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Existence of weak solutions to an anisotropic parabolic–parabolic chemotaxis system

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This work is devoted to the study of the sub-critical case of an anisotropic fully parabolic Keller–Segel chemotaxis system. We prove the existence of nonnegative weak solutions of (1.1) without restriction on the size of the initial data.

Keywords: Keller–Segel system; Nonlinear diffusion; Weak solutions

2020 *Mathematics Subject Classification:* 35K57; 35B33

1. Introduction

In this paper, we consider the following chemotaxis system with anisotropic porous medium-type diffusion:

$$\begin{cases} u_t = \displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(d_1 \frac{\partial u^{m_i}}{\partial x_i} - \chi \left(\frac{u}{\gamma+v}\right)^{q_i-1} \frac{\partial v}{\partial x_i}\right) & \text{in } \Omega_T, \\ v_t = d_2 \Delta v - v + u & \text{in } \Omega_T, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) & \text{on } \Omega \times \{0\}, \end{cases} \tag{1.1}$$

where $\Omega_T = \Omega \times (0,T)$, $T > 0$ is a fixed time, Ω is a bounded domain in $\mathbb{R}^N$, $N \geqslant 3$ with smooth boundary $\partial\Omega$, $q_i \geqslant 2$ and $m^- > q_i - \frac{2}{N}$ for all $i = 1, .., N$, such that

$$\begin{cases} m^+ = \max\limits_{1\leqslant i\leqslant N}\{m_i\}, \text{ and } q^+ = \max\limits_{1\leqslant i\leqslant N}\{q_i\}, \\ m^- = \min\limits_{1\leqslant i\leqslant N}\{m_i\}, \text{ and } q^- = \min\limits_{1\leqslant i\leqslant N}\{q_i\}. \end{cases}$$

The positive constant χ is called the chemotaxis coefficient, $d_1, d_2 > 0$ are the diffusion coefficients and $\gamma \geqslant 1$.

In general, organism or cell moves from a lower concentration towards a higher concentration of the chemo attractant, which is known as positive chemotaxis. In the same way, the opposite movement of the organisms is known as negative chemotaxis. In particular, microorganisms use chemotaxis to position themselves within the optimal portion of their habitats by monitoring the environmental concentration gradients of specific chemical attractant (positive chemotaxis) and repellent ligands (negative chemotaxis). Famous examples of biological species experiencing chemotaxis are the flagellated bacteria *Salmonella typhimurium* and *Escherichia coli*, the slime mould amoebae *Dictyostelium discoideum* and the human endothelial cells (see [**1**, **4**, **6**]). Theoretical and mathematical modelling of chemotaxis phenomena dates back to the pioneering works of Patlak in 1950s [**20**] and Keller–Segel in 1970s [**18**]. A general form of Patlak–Keller–Segel model for chemotaxis is given by

$$\begin{cases} u_t = \nabla .\left(\phi(u,v)\nabla u - \psi(u,v)\nabla v\right), \\ \tau v_t = d\Delta v + g(u,v)u - h(u,v)v, \end{cases} \tag{1.2}$$

where u denotes the density of cell population and v is the chemical attractant concentration. The mobility function $\phi(u,v)$ describes the diffusivity of the cells and $\psi(u,v)$ represents the chemotaxis sensibility, while the functions $g(u,v)$ and $h(u,v)$ are kinetic functions that describe production and degradation of the chemical signal, respectively. When $\phi(u,v) = 1$ and $\tau > 0$, system (1.2) becomes the classical parabolic–parabolic Keller–Segel system, such system has been extensively studied, see for example [**27**, **28**, **32**] and references therein. For the study of the parabolic-elliptic Keller–Segel system of quasilinear type, namely $\tau = 0$, with general $\phi(u,v)$ in (1.2), we refer to [**23**–**25**] and references therein.

Equation (1.1) with $m_i = m$ and $q_i = q$ is sometimes called the equation of isotropic diffusion. In the case of degenerate diffusion, the model $\phi(u,v) = mu^{m-1}$ and $\psi(u,v) = u^{q-1}$ in $\mathbb{R}^N$ was studied by several authors. The existence of the weak solutions was shown when $q - m < 0$ in [**25**] and when $q - m < \frac{2}{N}$ in [**13**]. When $q - m \geqslant \frac{2}{N}$ and the initial data (u_0, v_0) is small in some sense, the existence of the weak solutions was proved in [**14**], whereas, if $q - m > \frac{2}{N}$ then blow-up of solutions as in [**30**] was studied in [**15**, **16**].

