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FIRST HOMOLOGY OF IRREDUCIBLE 3-MANIFOLDS

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I. Introduction and definitions. In [**2**], J. Gross provides an infinite collection of topologically distinct irreducible homology 3-spheres. In this paper, we construct for any finitely generated abelian group A, an infinite collection $\{M_i\}$ of topologically distinct irreducible closed 3-manifolds such that $H_1(M_i) = A$ for each i.

The proof consists of first constructing a closed irreducible 3-manifold M_A with $H_1(M_A) = A$, and then providing a method for producing more such manifolds with the same first homology group.

All maps and spaces in this paper are assumed to be in the piecewise linear category, and all subspaces are assumed to be piecewise linear subspaces.

A 3-manifold M is *irreducible* if each 2-sphere in M bounds a 3-cell in M. A compact 2-manifold (or *surface*) F in a compact 3-manifold M is *properly embedded* in M if $F \cap \text{bd}M = \text{bd}F$. (bd$X$ is used to denote the boundary of the manifold X.) A surface F properly embedded in a compact 3-manifold M is *incompressible in* M if given any disk D in M with $D \cap F = \text{bd}D$, there exists a disk D' in F so that $D' \cap D = \text{bd}D' = \text{bd}D$. In this paper we agree that 2-spheres and disks are not incompressible. A well known consequence of the loop theorem (see [**6**]) states that a 2-sided properly embedded surface F in a compact 3-manifold M is incompressible in M if and only if $i_*: \pi_1(F) \to \pi_1(M)$ is an injection.

Haken [**3**] has shown that given a compact oriented irreducible 3-manifold M, there is a unique integer $P(M) \geqq 0$ associated with M with the following properties.

(1) There exists a collection $\{F_i | 1 \leqq i \leqq P(M)\}$ of mutually disjoint orientable incompressible surfaces properly embedded in M such that no pair of surfaces F_i, F_j cobound a product in M.

(2) If $\{G_i | 1 \leqq i \leqq k\}$ is a collection of mutually disjoint orientable incompressible surfaces properly embedded in M, and if $k > P(M)$, then some pair G_i, G_j cobounds a product in M.

We shall find $P(M)$ a convenient topological invariant to determine that certain manifolds are not homeomorphic.

We refer to Crowell and Fox [**1**] as a reference for the basic notions we shall use concerning knot spaces.

II. Construction of the manifolds.

LEMMA 1. *If M is an irreducible, orientable, closed, 3-manifold and if*

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$\pi_1(M) \neq 0$, *then there exists a closed, irreducible, orientable* 3-*manifold* M' *with* $P(M') > P(M)$, $\pi_1(M') \neq 0$, *and* $H_1(M') = H_1(M)$.

Proof. Let $\{F_i | 1 \leqq i \leqq P(M)\}$ be a mutually disjoint collection of closed orientable incompressible surfaces in M such that no pair F_i, F_j cobounds a product in M. Since $\pi_1(M) \neq 0$, no component of $M - \bigcup_{i=1}^{P(M)} F_i$ is simply connected. Let J be a simple closed surve in $M - \bigcup_{i=1}^{P(M)} F_i$ such that J is not contractible in M. Let $N(J)$ denote a regular neighbourhood of J in $M - \bigcup_{i=1}^{P(M)} F_i$, and let $M_1 = \mathrm{cl}(M - N(J))$. If M_1 is a solid torus, then M is a lens space and so $P(M) = 0$. It is easy to choose a new noncontractible curve J^* in M in such a fashion that M_1^* is not a solid torus. Thus in any case, we are able to choose J so that M_1 is not a solid torus.

Let S be a 2-sphere in M_1. Then S bounds a 3-cell B in M. Since J is not contractible and $J \cap S = \emptyset$, we must have $J \cap B = \emptyset$. If follows that M_1 is irreducible.

Suppose the boundary of M_1 is compressible in M_1. Let D be a disk in M_1 with $D \cap \mathrm{bd}M_1 = \mathrm{bd}D$, a nontrivial curve in $\mathrm{bd}M_1$. Since M_1 is irreducible, $\mathrm{cl}(M_1 - N(D))$ is a 3-cell where $N(D)$ denotes a regular neighbourhood of D in M_1. Then M_1 would be a solid torus contrary to our choice of J.

