More progress in congress

DEAR SIR,

There have been contributions in the *Gazette* recently over the number of weighings necessary or sufficient to arrange n objects in linear order. Cameron (No. 394, December 1971) produced a sequence which represents sufficient numbers. Broomhead (No. 398, December 1972) showed these numbers were not necessary by solving the 10 objects problem in 23 weighings, and he went on to mention the sequence $[\log_2 n!] + 1$ giving numbers which are necessary but not necessarily sufficient. My copy of Sprague's *Recreation in Mathematics* (translated by T. H. O'Beirne) in effect quotes the logarithmic sequence, but makes no claims regarding its sufficiency. The only words touching on this I quote:

> "However, it is not yet known whether W_n is always equal to the (value in the logarithmic sequence). It has actually been conjectured that 12 things need no less than 30 weighings, although the fact that $2^{29} > 12!$ suggests that 29 might suffice."

Some may interpret this as implying that the logarithmic value is known to be sufficient for all $n < 12$, but other interpretations are possible. 12 is a special value in that $\Delta([\log_2 n!] + 1) = 4$ for all n from 10 to 15 except $n = 11$; in other words $[\log_2 12!] + 1$ is the most critical value in this part of the range. Arguing with a mixture of looseness and intuition, it must be relatively harder to find a solution for 12 objects in 29 weighings than for 11 objects in 26 weighings. This would be an adequate reason for singling out $n = 12$ as a test case if one wanted to establish the insufficiency of the logarithmic numbers.

I do not know how Broomhead produced his solution of 23 weighings for 10 objects. Below is a way of producing a sequence of sufficient numbers which is always better than the Cameron sequence for $n \geqslant 10$. It contains the '23' solution.

If the Cameron sequence is C_n, we take

$$U_n = C_n \qquad \text{if } n < 10,$$
$$\left.\begin{aligned} U_{2n} &= 2U_n + 2n - 1, \\ U_{2n+1} &= U_n + U_{n+1} + 2n \end{aligned}\right\} \text{ if } n \geqslant 5.$$

Suppose U_r is a sufficient number of weighings for r objects for $r < 2n$. We shall now show that U_{2n} is sufficient for $2n$ and U_{2n+1} for $2n + 1$ objects.

Given $x + y$ objects, they can be partitioned arbitrarily into a pile of x and a pile of y. If $x < 2n$, the first pile can then be ordered within itself in U_x weighings, producing

$$A_1 < A_2 < \cdots < A_x.$$

Similarly if $y < 2n$ the second pile can be ordered within itself in U_y weighings producing

$$B_1 < B_2 < \cdots < B_y.$$

We now interlace the B terms into the A terms.

Suppose B_t is the first member of the B sequence to be heavier than A_x (though for the moment we do not know the value of t). The operations are then as follows:

Weigh B_1 against A_1. If it is lighter, we can place B_1 in its proper position in the A sequence immediately, and we have not moved any way along the A sequence.

If $B_1 > A_1$, we then weigh B_1 against A_2; if necessary we continue along the A sequence until B_1 is placed. Suppose b_1 weighings are required in this process to place B_1; then B_1 must have been compared with all As from A_1 to A_{b_1}, and finish up between A_{b_1-1} and A_{b_1}. We have therefore moved $b_1 - 1$ places along the sequence.

Suppose b_2 weighings are now required to place B_2, the first weighing obviously being against A_{b_1}; we then move a further $b_2 - 1$ places along the sequence. This continues down to b_t which is x places along the sequence. B_t is a further b_t (not $b_t - 1$) places along the sequence. Therefore

$$x = (b_1 - 1) + (b_2 - 1) + \cdots + (b_{t-1} - 1) + b_t,$$

so that

$$b_1 + b_2 + \cdots + b_t = x + t - 1,$$

which takes the maximum value of $x+y-1$ when $t=y$.

But when B_t is placed, all remaining Bs fit in without further weighings. Therefore

$$\text{total number of weighings required} \leqslant x+y-1.$$

If all the Bs are lighter than A_x, suppose that A_{z+1} is the lightest A to be heavier than B_y. As before,

B_1 moves $b_1 - 1$ places along,
B_2 moves a further $b_2 - 1$ places along,
... ...
... ...
B_{y-1} moves a further $b_{y-1} - 1$ places along,

but, now

B_y moves a further $b_y - 1$ places along.

Therefore

$$\begin{aligned} z &= (b_1 - 1) + (b_2 - 1) + \cdots + (b_y - 1) \\ &= \sum b - y, \end{aligned}$$

and the total number of weighings required is

$$\begin{aligned} \sum b &= y + z \\ &\leqslant y + x - 1. \end{aligned}$$

Hence in either case, $x+y-1$ weighings suffice for interlacing, and

$$\text{number of weighings required in all} \leqslant U_x + U_y + x + y - 1.$$

Applying this to $2n$ objects, take $x = n, y = n$; then the number of weighings required $\leqslant 2U_n + 2n - 1 = U_{2n}$. Again, taking $x = n, y = n+1$, the number of weighings required to order $2n+1$ objects $\leqslant U_n + U_{n+1} + 2n = U_{2n+1}$. Hence the U_n sequence gives sufficient numbers for all n.

Denoting the logarithmic sequence by W_n, the Cameron sequence by C_n, and the U sequence by U_n, we have:

n	W_n	C_n	U_n
8	16	16	16
9	19	20	20
10	22	24	23
11	26	28	27
12	29	32	31
13	33	36	35

If you continue the C_n and U_n sequences for n up to 81, you will observe some interesting comparisons.

Yours etc.,

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A rule for turning a generalised mattress

(see Classroom Note 269, October 1972)

DEAR MR. QUADLING,

We turn our mattress much less frequently (and less regularly) than once a week, with the result that each time I have forgotten which way we turned it on the last occasion. So one day I wondered whether there were not *one* operation that could be repeated each time and still send the mattress to all four possible positions in turn. I am ashamed to say, since I am a group theorist by training, that it took me a finite time (even though it was

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