In the present work, we are interested in the anisotropic case where the diffusion rates differ according to the direction x_i. Despite the resemblance with the isotropic cases presented in the previous mentioned works, the properties of the solutions to anisotropic equations are in striking contrast with the properties of the classical isotropic equations. The difficulties brought in by the anisotropy and the inhomogeneity of the diffusion operator are illustrated by the analysis of the self-similar solutions of anisotropic porous medium and $\vec{p}$-Laplacian types [**2**, **3**, **5**]. Unlike the isotropic case where the typical geometry is defined in terms of balls in $\mathbb{R}^N$, in the anisotropic case it is defined by parallelepipeds with the edge lengths related to the exponents m_i and q_i .

The chemotaxis model with anisotropic porous medium diffusion type is motivated from a biological point of view [**26**]. It is worthy of mentioning that the porous medium type diffusion can represent population pressure in cell invasion models [**21**], which initially arises from the ecology literature [**12**, **31**]. In fact, experimental investigation has shown that the diffusion coefficient depends on the bacterial density [**29**]. In the bacterial experiments done by Ohgiwari, Matsushita and Matsuyama [**19**], they recognized that cells located inside the bacterial colonies move actively, but cells became sluggish at the outermost front with apparently low cell density. This phenomenon indicates that bacteria become active as the cell density u increases. Thus, a natural choice of the bacterial diffusion coefficient is $\phi(u,v) = m_i u^{m_i - 1}$ with $m_i > 1$ for all $i = 1, .., N$, and this porous medium type bacterial diffusivity is based on the degenerate diffusion model proposed by Kawasaki *et al.* [17]. To our knowledge, Keller–Segel system with anisotropic porous medium diffusion models has not been studied specially and systematically.

2. Preliminary and main result

2.1. Imbedding and technical lemmas

To derive our existence and regularity results, we will need the following

THEOREM 2.1. [9], Theorem 1.1. *Let* $N \geqslant 2$, $\alpha_j \in (0,1)$, $1 \leqslant p < q$ *and* $p_j \geqslant 1$, $j = 1,..,N$, *be such that* $\sum_{j=1}^{N} \frac{1}{\alpha_j p_j} > 1$. *Then, for* $u \in W^{(\alpha),(p,\vec{p})}(\mathbb{R}^N)$ *(The fractional Sobolev–Liouville space) the inequality*

$$\|u\|_{L^q(\mathbb{R}^N)} \leqslant \sigma^{\frac{1}{q}} \|u\|_{L^p(\mathbb{R}^N)}^{1-\rho} \prod_{j=1}^{N} \left\| D_{x_j}^{\alpha_j} u \right\|_{L^{p_j}(\mathbb{R}^N)}^{\rho_j} \tag{2.1}$$

hold provided $M_N > 0$ *and*

$$q < q^* = \frac{\sum_{j=1}^{N} \frac{1}{\alpha_j}}{\sum_{j=1}^{N} \frac{1}{\alpha_j p_j} - 1},$$

where

$$M_N = 1 + \frac{1}{p} \sum_{j=1}^{N} \frac{1}{\alpha_j} - \sum_{j=1}^{N} \frac{1}{\alpha_j p_j}, \;\; \rho = \sum_{j=1}^{N} \rho_j \;\; \text{and} \;\; \rho_j = \frac{\frac{1}{p} - \frac{1}{q}}{\alpha_j M_N}.$$

The following lemma will show that theorem 2.1 holds true even for the case where $0 < p < 1$ and $p_j = 2, \; \forall j = 1,..,N$.

LEMMA 2.2. *Let* $\Omega \subset \mathbb{R}^N$ *with* $N \geqslant 3$ *be a bounded domain with smooth boundary, and* $0 < p < 1 \leqslant q < \frac{2N}{N-2}$. *Then, for all* $u \in H^1(\Omega)$ *we have*

$$\|u\|_{L^q(\Omega)}^{q} \leqslant \sigma^{\frac{1}{\beta}} \|u\|_{L^p(\Omega)}^{q(1-N\rho)} \prod_{j=1}^{N} \left\| \frac{\partial u}{\partial x_j} \right\|_{L^2(\Omega)}^{q\rho}, \tag{2.2}$$

where $\rho = \frac{2(q-p)}{q(p(2-N)+2N)}$, *and* $\frac{1}{\beta} = \frac{(q-p)(2+N)}{(q-1)(p(2-N)+2N)}$.

