

SHORT NOTES

THE LOCAL DISTRIBUTION OF STRESS NEAR A POINT OF ZERO SHEAR STRESS IN A RECTILINEAR FLOW FIELD

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ABSTRACT. The distribution of stress in the vicinity of a point at which shear stress magnitude is zero is investigated analytically for rectilinear flow of a fluid in a channel or pipe. For a fluid with non-linear power-law properties the contours of constant stress and velocity either approach circular or flat shapes near such a point, irrespective of the particular boundary conditions. There are no intermediate cases, although such intermediate behavior exists for linear fluids.

RÉSUMÉ. La répartition locale des contraintes près du point sans cisaillement dans un champ d'écoulement rectiligne. La distribution de la contrainte au voisinage d'un point où la contrainte de cisaillement est nulle, est étudiée analytiquement pour l'écoulement rectiligne d'un fluide dans un canal ou un tuyau. Pour un fluide dont l'écoulement suit une loi-puissance non linéaire, la forme des lignes d'égales contraintes et d'égales vitesses se rapproche soit du cercle, soit aplatie près d'un tel point, quelles que soient les conditions aux limites. Il n'y a pas de cas intermédiaires, bien que de tels comportements intermédiaires existent pour des fluides linéaires.

ZUSAMMENFASSUNG. Die lokale Spannungsverteilung in der Nähe eines Punktes mit verschwindender Scherspannung in einem geradlinigen Fließfeld. Die Spannungsverteilung in der Umgebung eines Punktes, an dem die Scherspannung verschwindet, wird analytisch für geradliniges Fließen einer Flüssigkeit in einem Kanal oder in einer Röhre untersucht. Bei einer Flüssigkeit mit nicht-linearen, potentiellen Eigenschaften nähern sich dort die Linien gleicher Spannung und Geschwindigkeit entweder kreisförmigen oder flachen Formen, ungeachtet der besonderen Randbedingungen. Es gibt keine Übergangszustände, obwohl ein solches Zwischenverhalten bei linearen Flüssigkeiten auftritt.

NYE (1965) has computed numerically the distribution of velocity and stress for the rectilinear flow of ice acted on by gravity in sloped cylindrical channels of rectangular, elliptical, and parabolic cross sections. In these calculations he assumed that no slip occurs at the channel boundary, that the ice is homogeneous and obeys a power-type flow law with exponent of 3, and that the flow is steady. One interesting result of the numerical calculations discussed by Nye concerns the distribution of stress and velocity in the neighborhood of the point of maximum velocity and zero shear-stress magnitude. He discovered that locally contours of constant velocity and shear-stress magnitude approach circular shape near this point for all of the channel shapes considered by him (Nye, 1965, p. 679).

In this note we demonstrate that there are two possibilities for the local distribution of stress and velocity around any point where the shear-stress magnitude is zero. Either the local flow field has circular symmetry, as in Nye's computations, or it is planar, i.e. the contours of constant velocity and shear-stress magnitude are flat. There are no other possibilities. This conclusion is independent of the cross-section shape or the distribution of slip velocity around the perimeter of the cross-section as long as the material obeys a power-type flow law with exponent greater than one. This is significant in terms of application to glaciers, which may have complex channel shapes and distribution of sliding velocity.

For rectilinear flow the equations of motion without acceleration and the relationship between stress and strain-rate are much simplified as discussed by Nye (1965). Choose the x-axis parallel to the flow direction and let the velocity be u. The equations of static equilibrium reduce to

$$\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f = 0 \tag{1}$$

where τ_{xy} and τ_{xz} are standard Cartesian components of stress and f is an effective body force related to the x component of gravity in a sloped channel or a pressure gradient in a pipe. A power-type flow law for simple shear reduces to

$$\frac{\partial u}{\partial y} = 2A\tau^{n-1}\tau_{xy}, \tag{2a}$$

$$\frac{\partial u}{\partial z} = 2A\tau^{n-1}\tau_{xz}, \tag{2b}$$

where $\tau = (\tau_{xy}^2 + \tau_{xz}^2)^{1/2}$ is the shear-stress magnitude. We assume that A and n are constants independent of position. These three equations must be satisfied throughout the cross-section and determine the three functions τ_{xy} , τ_{xz} and u when augmented by appropriate boundary conditions.

