PRIME IDEALS OF QUANTIZED WEYL ALGEBRAS

by M. AKHAVIZADEGAN† and D. A. JORDAN

(Received 1 March, 1995)

1.1. Introduction. The main object of study in this paper is the quantized Weyl algebra $A_n^{\bar{q},\Lambda}$ which arises from the work of Maltsiniotis [**10**] on noncommutative differential calculus. This algebra has been studied from the point of view of noncommutative ring theory by various authors including Alev and Dumas [**1**], the second author [**9**], Cauchon [**3**], and Goodearl and Lenagan [**5**]. In [**9**], it is shown that $A_n^{\bar{q},\Lambda}$ has n normal elements z_i and, subject to a condition on the parameters, the localization $B_n^{\bar{q},\Lambda}$ obtained on inverting these elements is simple of Krull and global dimension n. It is easy to show that each of these normal elements generates a height one prime ideal and that these are all the height one prime ideals of $A_n^{\bar{q},\Lambda}$. The purpose of this paper is to determine, under a stronger condition on the parameters, all the prime ideals of $A_n^{\bar{q},\Lambda}$ and to compare the prime spectrum with that of a related algebra $\mathcal{A}_n^{\bar{q},\Lambda}$. This algebra has more symmetric defining relations than those of $A_n^{\bar{q},\Lambda}$ but it shares the same simple localization $B_n^{\bar{q},\Lambda}$, which again is obtained by inverting n normal elements z_i. Like $A_n^{\bar{q},\Lambda}$, the alternative algebra can be regarded as an algebra of skew differential (or difference) operators on the coordinate ring of quantum n-space.

The construction of both $A_n^{\bar{q},\Lambda}$ and $\mathcal{A}_n^{\bar{q},\Lambda}$ involves parameters q_i, $1 \leq i \leq n$ and λ_{ji}, $1 \leq i < j \leq n$. Let $G(\Lambda, \bar{q})$ be the subgroup of the multiplicative group of the base field k generated by these parameters. The conditions that we impose on the parameters are that the ranks of certain subgroups of $G(\Lambda, \bar{q})$ should be maximal. Under these conditions, in both $A_n^{\bar{q},\Lambda}$ and $\mathcal{A}_n^{\bar{q},\Lambda}$, every nonzero prime is generated by a normalizing sequence of generators and we give an explicit description of these sequences. However there are some significant differences between the two cases. In $\mathcal{A}_n^{\bar{q},\Lambda}$, provided $n > 1$, the spectrum is finite and every nonzero prime ideal is generated by a subsequence of the sequence $z_1, z_2, \ldots, z_n$. Thus the maximal length of a chain of prime ideals is n. On the other hand, in $A_n^{\bar{q},\Lambda}$ there are always infinitely many maximal ideals and they have height $2n$. However the number of nonmaximal prime ideals is finite. One interesting feature is that there is a unique prime ideal of height $2n-1$. This is consistent with the known spectrum for the first quantized Weyl algebra A_1^q [**4**, 8.4] and it is also closer in nature to that for the coordinate ring of quantum space under the analogous conditions on the parameters.

In the remainder of Section 1, we establish the basic terminology for the rings to be considered later in the paper. Sections 2, 3 and 4 will discuss the prime ideals of the coordinate ring of quantum space, of $\mathcal{A}_n^{\bar{q},\Lambda}$ and $A_n^{\bar{q},\Lambda}$ respectively. The results on $A_n^{\bar{q},\Lambda}$ have been obtained independently by Rigal [**14**] under stronger conditions on the parameters. Although there are inevitable common features, Rigal's method, which is based on [**7**, Theorem 11.1], is essentially different. There are also features in common with the work of Goodearl and Lenagan [**5**] on catenarity and with the work of Oh [**13**] on the primitive ideals of the coordinate ring of quantum symplectic space.

† With the support of the Ministry of Culture and Higher Education of the Islamic Republic of Iran.

We thank the referee for some helpful suggestions, in particular, the inclusion of Corollary 3.5.

1.2. Quantum n-space. Throughout, k will be an algebraically closed field and k^* will denote the multiplicative group of k.

Let R be a k-algebra and let $r, s \in R$ and $q \in k^*$. We say that r q-commutes with s if $rs = qsr$. More symmetrically, we say that r and s semicommute (with each other) if r q-commutes with s for some $q \in k^*$.

Let $\Lambda = [\lambda_{ij}]$ be an $n \times n$ matrix of nonzero elements of k such that each $\lambda_{ii} = 1$ and λ_{ij} is always equal to λ_{ji}^{-1}. We denote by $A(\Lambda)$ the k-algebra generated by n indeterminates x_i, $1 \leq i \leq n$, subject to the semicommutation relations

$$x_j x_i = \lambda_{ji} x_i x_j.$$

This algebra is now usually called the coordinate ring of quantum n-space. Each x_i is normal in $A(\Lambda)$ and we denote by $P(\Lambda)$ the algebra obtained by inverting each x_i. These are the algebras considered in [**11**].

1.3. Iterated skew polynomial rings. Both $A_n^{\bar{q},\Lambda}$ and $\mathscr{A}_n^{\bar{q},\Lambda}$ arise from the construction of iterated skew polynomial rings studied in [**9**]. Here we describe this construction in the generality appropriate to this paper. For full details and for justification of the statements made below, see [**9**].

Let A be a k-algebra with two commuting k-automorphisms, α and γ, such that there exists a nonzero normal element v of A with $va = \gamma(a)v$ for all $a \in A$. Let $\beta = \alpha^{-1}\gamma$, let S denote the skew polynomial ring $A[y; \alpha]$ and extend β to S by setting $\beta(y) = \rho y$ for some nonzero $\rho \in k$. There is a β-derivation δ of S such that $\delta(A) = 0$ and $\delta(y) = v$. Let $R = R(A, \alpha, v, \rho)$ be the iterated skew polynomial ring $A[y; \alpha][x; \beta, \delta]$. Thus, for all $a \in A$,

$$ya = \alpha(a)y, \qquad xa = \beta(a)x, \qquad xy - \rho yx = v.$$

In this paper, we assume that $\rho \neq 1$ and also that $\alpha(v) = v$. The relation $xy - \rho yx = v$ can then be rewritten $xy - u = \rho(yx - u)$, where $u = \dfrac{1}{1-\rho} v$. The element

$$z = xy - yx = v + (\rho - 1)yx = \rho^{-1}(v + (\rho - 1)xy)$$

will be called the Casimir element of R. (This is a nonzero scalar multiple of the Casimir element $xy - u$ used in [**9**].) The following identities hold:

$$zy = \rho yz, \qquad zx = \rho^{-1}xz, \qquad za = \gamma(a)z \qquad \forall a \in A.$$

Consequently z is a normal element of R inducing a k-automorphism σ of R such that $\sigma(a) = \gamma(a)$ for all $a \in A$, $\sigma(y) = \rho y$ and $\sigma(x) = \rho^{-1}x$.

1.4. Quantized Weyl algebras. The quantized Weyl algebra $A_n^{\bar{q},\Lambda}$ in $2n$ variables $y_1, x_1, \ldots, y_n, x_n$ studied in [**1**], [**3**], [**5**], [**9**] is obtained from the base field k by n applications of the construction in 1.3. The first choice of v is 1 and, at each subsequent application, v is the Casimir element from the previous application.