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Proof. By Hölder's inequality , we get that

$$\begin{aligned}\|u\|_{L^1(\Omega)} &= \int_\Omega |u|^{\frac{q(1-p)}{q-p}} \, |u|^{\frac{(q-1)p}{q-p}} \, \mathrm{d}x \\ &\leqslant \|u\|_{L^p(\Omega)}^{\frac{(q-1)p}{q-p}} \, \|u\|_{L^q(\Omega)}^{\frac{q(1-p)}{q-p}} .\end{aligned}$$

Then, by using theorem 1 of section 5.4 in [10] and applying theorem 2.1 for $p=1$ and $p_j = 2,\ \forall j = 1, .., N$, we obtain

$$\begin{aligned}\|u\|_{L^q(\Omega)} &\leqslant \sigma^{\frac{1}{q}} \, \|u\|_{L^1(\Omega)}^{1-N\rho_0} \prod_{j=1}^{N} \left\| \frac{\partial u}{\partial x_j} \right\|_{L^2(\Omega)}^{\rho_0} \\ &\leqslant \sigma^{\frac{1}{q}} \left[\|u\|_{L^p(\Omega)}^{\frac{(q-1)p}{q-p}} \, \|u\|_{L^q(\Omega)}^{\frac{q(1-p)}{q-p}} . \right]^{1-N\rho_0} \prod_{j=1}^{N} \left\| \frac{\partial u}{\partial x_j} \right\|_{L^2(\Omega)}^{\rho_0} ,\end{aligned} \tag{2.3}$$

where $\rho_0 = \frac{1-\frac{1}{q}}{N(\frac{1}{N}+\frac{1}{2})}$. Then from (2.3) we get (2.2) with

$$\rho = \frac{2(q-p)}{q(p(2-N)+2N)}, \text{ and } \frac{1}{\beta} = \frac{(q-p)(2+N)}{(q-1)(p(2-N)+2N)}.$$

□

Possible references on the theory of anisotropic Sobolev spaces are in [7, 8] and references therein. Next, we give some fundamental estimates of solutions to the following Cauchy problem for inhomogeneous linear heat equations:

$$\begin{cases} z_t = \Delta z - z + f & \text{in } \Omega \times (0,T), \\ z(x,0) = z_0(x), & x \in \Omega. \end{cases} \tag{2.4}$$

The following lemma can be found in [14].

LEMMA 2.3. *Let $\Omega \subset \mathbb{R}^N$ with $N \in \mathbb{N}$ be a bounded domain with smooth boundary, $T > 0$, $1 \leqslant p \leqslant \infty$ and $z_0 \in L^p(\Omega)$. If $f \in L^1(0,T; L^p(\Omega))$, then (2.4) has a unique mild solution $z \in C([0,T]; L^p(\Omega))$ given by*

$$z(t) = e^{-t} e^{t\Delta} z_0 + \int_0^t e^{-(t-s)} e^{(t-s)\Delta} f(s) \, \mathrm{d}s, \ \ t \in [0,T], \tag{2.5}$$

where $(e^{t\Delta} f)(x,t) = (4\pi t)^{-\frac{N}{2}} \int_\Omega e^{-\frac{|x-y|^2}{4t}} f(y,t) \, dy$. Moreover, the following estimates hold.

- *Let $1 \leqslant q \leqslant p \leqslant \infty$ and $\frac{1}{q} - \frac{1}{p} < \frac{1}{N}$. Assume further that $z_0 \in W^{2,p}(\Omega)$ and $f \in L^\infty(0,T; W^{1,q}(\Omega))$. Then for every $t \in [0,T]$,*

$$\|z(t)\|_{L^p(\Omega)} \leqslant \|z_0\|_{L^p(\Omega)} + C_0 \|f\|_{L^\infty(0,T;L^q(\Omega))}, \tag{2.6}$$

$$\|\nabla z(t)\|_{L^p(\Omega)} \leqslant \|\nabla z_0\|_{L^p(\Omega)} + C_0 \|f\|_{L^\infty(0,T;L^q(\Omega))}, \tag{2.7}$$

$$\|\Delta z(t)\|_{L^p(\Omega)} \leqslant \|\Delta z_0\|_{L^p(\Omega)} + C_0 \|\nabla f\|_{L^\infty(0,T;L^q(\Omega))}, \tag{2.8}$$

where C_0 is a positive constant depending on p, q and N.