One requirement on any solution is that $\partial^2 u / \partial y \partial z = \partial^2 u / \partial z \partial y$. From Equations (2a) and (2b) this requirement expressed in terms of the stress components is

$$0 = 2A\tau^{n-1} \left\{ \frac{\partial \tau_{xy}}{\partial z} - \frac{\partial \tau_{xz}}{\partial y} + (n-1) \left[\frac{\tau_{xy}^2}{\tau^2} \frac{\partial \tau_{xy}}{\partial z} - \frac{\tau_{xz}^2}{\tau^2} \frac{\partial \tau_{xz}}{\partial y} + \frac{\tau_{xy}\tau_{xz}}{\tau^2} \left(\frac{\partial \tau_{xz}}{\partial z} - \frac{\partial \tau_{xy}}{\partial y} \right) \right] \right\}. \tag{3}$$

This is simply a statement of strain-rate compatibility. It is a necessary and sufficient condition for the existence of a single-valued u with continuous second partial derivatives which satisfies Equation (2).

Let us suppose that the cross-section and the distribution of slip velocity are such that there is at least one stationary point ($\partial u / \partial y = \partial u / \partial z = 0$ and $\tau_{xy} = \tau_{xz} = 0$) in the interior or on the boundary of the cross-section. Let us choose coordinates so that $y = z = 0$ at this point, and assume that the stress components have continuous first derivatives. We may write them in terms of an expansion to first order:

$$\left. \begin{aligned} \tau_{xy} &= -f[\beta y + \delta z + g(y, z)], \\ \tau_{xz} &= -f[\gamma z + \xi y + h(y, z)], \end{aligned} \right\} \tag{4}$$

where

$$\beta = -\frac{1}{f} \frac{\partial \tau_{xy}}{\partial y} \Big|_{y=z=0}, \quad \gamma = -\frac{1}{f} \frac{\partial \tau_{xz}}{\partial z} \Big|_{y=z=0}, \quad \delta = -\frac{1}{f} \frac{\partial \tau_{xy}}{\partial z} \Big|_{y=z=0} \quad \text{and} \quad \xi = -\frac{1}{f} \frac{\partial \tau_{xz}}{\partial y} \Big|_{y=z=0}.$$

In these expansions there are no zero-order terms because $\tau_{xy} = \tau_{xz} = 0$ at $y = z = 0$ by hypothesis. Further, the remainder terms g and h and their first derivatives go smoothly to zero as y and z go to zero. The first derivatives go as r^k with $k > 0$, which we indicate by the notation $O(k)$; g and h go as r^{1+k} or are $O(1+k)$.

We may notice immediately that static equilibrium, Equation (1), requires

$$1 - \beta - \gamma = 0. \tag{5}$$

Now consider whether compatibility, Equation (3), places any additional constraints on β , γ , δ and ξ . At $y = z = 0$ where $\tau = 0$, Equation (3) is automatically satisfied because the quantity in brackets is finite. At other points where $\tau \neq 0$, Equation (3) gives

$$0 = \{ [\beta^2 + \xi^2]y^2 + [\delta^2 + \gamma^2]z^2 + 2[\beta\delta + \gamma\xi]yz + O(2+k) \} \{ \delta - \xi + O(k) \} + \{ (n-1) [\beta^2 y^2 + \delta^2 z^2 + 2\beta\delta yz + O(2+k)] \} \{ \delta + O(k) \} - \{ \xi^2 y^2 + \gamma^2 z^2 + 2\gamma\xi yz + O(2+k) \} \{ \xi + O(k) \} + \{ \beta\xi y^2 + \gamma\delta z^2 + [\beta\gamma + \delta\xi]yz + O(2+k) \} \{ \gamma - \beta + O(k) \} \tag{6}$$

upon substitution of the stress components in the form of Equation (4). If this is to be satisfied at all locations, the coefficients for the y^2 , z^2 and yz terms must be individually zero. Namely

$$0 = (\beta^2 + \xi^2)(\delta - \xi) + (n-1)(\beta^2\delta - \xi^3 + \beta\xi[\gamma - \beta]), \tag{7a}$$

$$0 = (\delta^2 + \gamma^2)(\delta - \xi) + (n-1)(\delta^3 - \gamma^2\xi + \delta\gamma[\gamma - \beta]), \tag{7b}$$

$$0 = (\beta\delta + \gamma\xi)(\delta - \xi) + (n-1)(\beta\delta^2 - \gamma\xi^2 + [\beta\gamma - \delta\xi][\gamma - \beta]). \tag{7c}$$