The relations are as follows: for $1 \le i < j \le n$,

$$x_i x_j = q_i \lambda_{ij} x_j x_i, \qquad y_j y_i = \lambda_{ji} y_i y_j,$$
$$x_i y_j = \lambda_{ji} y_j x_i, \qquad x_j y_i = q_i \lambda_{ij} y_i x_j,$$
$$x_j y_j - q_j y_j x_j = 1 + \sum_{i=1}^{j-1} (q_i - 1) y_i x_i,$$

where Λ is as in 1.2 and $\bar{q} = (q_1, q_2, \ldots, q_n)$ is an n-tuple of elements of $k \setminus \{0, 1\}$. Thus each y_j (resp. x_j) semicommutes with each y_i (resp. x_i) and, provided $i \ne j$, with each x_i (resp. y_i).

There are n normal elements

$$z_i = x_i y_i - y_i x_i = 1 + \sum_{j=1}^{i} (q_j - 1) y_j x_j = q_i^{-1}\left(1 + \sum_{j=1}^{i} (q_j - 1) x_j y_j\right),$$

which semicommute with each of the generators and commute with each other:

$$z_j y_i = \begin{cases} y_i z_j \text{ if } j < i, \\ q_i y_i z_j \text{ if } j \ge i, \end{cases} \qquad z_j x_i = \begin{cases} x_i z_j \text{ if } j < i, \\ q_i^{-1} x_i z_j \text{ if } j \ge i, \end{cases} \qquad z_i z_j = z_j z_i.$$

The last of the listed relations for $A_n^{\bar{q},\Lambda}$ can be rewritten

$$x_j y_j - q_j y_j x_j = z_{j-1}.$$

In the notation of 1.3, $A_n^{\bar{q},\Lambda} = R(A_{n-1}^{\bar{q},\Lambda}, \alpha_n, z_{n-1}, q_n)$, where

$$\alpha_n : y_i \mapsto \lambda_{ni} y_i, \qquad x_i \mapsto \lambda_{in} x_i.$$

The automorphism $\gamma = \gamma_n$ of $A_{n-1}^{\bar{q},\Lambda}$ induced by z_{n-1} is given by

$$\gamma_n : y_i \mapsto q_i y_i, \qquad x_i \mapsto q_i^{-1} x_i.$$

There is abuse of notation here in the use of the same superscripts for $A_n^{\bar{q},\Lambda}$ and $A_{n-1}^{\bar{q},\Lambda}$. The matrices of parameters for $A_{n-1}^{\bar{q},\Lambda}$ are of course submatrices of $\bar{q}$ and Λ, so no confusion should arise.

The subalgebra $\mathcal{O}$ generated by $y_1, y_2, \ldots, y_n$ is the coordinate ring $A(\Lambda)$ of quantum n-space, the x_i's and z_i's act as partial q_i-difference operators and automorphisms respectively on $\mathcal{O}$, see [**9**, 2.9]. In particular, when $n = 2$, the actions of x_1, x_2, y_1, y_2, z_1 and z_2 on a typical monomial $y_1^i y_2^j$ in $\mathcal{O}$ are as follows:

$$x_1 : y_1^i y_2^j \mapsto \left(\frac{1 - q_1^i}{1 - q_1}\right) y_1^{i-1} y_2^j, \qquad x_2 : y_1^i y_2^j \mapsto \left(\frac{1 - q_2^j}{1 - q_2}\right) (q_1 \lambda_{12} y_1)^i y_2^{j-1},$$
$$y_1 : y_1^i y_2^j \mapsto y_1^{i+1} y_2^j, \qquad y_2 : y_1^i y_2^j \mapsto (\lambda_{21} y_1)^i y_2^{j+1},$$
$$z_1 : y_1^i y_2^j \mapsto (q_1 y_1)^i y_2^j, \qquad z_2 : y_1^i y_2^j \mapsto (q_1 y_1)^i (q_2 y_2)^j.$$

1.5. An alternative quantized Weyl algebra. The alternative quantized Weyl algebra $\mathcal{A}_n^{\bar{q},\Lambda}$ is also obtained from the base field k by n applications of the construction in 1.3 but with all choices of v taken to be 1. In the notation of 1.3, $\mathcal{A}_n^{\bar{q},\Lambda} =$

$R(\mathscr{A}_{n-1}^{\bar{q},\Lambda}, \alpha_n, 1, q_n)$, where the rules for α_n are as in 1.4. As v is always 1, γ_n is always the identity automorphism. The resulting relations are as follows: for $1 \le i < j \le n$,

$$x_i x_j = \lambda_{ij} x_j x_i, \qquad y_j y_i = \lambda_{ji} y_i y_j,$$
$$x_i y_j = \lambda_{ji} y_j x_i, \qquad x_j y_i = \lambda_{ij} y_i x_j,$$
$$x_j y_j - q_j y_j x_j = 1.$$

There are again n normal elements but with a more symmetric form

$$z_i = x_i y_i - y_i x_i = 1 + (q_i - 1) y_i x_i = q_i^{-1}(1 + (q_i - 1) x_i y_i),$$

and more symmetric identities

$$z_j y_i = \begin{cases} y_i z_j \text{ if } j \neq i, \\ q_i y_i z_j \text{ if } j = i, \end{cases} \qquad z_j x_i = \begin{cases} x_i z_j \text{ if } j \neq i, \\ q_i^{-1} x_i z_j \text{ if } j = i, \end{cases} \qquad z_i z_j = z_j z_i.$$

The subalgebra $\mathcal{O}$ generated by $y_1, y_2, \ldots, y_n$ is again $A(\Lambda)$ and the x_i's again act as partial q_i-difference operators on $\mathcal{O}$:

$$x_1 : y_1^i y_2^j \mapsto \left(\frac{1 - q_1^i}{1 - q_1}\right) y_1^{i-1} y_2^j, \qquad x_2 : y_1^i y_2^j \mapsto \left(\frac{1 - q_2^j}{1 - q_2}\right) (\lambda_{12} y_1)^i y_2^{j-1},$$
$$y_1 : y_1^i y_2^j \mapsto y_1^{i+1} y_2^j, \qquad y_2 : y_1^i y_2^j \mapsto (\lambda_{21} y_1)^i y_2^{j+1},$$
$$z_1 : y_1^i y_2^j \mapsto (q_1 y_1)^i y_2^j, \qquad z_2 : y_1^i y_2^j \mapsto y_1^i \mapsto y_1^i (q_2 y_2)^j.$$

In comparison with the actions given above for $A_2^{\bar{q},\Lambda}$, the only changes are in the scalar appearing in the action of x_2 and in the automorphism for z_2 which shows more symmetry.

1.6. Completely prime ideals. We list here two results which will be used to establish that certain ideals are completely prime. The first is well known; see for example [**6**, 2.1(vi)]. Given the restriction in 1.3 that $v \neq 0$, the second is a special case of [**9**, 2.7].

LEMMA. (i) *Let R be a ring with an automorphism α and an α-derivation δ and let $S = R[x; \alpha, \delta]$. Let I be an ideal of R such that $\alpha(I) = I$ and $\delta(I) \subseteq I$. Then IS is an ideal of S and there are an induced automorphism, also denoted α, of R/I and an induced α-derivation, also denoted δ, of R/I. There is an isomorphism $\theta : (R/I)[x; \alpha, \delta] \to S/IS$ with $\theta(x) = x$ and $\theta(r + I) = r + IS$.*

(ii) *In the notation of* 1.3, *if A is a domain and $R = R(A, \alpha, v, \rho)$ then zR is a completely prime ideal of R.*

1.7. Localizations. In [**9**] it is shown that, for $1 \le i \le n$, the set $\{y_i^j\}_{j \ge 1}$ is a right and left Ore set in $A_n^{\bar{q},\Lambda}$. The same is true, for the same reasons, in $\mathscr{A}_n^{\bar{q},\Lambda}$.