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- *Let $1 < p < \infty$ and $f \in L^p(0,T;L^p(\Omega))$. Then for every $t \in [0,T]$,*

$$\|\Delta z(t)\|_{L^p(0,t;L^p(\Omega))} \leqslant \|\Delta z_0\|_{L^p(\Omega)} \left(1 - e^{-pt}\right)^{\frac{1}{p}} + C\|f\|_{L^p(0,T;L^p(\Omega))}, \tag{2.9}$$

where C_0 is a positive constant depending on N and p.

2.2. Formulation of the problem and main result

Throughout this paper, we deal with weak solutions of (1.1). Our definition of the weak solutions now reads

DEFINITION 2.4. *A pair of nonnegative functions (u, v) is said to be a weak solution of (1.1) if and only if for all $i = 1, .., N$ we have*

$$u \in L^\infty(\Omega_T),\ u^{m_i} \in L^2(0,T;H^1(\Omega)),\ \textit{and } v \in L^\infty(0,T;H^1(\Omega)),$$

such that (u, v) satisfies the equations in the sense of distribution, i.e., that

$$\sum_{i=1}^{N} \int_0^T \int_\Omega \left\{ d_1 \frac{\partial u^{m_i}}{\partial x_i} . \frac{\partial \varphi}{\partial x_i} - \left(\frac{u}{\gamma + v}\right)^{q_i - 1} \frac{\partial v}{\partial x_i} . \frac{\partial \varphi}{\partial x_i} - u\varphi_t \right\} \, \mathrm{d}x\mathrm{d}t$$
$$= \int_\Omega u_0(x)\varphi(x, 0) \ \mathrm{d}x,$$
$$\int_0^T \int_\Omega \{d_2 \nabla v . \nabla \varphi + v\varphi - u\varphi - v\varphi_t\} \ \mathrm{d}x\mathrm{d}t = \int_\Omega v_0(x)\varphi(x, 0) \ \mathrm{d}x,$$

for any continuously differentiable function φ with compact support in $\Omega \times [0, T)$.

Motivated by the works mentioned in the previous section, our paper extends the results in [**14**, **25**] to the system (1.1) with anisotropic nonlinear diffusion. Now we state the main result of this paper. To be precise, we will assume the initial data (u_0, v_0) to satisfy

$$\begin{cases} u_0 \in C^0(\overline{\Omega}) \text{ with } u_0 \geqslant 0 \ \text{ in } \Omega, \\ v_0 \in W^{2,\infty}(\Omega), \text{ with } v_0 \geqslant 0 \text{ in } \Omega. \end{cases} \tag{2.10}$$

THEOREM 2.5. *Let $q_i \geqslant 2$ and $m^- > q_i - \frac{2}{N}$ for every $i = 1, ..N$, $\Omega \subset \mathbb{R}^N$ for $N \geqslant 3$ be a bounded domain with smooth boundary. Then for all (u_0, v_0) satisfying (2.10), the system (1.1) possesses at last one weak solution in the sense of definition 2.4.*

3. Approximated equations

The first equation of (1.1) is a quasilinear parabolic equation of degenerate type. Therefore, we cannot expect the system (1.1) to have a classical solution at the point where u vanishes. In order to prove theorem 2.5, we use a compactness method and

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introduce the following approximated equation of (1.1):

$$\begin{cases} u_{\varepsilon,t} = \displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(d_1 m_i (u_\varepsilon+\varepsilon)^{m_i-1}\frac{\partial u_\varepsilon}{\partial x_i} - \chi \frac{(u_\varepsilon+\varepsilon)^{q_i-2}u_\varepsilon}{(\gamma+v_\varepsilon)^{q_i-1}}\frac{\partial v_\varepsilon}{\partial x_i}\right) & \text{in } \Omega_T, \\ v_{\varepsilon,t} = d_2 \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon & \text{in } \Omega_T, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = \dfrac{\partial v_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ u_\varepsilon(x,0) = u_0(x),\ v_\varepsilon(x,0) = v_0(x) & \text{on } \Omega \times \{0\}, \end{cases} \tag{3.1}$$

where $\varepsilon \in (0,1)$.