Let K be a knot space (the complement of the interior of a regular neighbourhood of a simple closed curve in S^3), which contains a closed incompressible surface G of genus greater than one. H. Lyon [**5**] has shown that there exist knot spaces which contain closed incompressible surfaces of arbitrarily large genus. We assume that K is embedded in S^3. Let m be a nontrivial curve on $\mathrm{bd}K$ such that m bounds a disk in $\mathrm{cl}(S^3 - K)$. We shall refer to m as the *meridian* of K. Note that m generates the first homology of K. There is a unique curve l in $\mathrm{bd}K$ called a *longitude* of K such that l is not contractible in $\mathrm{bd}K$, but l is homologous to zero in K.

For notational convenience, let $T = \mathrm{bd}M_1$, $N(J) = D \times S^1$, $a = (\mathrm{bd}D) \times 0$, and $b = v \times S^1$, where v is a point on $\mathrm{bd}D$. Let $h\colon \mathrm{bd}K \to T$ be a homeomorphism that maps l onto a and m onto b, and let M' be the adjunction space $M_1 U_h K$. Since both K and M_1 have incompressible boundary, M' is again an irreducible 3-manifold.

Suppose G cobounds a product in M' with some surface F_i. Then since T separates G from F_i, we have $T \subset G \times [0, 1]$, and $i_* \pi_1(T) \to \pi_1(G \times [0,1])$ is an injection. But this is impossible since as proved in [**4**] the fundamental group of a closed surface of genus greater than one contains no abelian subgroups other than the infinite cyclic group. A similar argument shows that no pair F_i, F_j can cobound a product in M'. It follows that $P(M') > P(M)$.

We now claim that $H_1(M) = H_1(M')$. From the exact homology sequence of the pair (M_1, a) we obtain

$$H_1(a) \xrightarrow{j_1} H_1(M_1) \to H_1(M_1, a) \to 0.$$

The relative homology group $H_1(M_1, a)$ is isomorphic to $H_1(M_1 \cup Ca) \approx$

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$H_1(M)$ where Ca denotes the cone over the simple closed curve a. This together with the Mayer-Vietoris sequence yields the following commutative diagram whose rows are exact.

$$\begin{array}{ccccc} H_1(a) & \xrightarrow{j_1} & H_1(M_1) & \rightarrow H_1(M) \rightarrow 0 & \\ \downarrow j_2 & \searrow^{j_1 \oplus 1} & & & \\ H_1(T) & \xrightarrow{i_1 \oplus i_2} & H_1(M_1) \oplus H_1(K) & \rightarrow H_1(M') \rightarrow 0 & \end{array}$$

Commutativity of the diagram follows from the naturality of the maps i_1, i_2, j_1, j_2 and the fact that $i_2(a) = l$ which is homologically trivial in K. Define

$$f: \frac{H_1(M_1)}{j_1(H_1(a))} \rightarrow \frac{H_1(M_1) \oplus H_1(K)}{i_1 \oplus i_2(H_1(T))} \quad \text{by } f(\bar{x}) = \overline{(x, 1)}.$$

Recalling that $H_1(K)$ is the infinite cyclic group generated by $m = i_2(b)$, it is straightforward to verify that f is an isomorphism. This completes the proof of Lemma 1.

THEOREM. *Let A be a finitely generated abelian group. Then there are infinitely many closed orientable irreducible 3-manifolds $\{M_i\}$ with $H_1(M_i) = A$ for each i, and if $i \neq j$ then M_i is not homeomorphic to M_j.*

Proof. According to Lemma 1, it suffices to construct only one closed orientable irreducible 3-manifold M_A with $\pi_1(M_A) \neq 0$, and $H_1(M_A) = A$ for each finitely generated abelian group A. We shall need three basic manifolds for the required constructions.

The first manifold we need is a Seifert fibre space M_1 over a surface of genus n with one boundary component r, and k singular fibres $h_1, h_2, \ldots, h_k$. M_1 can be constructed as follows. Let F be a closed surface of genus n and let F_1 be the surface obtained from F by removing the interiors of $k + 1$ disjoint disks whose boundaries we label $r, h_1, h_2, \ldots, h_k$. Set $M_0 = F_1 \times S^1$. The boundary of M_0 consists of tori $r \times S^1$, $h_1 \times S^1, \ldots, h_k \times S^1$. Choose points $x_0, x_1, \ldots, x_k$ on the curves $r, h_1, \ldots, h_k$ and put $w_i = x_i \times S^1$, $0 \leqq i \leqq k$. M_1 is completed by sewing solid tori $D_i \times S^1$ onto the boundary components $h_1 \times S^1, \ldots, h_k \times S^1$ of M_0, so that the curve $(\mathrm{bd}D_i) \times 0$ goes to the curve $h_i^{\alpha_i} w_k^{-1}$ where α_i is any nonzero integer. The integer α_i will be refered to as the *index of the fiber* h_i. Van Kampen's theorem yields the following presentation for $\pi_1(M)$.