We denote by $B_n^{\bar{q},\Lambda}$ the localization of $A_n^{\bar{q},\Lambda}$ obtained by inverting the n normal elements z_i, $1 \le i \le n$. It is shown in [**9**] that, provided no q_i is a root of unity, $B_n^{\bar{q},\Lambda}$ is simple with Krull and global dimension n. We denote the corresponding localization of $\mathscr{A}_n^{\bar{q},\Lambda}$ by $\mathscr{B}_n^{\bar{q},\Lambda}$. The methods of [**9**] can be adapted to show that, provided no q_i is a root of unity, $\mathscr{B}_n^{\bar{q},\Lambda}$ is simple with Krull and global dimension n. Alternatively, this follows from the corresponding result for $B_n^{\bar{q},\Lambda}$ because there is an isomorphism $\theta:\mathscr{B}_n^{\bar{q},\Lambda}\to B_n^{\bar{q},\Lambda}$ with $\theta: y_i\mapsto y_i$; $x_i\mapsto z_{i-1}^{-1}x_i$, where $z_0=1$.

1.8. Skew commutator formula. The following formula, which holds for $d\geq 1$, is the special case of [**9**, 2.6(i)] appropriate to those rings of the form $R(A,\alpha,v,\rho)$ in 1.3 with $\alpha(v)=v$ and $\rho\neq 1$.

$$xy^d-\rho^d y^d x=\frac{1-\rho^d}{1-\rho}vy^{d-1}.$$

1.9. The Noetherian condition. All rings considered in this paper are either iterated skew polynomial rings over a field k or are formed from such rings by taking homomorphic images and/or localizations. By well-known results, see [**7**] or [**12**], they are all right and left Noetherian.

2. Prime ideals of $A(\Lambda)$**.** Proposition 2.2 below is very easy to prove. However it establishes patterns for the hypotheses, results and methods in the subsequent sections on quantized Weyl algebras. The method in 2.2 involves localization and the following simplicity criterion for $P(\Lambda)$ due to McConnell and Pettit [**11**].

2.1. Proposition. *If $n\geqslant 2$ then $P(\Lambda)$ is simple if and only if the only integers t_i, $1\leq i\leq n$, such that $\prod_{i=1}^{n}\lambda_{ij}^{t_i}=1$ for all j are $t_i=0$.*

2.2. Proposition. *For $1\leq i\leq n$, let $G_i(\Lambda)=\langle\lambda_{ij}:1\leq j\leq n\rangle$, the subgroup of k^* generated by the entries on the i-th row of Λ (or, equivalently, by the entries in the i-th column of Λ). Suppose that each $G_i(\Lambda)$ has rank $n-1$. Then every nonzero nonmaximal prime ideal of $A(\Lambda)$ is generated by a nonempty proper subset of $\{x_i:1\leq i\leq n\}$ and every maximal ideal of $A(\Lambda)$ is generated by $x_1,x_2,\dots,x_i-\mu,\dots,x_n$ for some i, $1\leq i\leq n$, and some $\mu\in k$.*

Proof. The result is certainly true when $n=1$ in which case $A(\Lambda)=k[x_1]$. Let $n>1$. Under the stated hypothesis on $G_i(\Lambda)$, 2.1 applies to show that $P(\Lambda)$ is simple. Thus every nonzero prime ideal of $A(\Lambda)$ must contain one of the normal elements x_i. It is clear that $A(\Lambda)/x_iA(\Lambda)\simeq A(\Lambda')$ where Λ' is the $(n-1)\times(n-1)$ matrix obtained from Λ by deleting row i and column i. Also, each $G_j(\Lambda')$ must have rank $n-2$. The result follows easily by induction.

2.3. Remark. The condition that the rank of each $G_i(\Lambda)$ is $n-1$ is significantly weaker than the condition that the rank of the subgroup $G(\Lambda)$ generated by all the entries in Λ takes its maximal value, namely $\frac{1}{2}n(n-1)$. It is possible for the rank of each $G_i(\Lambda)$ to be $n-1$ but for the rank of $G(\Lambda)$ to be $n-1$ if n is even and n if n is odd. Examples for $n=4$ and 5 are as follows:

$$\begin{bmatrix}1&a&b&c\\a^{-1}&1&c&b\\b^{-1}&c^{-1}&1&a\\c^{-1}&b^{-1}&a^{-1}&1\end{bmatrix},\quad\begin{bmatrix}1&a&b&c&d\\a^{-1}&1&c&d&e\\b^{-1}&c^{-1}&1&e&a\\c^{-1}&d^{-1}&e^{-1}&1&b\\d^{-1}&e^{-1}&a^{-1}&b^{-1}&1\end{bmatrix}.$$

Here a, b, c, d and e are elements of k^* generating a subgroup of rank 5.

For arbitrary n, an example can be constructed using an edge-colouring $\mathscr{C}$ of the complete graph K_n on n vertices with a minimal number of colours. This number, m_n, say, is $n-1$ if n is odd, see [**2**, p. 82]. Let c_i, $1 \leq i \leq m_n$, be the colours in $\mathscr{C}$ and let a_i, $1 \leq i \leq m_n$, be elements of k^* generating a subgroup of rank m_n. Let Λ be the $n \times n$ matrix satisfying the rules: each $\lambda_{ii} = 1$ and, whenever $j > i$, $\lambda_{ij}^{-1} = \lambda_{ji} = a_r$ where c_r is the colour of the edge joining vertices i and j. Then, as no two edges of the same colour meet at a vertex, none of the a_i's can appear twice in any column. Thus each $G_i(\Lambda)$ has rank $n-1$. We thank Victor Bryant for pointing out the connection with edge-colourings.

3. Prime ideals of $\mathscr{A}_n^{\bar{q},\Lambda}$.

3.1. NOTATION. For $\mathscr{A}_n^{\bar{q},\Lambda}$, the analogous condition on the parameters to that in 2.2 involves the subgroups $H_i = H_i(\Lambda, \bar{q})$ generated by G_i and q_i. The maximal rank for each of these is n whereas the maximal rank of the group $H = H(\Lambda, \bar{q})$ generated by all the parameters λ_{ij} and q_i is $\frac{1}{2}n(n+1)$. We shall show that if $n > 1$ and each $H_i(\Lambda, \bar{q})$ has rank n then every nonzero prime ideal of $\mathscr{A}_n^{\bar{q},\Lambda}$ is generated by a subset of the set $\{z_i : 1 \leq i \leq n\}$ of normal elements. When considering such a subset Z, the symmetry in the defining relations for $\mathscr{A}_n^{\bar{q},\Lambda}$ allows us to assume, without loss of generality, that $Z = \{z_1, z_2, \ldots, z_m\}$. Throughout this section, we fix Λ and $\bar{q}$ and write $\mathscr{A}_n$ for $\mathscr{A}_n^{\bar{q},\Lambda}$. We let P_m denote the ideal $z_1\mathscr{A}_n + z_2\mathscr{A}_n + \ldots + z_m\mathscr{A}_n$ of $\mathscr{A}_n$.

