality, as $x \to 0$). The proof of Theorem 3.7 is complete. □

REMARK 3.8. The upper pointwise bound in (3.1) in terms of the classical modulus of continuity $\omega_2(g, \sqrt{x(1-x)/n})$ was first established in [13]. As already mentioned, this modulus is not appropriate as a lower bound.

It is known that for the 'test' function $f_1(x) = x^2$, $x \in [0,1]$,

$$B_n(f_1,x) - f_1(x) = \frac{x(1-x)}{n} \approx \omega_2\bigg(f_1, \sqrt{\frac{x(1-x)}{n}}\bigg).$$

What is the situation for all other continuous functions $f(x)$? In response to this question, we formulate the following result.

THEOREM 3.9. *There are no constants $c(f)$ and $C(f)$ such that*

$$c(f)\Omega_2(f,\sigma(n,x)) \le |B_n(f,x) - f(x)| \le C(f)\Omega_2(f,\sigma(n,x)) \tag{3.4}$$

holds true for all $f \in C[0,1]$, all $x \in [0,1]$ and all $n \in \mathbb{N}$ with appropriate constructive characteristic $\Omega_2(f,\cdot)$, where $\Omega_2(f,\cdot)$ satisfies the properties of second-order modulus of smoothness (or related K-functional) and argument $\sigma(n,x)$.

PROOF. The proof follows immediately from Theorem A and (2.2) for $g(x)$. We fix $n \in \mathbb{N}$ and take $x \to 0$. If we suppose that (3.4) holds true, this would imply simultaneously that

$$\Omega_2(g,\sigma(n,x)) \le k_1\frac{x(1-x)}{n} \quad \text{as } x \to 0,$$

$$\Omega_2(g,\sigma(n,x)) \ge K_2\sqrt{\frac{x(1-x)}{n}} \quad \text{as } x \to 0,$$

with some positive constants k_1, K_2 independent of n, x, which is not possible. Hence (3.4) fails for $g(x)$. □

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REMARK 3.10. The case of 'norm' estimates is quite different. We mention here the well-known equivalence result of Knoop and Zhou for Bernstein operators, namely

$$c\omega_2^{\varphi}\Big(f, \frac{1}{\sqrt{n}}\Big) \le \|B_n f - f\|_{C[0,1]} \le C\omega_2^{\varphi}\Big(f, \frac{1}{\sqrt{n}}\Big),$$

established in 1994 in [7]. Similar strong converse inequalities are valid for many other linear positive operators.

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References

[1] J.-D. Cao, H. Gonska and D. Kacsó, 'On the impossibility of certain lower estimates for sequences of linear operators', *Math. Balkanica* **19** (2005), 39–58.
[2] J.-D. Cao, H. Gonska and D. Kacsó, 'On the second order classical and (weighted) Ditzian–Totik moduli of smoothness', in: *Mathematical Analysis and Approximation Theory, Proc. 6th Romanian–German Sem. Approx. Theory and its Appl.,* Baisoara, 2004 (eds. I. Gavrea *et al.*) (Mediamira Science Publisher, Cluj-Napoca, 2005), pp. 35–42.
[3] R. A. DeVore and G. G. Lorentz, *Constructive Approximation* (Springer, Berlin, 1993).
[4] Z. Ditzian, 'Direct estimate for Bernstein polynomials', *J. Approx. Theory* **79** (1994), 165–166.
[5] Z. Ditzian and V. Totik, *Moduli of Smoothness* (Springer, New York, 1987).
[6] M. Felten, 'Direct and inverse estimates for Bernstein polynomials', *Constr. Approx.* **14** (1998), 459–468.
[7] H. B. Knoop and X.-l. Zhou, 'The lower estimate for linear positive operators (II)', *Res. Math.* **25** (1994), 315–330.
[8] A. Lupaş, 'On a problem proposed by G. Tachev', in: *Mathematical Analysis and Approximation Theory, Proc. 5th Romanian–German Seminar* (eds. A. Lupaş *et al.*) (Burg, Sibiu, 2002), p. 326.
[9] A. Lupaş, L. Lupaş and V. Maier, 'The approximation of a class of functions', in: *Mathematical Analysis and Approximation Theory, Proc. 5th Romanian–German Seminar* (eds. A. Lupaş *et al.*) (Burg, Sibiu, 2002), pp. 155–168.
[10] V. Maier, 'The L_1-saturation class of the Kantorovich operator', *J. Approx. Theory* **22** (1978), 227–232.
[11] V. Maier, 'A new proof for the approximation of the Log-function by Kantorovich polynomials in the L_p-norm', *Rev. Anal. Numér. Théor. Approx.* **28**(2) (1999), 173–177.
[12] P. E. Parvanov and B. D. Popov, 'The limit case of Bernstein's operators with Jacobi-weights', *Math. Balkanica* **8**(2–3) (1994), 165–177.
[13] T. Popoviciu, 'Sur l'approximation des fonctions convexes d'ordre superieur', *Math. Cluj* **10** (1935), 49–54.
[14] G. Tachev, 'Three open problems', in: *Mathematical Analysis and Approximation Theory, Proc. 5th Romanian–German Seminar* (eds. A. Lupaş *et al.*) (Burg, Sibiu, 2002), p. 329.
[15] G. Tachev, 'Approximation of logarithmic function by Bernstein operator', manuscript, submitted, 2011.

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