$$\pi_1(M_1) = \Big(p_i, q_i, r, h_j, w \,|\, r \prod_{i=1}^{k} h_i = \prod_{i=1}^{n} [p_i, q_i], h_j^{\alpha_i} = w, [w, p_i]$$
$$= [w, q_i] = [w, h_j] = [w, r] = 1\Big), 1 \leqq i \leqq n, 1 \leqq j \leqq k.$$

Note that in case the genus of M_1 is zero, the relation $r \prod_{i=1}^{k} h_i = \prod_{i=1}^{n}[p_i q_i]$ is to be replaced by the relation $r \prod_{i=1}^{k} h_i = 1$. We shall need the following information about M_1.

LEMMA 2. *If the genus of M_1 is greater than zero, or if the genus of M_1 is zero and M_1 has at least two singular fibres of index greater than one, then* bdM_1 *is incompressible.*

Proof. Since M_1 is irreducible and orientable we need only show that M_1 is not a solid torus. We have that

$$H_1(M_1) = 2n\mathbf{Z} \oplus \left(r, h_j, w | r \prod_{i=1}^{k} h_i = 1, h_i^{\alpha i} = w,\right.$$

$$\left.[h_i, h_j] = [r, h_j] = [w, r] = [w, h_j] = 1\right), 1 \leqq j \leqq k, 1 \leqq i \leqq k.$$

In particular we see that if $n \neq 0$, then $H_1(M_1) \neq \mathbf{Z}$, the first homology group of a solid torus. If the genus of M_1 is zero, then

$$\pi_1(M_1) = \left(h_j, r, w | r \prod_{i=1}^{k} h_i = 1, h_j^{\alpha_j} = w, [w, h_j] = [w, r] = 1\right), \quad 1 \leqq j \leqq k.$$

Let G be the smallest normal subgroup in $\pi_1(M_1)$ containing the element w. Then

$$\begin{aligned}\pi_1(M_1)/G &= \left(h_j, r | r = \left(\prod_{i=1}^{k} h_i\right)^{-1}, h_j^{\alpha_j} = 1\right) \\ &= (h_j | h_j^{\alpha_j} = 1) \quad 1 \leqq j \leqq k \\ &= \mathbf{Z}_{\alpha_1} * \mathbf{Z}_{\alpha_2} * \ldots * \mathbf{Z}_{\alpha_k}.\end{aligned}$$

Since at least two of the α_i's are not equal to one, we have that $\pi_1(M_1)$ has a factor group that is a nontrivial free product. Thus $\pi_1(M_1) \neq \mathbf{Z}$ since each factor group of $\mathbf{Z}$ is abelian. This completes the proof of Lemma 2.

M_2 is a trefoil knot space so that $\pi_1(M_2) = (x_1, x_2 | x_1 x_2 x_1 = x_2 x_1 x_2)$. The generators x_1, x_2 of $\pi_1(M_2)$ are chosen so that a meridian m is homotopic to x_1, and the longitude l is homotopic to $x_1^{-4} x_2 x_1^2 x_2$.

M_3 is a bundle with base S^1 and fibre H a torus with the interior of a disk removed. Let s denote the boundary curve of H. Choose generators a, b for $\pi_1(H)$ such that $[a, b]$ is homotopic in H to s. Let $f\colon H \rightarrow H$ be an orientation preserving homeomorphism that maps a onto a and b onto ab. We require further that f fixes a point u on s. M_3 is then the identification space obtained from $H \times [0, 1]$ by identifying $h \times 0$ with $f(h) \times 1$ for each h in H. Let y be the simple closed curve $u \times [0, 1]$ with $u \times 0$ identified with $f(u) \times 1 = u \times 1$. By Van Kampen's theorem,

$$\pi_1(M_3) = (a, b, v | vav^{-1} = a, vbv^{-1} = ab).$$

Case 1: *A is of the form $2n\mathbf{Z} \oplus \mathbf{Z}_{t_1} \oplus \ldots \oplus \mathbf{Z}_{t_k}$, $t_i \neq 1$, $1 \leqq i \leqq k$ with $n \geqq 1$ or $k \geqq 2$.* We construct M_A as follows. Take M_1 to be of genus n with k