It is cumbersome to use the standard overlining notation for images in factor rings of A_n and $\mathscr{A}_n$ so we shall often abuse notation and write, for example, y_i for the image of y_i in a factor ring. We shall occasionally indicate such abuses by inserting the phrase "the images of" in brackets.

3.2. LEMMA. *For $1 \leq m \leq n$, P_m is a completely prime ideal of $\mathscr{A}_n$.*

Proof. We first consider P_n. For $1 \leq i \leq n$, $1 + (q_i - 1)y_i x_i = z_i \in P_n$ and $1 + (q_i - 1)x_i y_i = q_i z_i \in P_n$ so, modulo P_n, each y_i is invertible with inverse $(1 - q_i)x_i$. Consequently, $\mathscr{A}_n/P_n$ is isomorphic to $P(\Lambda)$ which is certainly a domain. For $1 \leq m < n$, the elements $z_1, z_2, \ldots, z_m$ generate a completely prime ideal of $\mathscr{A}_m$ and they are fixed (resp. annihilated) by each of the automorphisms (resp. derivations) used in the construction of $\mathscr{A}_n$ from $\mathscr{A}_m$. It follows from 1.6(i) that P_m is completely prime.

3.3. PROPOSITION. *Suppose that $1 \leq m \leq n$ and let $T = \mathscr{A}_n/P_m$. If each of the groups H_i has rank n then the localization T_1 of T at the multiplicatively closed set generated by the images in T of the normal elements $z_{m+1}, z_{m+2}, \ldots, z_n$ is simple.*

Proof. Let $T = \mathscr{A}_n/P_m$. Each of the sets $\{y_i^j\}_{j \geq 1}$ and $\{z_i^j\}_{j \geq 1}$ is a right and left Ore set in $\mathscr{A}_n$, see 1.7, and the $2n$ elements y_i, z_j semicommute with each other. It follows that the multiplicatively closed set generated by these elements y_i, z_i, $1 \leq i \leq n$,is a right and left Ore set in $\mathscr{A}_n$. The image $\mathscr{C}$ in T of this Ore set is a right and left denominator set in T. Let T_2 denote the localization of T at $\mathscr{C}$. For $1 \leq i \leq m$, the image of y_i is already invertible in T and the image of z_i is zero. Thus T_2 is obtained from T by inverting the images of y_i and z_i for $m+1 \leq i \leq n$. It is generated, as a k-algebra, by (the images of) $y_1^{\pm 1}, y_2^{\pm 1}, \ldots, y_n^{\pm 1}, z_{m+1}^{\pm 1}, z_{m+2}^{\pm 1}, \ldots$ and $z_n^{\pm 1}$ subject to the relations

$$y_j y_i = \lambda_{ji} y_i y_j, \qquad z_j y_i = \begin{cases} y_i z_j \text{ if } j \neq i, \\ q_i y_i z_j \text{ if } j = i \end{cases}.$$

Thus $T_2 \simeq P(\Lambda^\dagger)$ where $\Lambda^\dagger$ is a $(2n-m) \times (2n-m)$ matrix of the form

$$\begin{bmatrix} \Lambda & \Gamma \\ \Gamma^* & \Theta \end{bmatrix}.$$

Here Γ is $n \times (n-m)$ with all entries equal to 1 except that, for $m+1 \leq i \leq n$, the $(i, n+1)$-entry is q_i^{-1}, Γ^* is obtained from Γ by inverting all the entries and taking the transpose and Θ is $(n-m) \times (n-m)$ with all entries equal to 1. As each H_i has maximal rank, it follows from 2.1 that T_2 is simple. For $0 \leq j \leq n-m$, let U_j be the localization of T_1 obtained by inverting $y_{m+1}, y_{m+2}, \ldots, y_{m+j}$. Thus $U_0 = T_1$ and $U_{n-m} = T_2$ is simple. Let $j > 0$ and suppose that U_j is simple. Let I be a nonzero ideal of U_{j-1} and let $y = y_{m+j}$, $x = x_{m+j}$ and $q = q_{m+j}$. By the simplicity of U_j, which is obtained from U_{j-1} by inverting the powers of y, $y^d \in I$ for some $d \geq 0$ and we choose the minimal such d. The formula 1.8 gives

$$xy^d - q^d y^d x = \frac{(1-q^d)}{(1-q)} y^{d-1}.$$

As q cannot be a root of unity, it follows that $y^{d-1} \in I$, contradicting the minimality of d unless $d = 0$. Thus $1 \in I$ and so U_{j-1} is simple. By induction, each U_i is simple and in particular $U_0 = T_1$ is simple.

3.4. Proposition. *If $n > 1$ and each of the groups $H_i(\Lambda, \bar{q})$ has rank n then every nonzero prime ideal P of $\mathcal{A}_n^{\bar{q},\Lambda}$ is generated by a non-empty subset of the set of normal elements $\{z_i : 1 \leq i \leq n\}$.*

Proof. In 1.7, we observed that the localization $\mathcal{B}_n^{\bar{q},\Lambda}$ is simple. Hence $z_i \in P$ for some i. By symmetry, we can renumber so that, for some m, $z_1, z_2, \ldots, z_m \in P$ and $z_{m+1}, z_{m+2}, \ldots, z_n \notin P$. Thus $P_m \subseteq P$. By 3.3, the localization of $\mathcal{A}_n/P_m$ obtained by inverting the (images of) the normal elements $z_{m+1}, z_{m+2}, \ldots, z_n$ is simple so if $P \neq P_m$ then P must contain z_j for some j with $m < j \leq n$, contradicting the choice of m. Thus $P = P_m$ and, undoing the reordering, P is generated by a non-empty subset of $\{z_i : 1 \leq i \leq n\}$.

3.5. Corollary. *Under the hypotheses of* 3.4, *every prime ideal of $\mathcal{A}_n^{\bar{q},\Lambda}$ is (right and left) localizable.*

Proof. As the prime ideals are all generated by normal elements they have the (right and left) Artin-Rees property, see [**12**, Theorem 4.2.6]. By Noetherian induction all ideals have the Artin-Rees property and the result follows from [**12**, Theorem 4.2.11(i)].

We shall see in 4.13(v) that the analogous result for $A_n^{\bar{q},\Lambda}$ is false.

4. Prime ideals of $A_n^{\bar{q},\Lambda}$. Throughout this section, we fix Λ and $\bar{q}$ and write A_n for $A_n^{\bar{q},\Lambda}$. The prime spectrum of A_n is more complex than that of $\mathcal{A}_n^{\bar{q},\Lambda}$. Considerations of symmetry, as in Section 3, do not apply to $A_n^{\bar{q},\Lambda}$. Moreover, the ideals analogous to those

https://doi.org/10.1017/S0017089500031712 Published online by Cambridge University Press

in 3.2 need not be prime. For example, the ideal generated by the normal elements z_1 and z_2 is not prime.