Of course, these alone are not sufficient to guarantee that Equation (6) is satisfied, since there are additional conditions associated with higher order terms. However, Equations (7) are necessary for the stress distribution of form of Equations (4) to be compatible.

To find the possible values for β , γ , δ and ξ , we first note that $\delta = \xi$. This can be demonstrated by adding Equations (7a) and (7b) to get

$$0 = (\delta - \xi) \{ \beta^2 + \gamma^2 + \delta^2 + \xi^2 + \frac{1}{2}(n-1)[(\beta - \gamma)^2 + \beta^2 + \gamma^2 + (\delta + \xi)^2 + \delta^2 + \xi^2] \}. \tag{8}$$

The quantity in the bracket is necessarily non-zero and positive for real values of β , γ , δ and ξ , since at least one of these must be non-zero in order to satisfy Equation (5) and $n \geq 1$. We note next that we may orient the coordinate system such that $\partial \tau_{xy} / \partial z = 0$ at $y = z = 0$, and consequently $\delta = 0$ (and

also $\xi = 0$). This is obvious if the overall flow has a plane of reflection symmetry. In this case we need only choose the y -axis perpendicular to this plane, so $\tau_{xy}(0, z) = 0$. It is not so obvious for general boundary conditions but it can be demonstrated by considering the transformation properties of the third order tensor $\partial\tau_{ij}/\partial x_k$ and using the fact that $\delta = \xi$. For such a choice of coordinate orientation, we see that Equations (7a) and (7b) are trivially satisfied; Equation (7c) becomes

$$0 = (n-1)\beta\gamma(\gamma-\beta). \tag{9}$$

If we now combine the constraints on β , γ , δ and ξ associated with Equation (1) and Equation (3) as expressed in Equation (5) and Equation (9), we see the possible local stress distribution can be written

$$\left. \begin{aligned} \tau_{xy} &= -f[\beta\gamma + g(y, z)], \\ \tau_{xz} &= -f[(1-\beta)z + h(y, z)], \end{aligned} \right\} \tag{10}$$

where β must satisfy

$$0 = (n-1)\beta(1-\beta)(1-2\beta). \tag{11}$$

For linear rheology ($n = 1$), there are no restrictions on the value β may assume. However, for non-linear rheology ($n > 1$), β may have only three possible values $\frac{1}{2}$, 1 or 0, which correspond respectively to contours of constant τ with semicircular shape, flat shape perpendicular to y , or flat shape perpendicular to z .

The form of the local velocity field can be deduced by integration of Equations (2). For $n = 1$,

$$u(y, z) = u(0, 0) - Af[\beta y^2 + (1-\beta)z^2] + O(2+k).$$

Since there is no restriction on β , there is a continuum of possible contour shapes including locally elliptical, flat, or hyperbolic patterns. This is not so for $n > 1$. With $\beta = \frac{1}{2}$

$$u(y, z) = u(0, 0) - \frac{2A}{n+1} \left(\frac{f}{2}\right)^n r^{n+1} + O(n+1+k) \tag{12a}$$

where $r = (y^2 + z^2)^{\frac{1}{2}}$ and the contours are locally semicircular. With $\beta = 1$

$$u(y, z) = u(0, 0) - \frac{2A}{n+1} f^n y^{n+1} + O(n+1+k) \tag{12b}$$

and the contours are locally flat. The result for $\beta = 0$ is the same as Equation (12b), but with z replacing y . There are no cases intermediate between locally semicircular and flat contour patterns.

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REFERENCE

Nye, J. F. 1965. The flow of a glacier in a channel of rectangular, elliptic or parabolic cross-section. *Journal of Glaciology*, Vol. 5, No. 41, p. 661-90.