singular fibres $h_1, \ldots, h_k$ of indexes $t_1, \ldots, t_k$. Sew the boundary of the knot space M_2 onto the boundary of M_1 by a homeomorphism that takes the longitude of M_2 onto w_0, and the meridian of M_2 onto r. M_A is irreducible because it is constructed by sewing together irreducible manifolds with incompressible boundary. Appealing once again to Van Kampen's theorem,

$$\pi_1(M_A) = \Big(p_i, q_i, h_j, r, w, x_1, x_2 | r \prod_{i=1}^{k} h_i = \prod_{i=1}^{n} [p_i, q_i], x_1x_2x_1 = x_2x_1x_2,$$
$$h_j^{t_j} = w, [w, p_i] = [w, q_i] = [w, r] = [w, h_j] = 1,$$
$$x_1 = r, w = x_1^{-4}x_2x_1^{2}x_2\Big), \quad 1 \leqq i \leqq n, 1 \leqq j \leqq k.$$

Abelianizing we obtain

$$H_1(M_A) = 2n\mathbf{Z} \oplus (h_1, \ldots, h_k | h_i^{t_i} = 1, [h_i, h_j] = 1) = A,$$
$$1 \leqq i \leqq k, 1 \leqq j \leqq k$$

Case 2: *A is of the form* $(2n + 1)\mathbf{Z} \oplus \mathbf{Z}_{t_1} \oplus \ldots \oplus \mathbf{Z}_{t_k}$, $n \geqq 1$ *or* $k \geqq 2$. We begin as in Case 1 with M_1 of genus n with k singular fibres $h_1, \ldots, h_k$ with indexes $t_1, \ldots, t_k$. M_A is obtained by sewing the boundary of M_3 onto the boundary of M_1 by a homeomorphism that takes y onto r and s onto w_0.

$$\pi_1(M_A) = \Big(p_i, q_i, h_j, w, r, a, b, v | r \prod_{i=1}^{k} h_i = \prod_{i=1}^{n} [p_i, q_i], r = v,$$
$$h_j^{t_3} = w, [w, p_i] = [w, q_i] = [w, r] = [w, h_j] = 1,$$
$$[a, b] = w, vav^{-1} = a, vbv^{-1} = ab\Big), \quad 1 \leqq i \leqq n, 1 \leqq j \leqq k.$$

Abelianizing we arrive at

$$\begin{aligned} \mathrm{H}_1(M_A) &= 2n\mathbf{Z} \oplus (h_j, b | [h_i, h_j] = [h_j, b] = 1, h_i^{t_i} = 1), 1 \leqq i \leqq k, 1 \leqq j \leqq k \\ &= (2n + 1)\mathbf{Z} \oplus \mathbf{Z}_{t_1} \oplus \ldots \oplus \mathbf{Z}_{t_k} = A \end{aligned}$$

M_A is irreducible just as in Case 1.

Case 3: *A is of the form* $\mathbf{Z} \oplus \mathbf{Z}_t$, $t \geqq 1$. (*This includes the case* $A = \mathbf{Z}$.) M_A is a bundle with fibre T a torus and base S^1.

Let a and b be simple closed curves on T that meet in exactly one transverse intersection. Let $h: T \to T$ be an orientation preserving homeomorphism that maps a onto b and b onto the curve $a^{-1}b^{t+2}$. Van Kampen's theorem provides the following presentation for $\pi_1(M_A)$.

$$\begin{aligned} \pi_1(M_A) &= (a, b, v | vav^{-1} = b, vbv^{-1} = a^{-1}b^{t+2}) \\ H_1(M_A) &= (a, b, v | a = b, b = a^{-1}b^{t+2}, [a, b] = [a, v] = [b, v] = 1) \\ &= (b, v | b^t = 1, [b, v] = 1) \\ &= \mathbf{Z} \oplus \mathbf{Z}_t. \end{aligned}$$

Case 4: $A = \mathbf{Z}_t$, $t \neq 1$. Choose M_A to be a lens space (the union of two solid tori) with fundamental group $\mathbf{Z}_t$.

Case 5: *A is the trivial group*. Many well known examples are available. For example, M_A can be constructed by sewing together two copies M_2', M_2'' of M_2 by a homeomorphism that maps the longitude of M_2' onto the meridian of M_2'' and the meridian of M_2' onto the longitude of M_2''.

This completes the proof of the Theorem.

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