4.1. Lemma. *Let I be an ideal of A_n and let $1 \le i \le n$.*
(i) *If $y_i \in I$ or $x_i \in I$ then $z_i \in I$ and, provided $i > 1$, $z_{i-1} \in I$.*
(ii) *If $i > 1$, $z_{i-1} \in I$ and $z_i \in I$ then, modulo I, $x_i y_i = 0 = y_i x_i$.*
(iii) *If $z_i \in I$ or if $i > 1$ and $z_{i-1} \in I$ then x_i and y_i are normal modulo I.*
(iv) *Suppose that I is prime and that $i > 1$. If $z_i \in I$ and $z_{i-1} \in I$ then $y_i \in I$ or $x_i \in I$.*

Proof. (i) and (ii) are consequences of the identities

$$x_i y_i - q_i y_i x_i = z_{i-1},$$
$$z_i = x_i y_i - y_i x_i = z_{i-1} + (q_i - 1) y_i x_i,$$
$$q_i z_i = z_{i-1} + (q_i - 1) x_i y_i.$$

These also show that if $z_i \in I$ then x_i and y_i commute modulo I and if $z_{i-1} \in I$ then x_i and y_i semicommute modulo I. (iii) follows because y_i (resp. x_i) semicommutes with each of the generators of A_n except x_i (resp. y_i). Finally, (iv) is immediate from (ii) and (iii).

We shall see that every nonzero prime ideal of A_n has a normalizing sequence of generators, that, as was the case in Sections 2 and 3, there are only finitely many nonmaximal primes and that, as was the case in Section 2 but not in Section 3, there are infinitely many maximal ideals. We begin with a definition which describes the normalizing sequences for the nonmaximal primes.

4.2. Definition. For a subsequence S of the sequence

$$N := \{z_1, y_1, x_1, z_2, y_2, x_2, \dots, z_n, y_n, x_n\}$$

and an integer i with $1 \le i \le n$, we say that i *features* in S if at least one of z_i, y_i, x_i is in S. By a *p-sequence*, we mean a subsequence S, possibly empty, of N with the following properties:
(i) if $z_i \in S$ then $i - 1$ does not feature in S;
(ii) if $y_i \in S$ or $x_i \in S$ then $i - 1$ does feature in S.
Thus a p-sequence is the concatenation of a set of subsequences S_i such that the set of subscripts featuring in S_i is consecutive, each S_i begins with z_{r_i} but contains no other z_j and there is a gap between the largest subscript featuring in S_i and the smallest one in S_{i+1}. An example is $\{z_2, x_3, y_4, x_4, z_6, y_7\}$. The number of terms in a p-sequence S will be called the *length* of S and will be written $\operatorname{len}(S)$.

4.3. Lemma. *Let S be a p-sequence and let P be the right ideal of A_n generated by S.*
(i) *For $1 \le i \le n$, if i features in S then $z_i \in P$.*
(ii) *P is an ideal of A_n and S is a normalizing sequence of generators for P.*

Proof. If i features in S then there exist integers $j \ge 1$ and $m \ge 0$ such that $j - 1$ does not feature in S (or $j = 1$), $j, j+1, \dots, j+m$ feature in S and $i = j + m$. Thus $z_j \in S$ and, for $j < s \le j + m$, $x_s \in S$ or $y_s \in S$. As $z_s = z_{s-1} + (q_s - 1) y_s x_s$ and $q_s z_s = z_{s-1} + (q_s - 1) x_s y_s$, it follows that each $z_s \in P$, in particular $z_i \in P$. Thus (i) holds. (ii) is immediate from (i), 4.1(ii) and the normality of the elements z_i.

4.4. Inductive methods. Suppose that $n>1$ and let $S'=S\cap A_{n-1}$ which is a p-sequence in A_{n-1} and hence, by 4.3(ii) is a normalizing sequence in A_{n-1}. Let Q be the ideal of A_{n-1} generated by S' and let $B=A_{n-1}/Q$. The ring A_n has the form $R(A_{n-1},\alpha,z_{n-1},q_n)=A_{n-1}[y_n;\alpha][x_n;\beta,\delta]$ of 1.3 with α specified in 1.4. In particular, each y_i, x_i and z_i are eigenvectors for both α and β, $\delta(A_{n-1})=0$ and $\delta(y_n)=z_{n-1}$. It follows that Q is invariant under each of α, the automorphism γ induced by z_{n-1}, $\beta=\gamma\alpha^{-1}$ and δ. By two applications of 1.6(ii), QA_n is an ideal of A_n and, with the convention on induced maps as in 1.6, $A_n/QA_n\simeq B[y_n;\alpha][x_n;\beta,\delta]$. The next lemma makes explicit the relationship between the rings A_n/P and A_{n-1}/Q and is the basis for inductive proofs on the prime ideals of A_n.

LEMMA. *Let S be a p-sequence in A_n.*

(i) *If n does not feature in S then $P=QA_n$, $A_n/P\simeq B[y_n;\alpha][x_n;\beta,\delta]$, $z_n\notin P$, $y_n\notin P$ and $x_n\notin P$.*

(ii) *If $x_n\in S$ and $y_n\notin S$ then $P=QA_n+x_nA_n$, $A_n/P\simeq B[y_n;\alpha]$ and $y_n\notin P$.*

(iii) *If $y_n\in S$ and $x_n\notin S$ then $P=QA_n+y_nA_n$, $A_n/P\simeq B[x_n;\beta]$ and $x_n\notin P$.*

(iv) *If $y_n\in S$ and $x_n\in S$ then $P=QA_n+y_nA_n+x_nA_n$ and $A_n/P\simeq B$.*

(v) *If $z_n\in S$ then $P=QA_n+z_nA_n$, $y_n\notin P$ and $x_n\notin P$ and $A_n/P\simeq R/z_nR$, where, in the notation of* 1.3, $R=R(B,\alpha,z_{n-1},q_n)$.

Proof. The relationships between P and Q are immediate from the definitions and (i)–(iv) follow easily using the isomorphism $A_n/QA_n\simeq B[y_n;\alpha][x_n;\beta,\delta]$. For (v), suppose that $z_n\in S$. Then, by definition of p-sequence, $n-1$ cannot feature in S'. By (i), $z_{n-1}\notin Q$. Thus (the image of) z_{n-1} is a nonzero normal element in B and $A_n/QA_n\simeq B[y_n;\alpha][x_n;\beta,\delta]=R(B,\alpha,z_{n-1},q_n)=R$. Hence $A_n/P=A_n/(QA_n+z_nA_n)\simeq R/z_nR$. Viewed as a polynomial in x_n over $B[y_n;\alpha]$, z_n has degree 1 and a noninvertible leading coefficient $(q_i-1)y_n$. Hence $x_n\notin z_nR$ and, similarly, $y_n\notin z_nR$. Applying the isomorphism $R/z_nR\simeq A_n/P$, $y_n\notin P$ and $x_n\notin P$.

4.5. PROPOSITION. *Let S be a p-sequence in A_n and let P be the ideal of A_n generated by S. Then P is completely prime.*

Proof. This is certainly true when $n=1$, in which case $\{z_1\}$ is the only nonempty p-sequence and z_1A_1 is a completely prime ideal, with factor isomorphic to $k[y_1^{\pm1}]$. Inductively, we may assume that if $S'=S\cap A_{n-1}$ and $Q=S'A_{n-1}$ then $B=A_{n-1}/Q$ is a domain. In each of the cases (i)–(iv) in 4.4 it is immediate that A_n/P is a domain while in 4.4(v), 1.6(ii) applies to give the same conclusion. By the definition of p-sequence these five cases are exhaustive so, by induction, P is always completely prime.

4.6. LEMMA. *In each of the cases listed in* 4.4, $P\cap A_{n-1}=Q$. *Hence, for* $1\le i\le n$,

(i) *$z_i\in P$ if and only if i features in S,*

(ii) *$y_i\in P$ if and only if $y_i\in S$,*

(iii) *$x_i\in P$ if and only if $x_i\in S$.*

Proof. It is routine to check that $P\cap A_{n-1}=Q$ in each case. Hence we have $P\cap A_i=(S\cap A_i)A_i$ for $1\le i\le n$. For (i) the "if" part is 4.3(i) and for (ii) and (iii) the "if" parts are obvious. The "only if" parts hold for the case $i=n$ by 4.4 and for lower values of i because $P\cap A_i=(S\cap A_i)A_i$.

https://doi.org/10.1017/S0017089500031712 Published online by Cambridge University Press

4.7. The maximal p-sequence. Let S be the p-sequence $z_1, y_2, x_2, y_3, x_3, \ldots, y_n, x_n$ and let $M = SA_n$. By 4.1(i), $z_i \in M$ for all i. Hence if T is any p-sequence and P is the ideal generated by T then $P \subseteq M$. We shall call S the *maximal* p-sequence in A_n. Note that, by $n-1$ applications of 4.4(iv) and one of 4.4(v), $A_n/M \simeq k[y_1^{\pm 1}]$. It follows that there are infinitely many maximal ideals of A_n of the form $M + (y_1 - \mu)A_n$, $\mu \in k^*$ and that these are the only prime ideals of A_n strictly containing M.

4.8. Localizations. We aim to show that, under conditions on the parameters, similar to those in 3.4, every nonzero nonmaximal prime ideal of A_n is generated by a p-sequence. Our method is similar to that in Section 3 and involves certain localizations T_1 and T_2 of A_n/P where P is as in 4.3. The conditions on the parameters will be stronger than in 3.4 in that the subgroup $G(\bar{q})$ of k^* generated by the parameters q_i, $1 \le i \le n$, should also have rank n.

Let S be a p-sequence in A_n, let P be the prime ideal of A_n generated by S and let T be the domain A_n/P. Let

$$C(S) = \{z_i : z_i \notin P\} \cup \{y_i : y_i \notin P \text{ and } z_i \in P\} \cup \{x_i : x_i \notin P \text{ and } z_i \in P\}$$

and let $D(S) = C(S) \cup \{y_i : y_i \notin P\}$. If $z_i \in C(S)$ then, as z_i semicommutes with each element of S and is normal in A_n, z_i is normal modulo P. Also, if $y_i \in C(S)$ then y_i is normal modulo P by 4.1(iii). The same is true of x_i. Thus each element of $C(S)$ is normal modulo P. As x_i can only be in $C(S)$ when $z_i \in P$, in which case x_i commutes with y_i modulo P, the elements of $C(S)$ all semicommute with each other modulo P. Therefore the set of images in T of elements of the form $\mu c_1^{j_1} c_2^{j_2} \ldots c_s^{j_s}$, $\mu \in k^*$, $c_i \in C(S)$, $j_i \ge 0$ is a right and left Ore set in T. Let T_1 denote the localization of T at this set. Thus T_1 is obtained from T by inverting the elements of $C(S)$, each of which is normal modulo P.

Now consider $y_i \in D(S) \backslash C(S)$. By 1.8, $\{y_i^j\}_{j \ge 1}$ is a right and left Ore set in A_n and so its image is a right and left Ore set in the domain T. As with $C(S)$, the elements of $D(S)$ semicommute with each other modulo P and the set of images in T of elements of the form $\mu d_1^{j_1} d_2^{j_2} \ldots d_s^{j_s}$, $\mu \in k^*$, $d_i \in D(S)$, $j_i \ge 0$ is a right and left Ore set in T. Let T_2 denote the localization of T at this set. Thus T_2 is obtained from T by inverting the elements of $D(S)$ and from T_1 by inverting those elements y_i such that $y_i \notin P$ and $z_i \notin P$.

4.9. PROPOSITION. *Suppose that each of the groups $H_i(\Lambda, \bar{q})$ and $G(\bar{q})$ has rank n. Let S be a p-sequence in A_n, let P be the prime ideal of A_n generated by S and let T be the domain A_n/P. If S is nonmaximal then the localization T_2 specified in* 4.8 *is simple.*

Proof. As T is generated by (the images of) those elements y_i and x_i not belonging to P, the localization T_2 is generated by these elements together with the inverses of the elements in $D(S)$. Let T_2' be the subalgebra of T_2 generated by (the image of) $E(S) \cup \{e^{-1} : e \in E(S)\}$, where

$$E(S) = \{y_i : y_i \notin P\} \cup \{z_i : z_i \notin P\} \cup \{x_i : x_i \notin P \text{ and } y_i \in P\}.$$

Note that if $x_i \notin P$ but $y_i \in P$ then $z_i \in P$ so $x_i \in D(S)$ and therefore T_2' is indeed a subalgebra of T_2. We claim that $T_2 = T_2'$. Clearly (the images of) each z_i and y_i are in T_2', as are z_i^{-1} if $z_i \in D(S)$, y_i^{-1} if $y_i \in D(S)$ and x_i if $x_i \notin P$ but $y_i \in P$. Suppose that $x_i \notin P$ and $y_i \notin P$. Then y_i^{-1}, z_i, $z_{i-1} \in T_2'$ so $x_i = ((q_i - 1)y_i)^{-1}(z_i - z_{i-1}) \in T_2'$. Furthermore, if

$x_i \in D(S)$ then $z_i \in P$ and, by 4.1(iv), $z_{i-1} \notin P$, so $x_i = ((q_i - 1)y_i)^{-1}(-z_{i-1})$ and $x_i^{-1} = (1-q_i)z_{i-1}^{-1}y_i \in T_2'$. It follows that $T_2 = T_2'$.

Let $S' = S \cap A_{n-1}$ and let $Q = S'A_{n-1}$ as in 4.4. Note that, by 4.6, $E(S') = E(S) \cap A_{n-1}$. We rewrite the elements of $E(S)$ as $w_1, w_2, \ldots, w_m$, say, where $w_1 > w_2 > \ldots > w_m$ in the ordering

$$y_1 > z_1 > x_1 > y_2 > z_2 > x_2 > \ldots > y_n > z_n > x_n.$$

Then $E(S') = \{w_1, w_2, \ldots, w_r\}$ for some $r \leq m$. Note that y_1 cannot be in P so $w_1 = y_1$.

The elements of $E(S)$ semicommute according to the rules

$$x_ix_j = q_i\lambda_{ij}x_jx_i, \qquad y_jy_i = \lambda_{ji}y_iy_j,$$
$$x_iy_j = \lambda_{ji}y_jx_i, \qquad x_jy_i = q_i\lambda_{ij}y_ix_j \ (j > i)$$

and

$$z_jy_i = \begin{cases} y_iz_j \text{ if } j < i, \\ q_iy_iz_j \text{ if } j \geq i, \end{cases} \qquad z_jx_i = \begin{cases} x_iz_j \text{ if } j < i, \\ q_i^{-1}x_iz_j \text{ if } j \geq i, \end{cases} \qquad z_iz_j = z_jz_i$$

from 1.4. Note that y_i and x_i cannot both appear in $E(S)$.

Let $\Delta = [d_{ij}]$ be the $m \times m$ matrix determined by the rule

$$w_jw_i = d_{ji}w_iw_j$$

and let $\Delta^\dagger$ be the corresponding $r \times r$ matrix for $E(S')$. We claim that the algebra $P(\Delta)$ is simple. As $T_2 = T_2'$ is a homomorphic image of $P(\Delta)$, it will follow that T_2 is simple. We shall establish the simplicity of $P(\Delta)$ by induction on n. When $n = 1$ the only nonmaximal p-sequence S is empty, $E(S) = \{y_1, z_1\}$ and

$$\Delta = \begin{bmatrix} 1 & q_1^{-1} \\ q_1 & 1 \end{bmatrix}.$$

In this case, $P(\Delta)$ is certainly simple.

Now suppose that $n > 1$. Note that the condition on the parameters must hold in A_{n-1} otherwise it would fail in A_n. By induction, we can assume that either $P(\Delta^\dagger)$ is simple or that S' is the maximal p-sequence in A_{n-1}, in which case $r = 1$ and $\Delta^\dagger = [1]$. With a view to applying 2.1, let t_i, $1 \leq i \leq m$, be integers such that $\prod_{i=1}^{m} d_{ij}^{t_i} = 1$ for all j. We consider separately the cases (iv)–(v) of 4.4.

(i) If n does not feature in S then $E(S) = E(S') \cup \{y_n, z_n\}$, $m = r + 2$ and

$$\Delta = \begin{bmatrix} \Delta^\dagger & \Gamma \\ \Gamma^* & \Omega \end{bmatrix}, \quad \text{where} \quad \Omega = \begin{bmatrix} 1 & q_n^{-1} \\ q_n & 1 \end{bmatrix},$$

Γ is $r \times 2$ and Γ^* is obtained from Γ by inverting the entries and taking transpose. Each of the entries $d_{i,m-1}$, $1 \leq i \leq r$, arises from the semicommutation of y_n with either y_i, z_i or x_i for some $i < n$ and so is one of λ_{in}, 1, λ_{ni}, $i < n$. Also $d_{m-1,m-1} = 1$ and $d_{m,m-1} = q_n$. As H_n has maximal rank, it follows that $t_m = 0$. Each of the entries d_{im}, $1 \leq i \leq r$, arises from the semicommutation of z_n with either y_i, z_i or x_i for some $i < n$ and is one of q_i^{-1}, 1, q_i, $i < n$. As $d_{m-1,m} = q_n^{-1}$, $d_{mm} = 1$ and $G(\bar{q})$ has maximal rank, it follows that $t_{m-1} = 0$. Hence

$\prod_{i=1}^{r} d_{ij}^{t_i} = 1$ for each j, $1 \le j \le r$. In the case where S' is nonmaximal it follows from the simplicity of $P(\Delta^\dagger)$ that each $t_j = 0$ and hence that $P(\Delta)$ is simple. When S' is maximal, $m = 3$, $E(S) = \{y_1, y_n, z_n\}$ and

$$\Delta = \begin{bmatrix} 1 & \lambda_{1n} & q_1^{-1} \\ \lambda_{n1} & 1 & q_n^{-1} \\ q_1 & q_n & 1 \end{bmatrix}.$$

The maximality of the rank of $G(\bar{q})$ shows that $t_1 = t_2 = 0 = t_3$ and hence that $P(\Delta)$ is again simple.

(ii) If $x_n \in S$ and $y_n \notin S$ then $E(S) = E(S') \cup \{y_n\}$, $m = r + 1$ and

$$\Delta = \begin{bmatrix} \Delta^\dagger & \Gamma \\ \Gamma^* & 1 \end{bmatrix},$$

where Γ is $r \times 1$ and Γ^* is obtained from Γ by inverting the entries and taking transpose. Here $d_{m1} = \lambda_{n1}$. Each entry d_{i1}, $1 \le i \le r$, arises from the semicommutation of y_1 with either y_i, z_i or x_i for some $i < n$ and is one of λ_{i1}, q_1, $q_1\lambda_{1i}$, $i < n$. As H_1 has maximal rank, it follows that $t_m = 0$. In the case where S' is nonmaximal it follows, as in (i), that $P(\Delta)$ is simple. When S' is maximal, $m = 2$, $E(S) = \{y_1, y_n\}$,

$$\Delta = \begin{bmatrix} 1 & \lambda_{1n} \\ \lambda_{n1} & 1 \end{bmatrix}$$

and it is easy to check that $P(\Delta)$ is again simple.

(iii) If $y_n \in S$ and $x_n \notin S$ then $E(S) = E(S') \cup \{x_n\}$, $m = r + 1$. In this case $P(\Delta)$ is again simple, the details being much as in (ii) but with $d_{m1} = q_1\lambda_{1n}$.

(iv) If $y_n \in S$ and $x_n \in S$ then $E(S) = E(S')$, $m = r$ and $P(\Delta)$ is simple except in the case where S' is the maximal p-sequence in A_{n-1}. In the exceptional case, $S = S' \cup \{y_n, x_n\}$ is the maximal p-sequence in A_n.

(v) If $z_n \in S$ then again $E(S) = E(S') \cup \{y_n\}$, $m = r + 1$ and, as in (ii) $P(\Delta)$ is simple.

This completes the inductive proof that, provided S is nonmaximal, $P(\Delta)$ is simple. It follows that $T_2 \simeq P(\Delta)$ is simple.

4.10. Proposition. *Under the hypotheses of* 4.9, *if* S *is nonmaximal then the localization* T_1 *specified in* 4.8 *is simple.*

Proof. The ring T_2 is obtained from T_1 by inverting those elements y_i such that $y_i \notin P$ and $z_i \notin P$. List these as $y_{i_1}, y_{i_2}, \ldots, y_{i_s}$, say. Note that, by 4.1(i) and 4.1(iv), $x_{i_j} \notin P$ and $z_{i_j-1} \notin P$ for $1 \le j \le s$. Thus each z_{i_j-1} is invertible in T_1.

The proof that T_1 is simple is similar to the last step in the proof of 3.3, with y_{m+1}, $y_{m+2}, \ldots, y_n$ replaced by $y_{i_1}, y_{i_2}, \ldots, y_{i_s}$. Thus, if U_j is the localization of T_1 obtained by inverting $y_{i_1}, y_{i_2}, \ldots, y_{i_j}$ then $U_0 = T_1$, $U_s = T_2$ is simple by 4.9 and 1.8 can be applied to show that each U_i is simple and in particular that $U_0 = T_1$ is simple. The appropriate case of 1.8 is

$$xy^d - q^d y^d x = \frac{(1 - q^d)}{(1 - q)} z_{i_j-1} y^{d-1},$$

where $y = y_{i_j}$, $x = x_{i_j}$ and $q = q_{i_j}$. As q cannot be a root of unity and z_{i_j-1} is invertible in T_1, this can be applied in the same way as the corresponding formula in 3.3.

4.11. PROPOSITION. *Under the hypotheses of* 4.9, *let S be a nonmaximal p-sequence and let P be the prime ideal of A_n generated by S. Let P' be a prime ideal of A_n strictly containing P. Then there exists a p-sequence S' with* $\operatorname{len} S' = \operatorname{len} S + 1$ *and with* $P \subset S'A_n \subseteq P'$.

Proof. By 4.10, the localization T_1 of A_n/P is simple. This localization is obtained by inverting the elements of $C(S)$ which are all normal. Hence P' contains an element r of one of the forms z_i, where $z_i \notin P$, y_i, where $y_i \notin P$ and $z_i \in P$ or x_i, where $x_i \notin P$ and $z_i \in P$.

Suppose that $r = z_i$. Then i does not feature in S by 4.3(i). A p-sequence S' with the required properties can be obtained in one of the following ways.

(i) If neither $i-1$ nor $i+1$ features in S then insert z_i in S.

(ii) If $i-1$ features in S but $i+1$ does not then $z_{i-1} \in S$ and, by 4.1(iv), P' contains either x_i or y_i. Insert either x_i or y_i in S.

(iii) If $i+1$ features in S but $i-1$ does not then $z_{i+1} \in S$ and, by 4.1(iv), P' contains either x_{i+1} or y_{i+1}. Replace z_{i+1} by z_i, y_{i+1} or by z_i, x_{i+1} as appropriate.

(iv) If $i-1$ and $i+1$ both feature in S then, by 4.1(iv), P' contains either x_{i+1} or y_{i+1} and either x_i or y_i. Replace z_{i+1} by x_i, x_{i+1}, by y_i, x_{i+1}, by x_i, y_{i+1} or by y_i, y_{i+1} as appropriate.

Now suppose that $r = y_i$ or $r = x_i$ where $z_i \in P$. Thus i features in S by 4.6(i) and $z_{i-1} \in P'$ by 4.1(i). A p-sequence S' with the required properties can be obtained in one of the following ways.

(i) If neither $i-1$ nor $i-2$ features in S then $z_i \in S$. Replace it by z_{i-1}, r.

(ii) If $i-2$ features in S but $i-1$ does not then $z_i \in S$ and, by 4.1(iv), P' contains either x_{i-1} or y_{i-1}. Replace z_i by x_{i-1}, r or by y_{i-1}, r as appropriate.

(iii) If $i-1$ features in S then insert r in S.

4.12. PROPOSITION. *Suppose that each of the groups $H_i(\Lambda, \bar{q})$ and $G(\bar{q})$ has rank n. Every nonmaximal prime ideal of A_n is generated by a p-sequence and every maximal ideal of A_n has the form $M + (y_1 - \mu)A_n$ where M is generated by the maximal p-sequence in A_n and $\mu \in k^*$.*

Proof. Let P' be a prime ideal of A_n. Let S be a p-sequence of maximal length contained in P' and let P be the prime ideal of A_n generated by S. Thus $P \subseteq P'$. If S is not the maximal p-sequence then by 4.11 and the maximality of $\operatorname{len}(S)$, $P' = P$. On the other hand, if S is the maximal p-sequence then, by 4.8, either $P = M$ or $P = M + (y_1 - \mu)A_n$ for some $\mu \in k^*$.

4.13. REMARKS. (i) Note that, unlike the situation in Section 3, the spectrum in the case $n = 1$ is typical of the general case.

(ii) Under the hypotheses of 4.12, the maximal ideals of A_n have height $2n$, whereas, in 3.4, those of $\mathscr{A}_n$ have height n. In A_n the ideal generated by the maximal p-sequence is the unique prime of height $2n-1$ and contains all the nonmaximal primes. In $\mathscr{A}_n$, there are 2^n prime ideals including a unique maximal ideal. Rigal [**14**, 3.16] computes the number of nonmaximal primes in A_n which is $\frac{1}{2}((2+\sqrt{2})^n+(2-\sqrt{2})^n)$.

(iii) Let $P_0 \subset P_1 \subset \ldots \subset P_m$ be a saturated chain of prime ideals of A_n. Under the hypotheses of 4.12, P_0 is generated by a p-sequence of length l, say, and P_m is generated either by a p-sequence of length h, say, or by the maximal p-sequence together with $y_1-\mu$ for some μ, in which case we set $h=2n$. In the latter case, P_{m-1} must be generated by the maximal p-sequence, which has length $2n-1$. By 4.11, each P_i, $1 \leq i < m$ is generated by a p-sequence of length $l+i$ and so the chain has length $h-l$. Thus A_n is catenary. Goodearl and Lenagan [**5**, 3.13] have proved the catenarity of A_n under the more general hypotheses that no q_i is a root of unity.

(iv) We have seen that, under the hypotheses of 4.12, every prime factor A_n/P of A_n is a Noetherian domain with a localization isomorphic to the k-algebra $P(\Lambda)$ for some $m \times m$ matrix Λ and some $m \geq 1$. Hence the quotient division ring D of A_n/P is isomorphic to the quotient division ring of the coordinate ring $A(\Lambda)$ of quantum m-space. Cauchon [**3**, II.2.1] has shown that if the group H is torsion-free and D is the quotient division ring of a prime factor of A_n then there is a field extension K of k such that D is isomorphic either to K or the quotient division ring of the coordinate ring of quantum m-space over K for some m.

(v) The analogue of Corollary 3.5 for A_n is false. As the elements appearing in a p-sequence semicommute with each other, the nonmaximal prime ideals have the Artin-Rees property by [**12**, Theorem 4.2.7(i)]. However the maximal ideals do not have the Artin-Rees property and are not localizable. To see this, let $P=M+(y_1-\mu)A_n$ be a maximal ideal as in 4.12, and let α be the automorphism of A_n induced by z_1, that is $z_1 a=\alpha(a)z_1$ for all $a \in A_n$. Then $\alpha(y_1-\mu)=q_1 y_1-\mu \notin P$. It follows from [**8**, Proposition 1] that P does not have the Artin-Rees property and is not localizable.

REFERENCES

1. J. Alev and F. Dumas, Sur le corps des fractions de certaines algèbres quantiques, *J. Algebra* **170** (1994), 229–265.

2. V. W. Bryant, Aspects of combinatorics (Cambridge University Press, 1993).

3. G. Cauchon, Quotient premiers de $\mathcal{O}_q(\mathcal{M}_n(k))$, preprint, Université de Reims.

4. K. R. Goodearl, Prime ideals in skew polynomial rings and quantized Weyl algebras, *J. Algebra* **150** (1992), 324–377.

5. K. R. Goodearl and T. H. Lenagan, Catenarity in quantum algebras, preprint, Universities of California (Santa Barbara) and Edinburgh.

6. K. R. Goodearl and E. S. Letzter, *Prime ideals in skew and q-skew polynomial rings*, *Mem. Amer. Math. Soc.* **109** (1994), *no.* 521.

7. K. R. Goodearl and R. B. Warfield, Jr., *An introduction to noncommutative Noetherian rings* (Cambridge University Press, 1989).

8. D. A. Jordan, Normal elements and completions of non-commutative Noetherian rings, *Bull. London Math. Soc.* **19** (1987), 417–424.

9. D. A. Jordan, A simple localization of the quantized Weyl algebra, *J. Algebra* **174** (1995), 267–281.

10. G. Maltsiniotis, Groupes quantique et structures différentielles, *C.R. Acad. Sci. Paris Sér. I Math.* **311** (1990), 831–834.

11. J. C. McConnell and J. J. Pettit, Crossed products and multiplicative analogues of Weyl algebras, *J. London Math. Soc.* (2) **38** (1988), 47–55.

12. J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings* (Wiley, 1987).

13. S-Q. Oh, Primitive ideals of the coordinate ring of quantum symplectic space, *J. Algebra* **174** (1995), 531–552.

14. L. Rigal, Spectre de l'algèbre de Weyl quantique, preprint, Université de Paris VI (1994).

SCHOOL OF MATHEMATICS AND STATISTICS
PURE MATHEMATICS SECTION
UNIVERSITY OF SHEFFIELD
THE HICKS BUILDING
SHEFFIELD S3 7RH
UK

Present address of first author:

MATHEMATICS DEPARTMENT
IMAM KHOMEINI INTERNATIONAL UNIVERSITY
GHAZVIN
ISLAMIC REPUBLIC OF IRAN