3.1. Existence of weak solutions of (3.1)

We are going to give an existence result of (3.1) under the condition that there exists a positive constant k such that

$$d = \min\{d_1 m_i \varepsilon^{m_i-1}, d_2\} \geqslant \frac{K}{\gamma^{q_i-1}}, \quad \forall i = 1,..,N. \tag{3.2}$$

THEOREM 3.1. *Assume that* (3.2) *holds. If* $u_0, v_0 \in L^2(\Omega)$, *then* (3.1) *possesses a nonnegative weak solution* $(u_\varepsilon, v_\varepsilon)$ *such that*

$$u_\varepsilon, v_\varepsilon \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),\ u_{\varepsilon,t}, v_{\varepsilon,t} \in L^2(0,T;(W^{1,\infty}(\Omega))'),$$

such that u_ε *has the conservation law*

$$\|u_\varepsilon(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}, \quad t \in [0,T]. \tag{3.3}$$

Proof. The existence of the weak solution to (3.1) can be obtained by using Schauder's fixed point theorem, a priori estimates and using the compactness results. We start by introducing for a small number $\delta > 0$ the following

$$F_\delta = \frac{F}{1+\delta F}, \text{ with } F(s,t) = -t + s,$$
$$f_\delta(s) = \frac{s^+}{1+\delta s^+}, \text{ with } s^+ = \max\{0,s\},$$

such that we have $0 \leqslant f_\delta(s) \leqslant \min\{s^+, \frac{1}{\delta}\}$ for any $s \in \mathbb{R}$ and $f_\delta(s) \longrightarrow s$ pointwise in $\mathbb{R}$ as $\delta \longrightarrow 0$. Therefore, we can conclude that there exists a positive constant K such that

$$\begin{aligned} \frac{(f_\delta(u_\varepsilon)+\varepsilon)^{q_i-2} f_\delta(u_\varepsilon)}{(\gamma + f_\delta(v_\varepsilon))^{q_i-1}} &\leqslant \frac{1}{\gamma^{q_i-1}}(\min\{u_\varepsilon^+, \delta^{-1}\}+1)^{q_i-2}\min\{u_\varepsilon^+, \delta^{-1}\} \\ &\leqslant \frac{K}{\gamma^{q_i-1}}. \end{aligned}$$

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Let $\overline{u}_\varepsilon, \overline{v}_\varepsilon \in L^2(\Omega_T)$ be given and consider the linear problem

$$\begin{cases} u_{\varepsilon,t} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_{11}\frac{\partial u_\varepsilon}{\partial x_i} + a_{12}\frac{\partial v_\varepsilon}{\partial x_i}\right) = 0, \\ v_{\varepsilon,t} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_{21}\frac{\partial u_\varepsilon}{\partial x_i} + a_{22}\frac{\partial v_\varepsilon}{\partial x_i}\right) = F_\delta(\overline{u}_\varepsilon^+, \overline{v}_\varepsilon^+), \end{cases}$$

where the diffusion matrix $A_i = (a_{jk})_i$ is given by

$$A_i(\overline{u}_\varepsilon, \overline{v}_\varepsilon) = \begin{pmatrix} d_1 m_i (f_\delta(\overline{u}_\varepsilon) + \varepsilon)^{m_i - 1} & -\chi \frac{(f_\delta(\overline{u}_\varepsilon)+\varepsilon)^{q_i-1} f_\delta(\overline{u}_\varepsilon)}{(\gamma + f_\delta(\overline{v}_\varepsilon))^{q_i-1}} \\ 0 & d_2 \end{pmatrix}.$$

Moreover, the matrix A_i is uniformly positive definite, since for any $X = (x, y) \in \mathbb{R}^2$ we have

$$\begin{aligned} X^T A_i X &= x^2 d_1 m_i (f_\delta(\overline{u}_\varepsilon) + \varepsilon)^{m_i-1} + d_2 y^2 - \chi \frac{(f_\delta(\overline{u}_\varepsilon) + \varepsilon)^{q_i-1} f_\delta(\overline{u}_\varepsilon)}{(\gamma + f_\delta(\overline{v}_\varepsilon))^{q_i-1}} xy \\ &\geqslant d(x^2 + y^2) - \chi \frac{(f_\delta(\overline{u}_\varepsilon) + \varepsilon)^{q_i-1} f_\delta(\overline{u}_\varepsilon)}{2(\gamma + f_\delta(\overline{v}_\varepsilon))^{q_i-1}} (x^2 + y^2) \\ &\geqslant (d - \frac{K}{\gamma^{q_i-1}})(x^2 + y^2) \geqslant 0, \end{aligned}$$

where we used (3.2) and the fact that $xy = \frac{1}{2}(x+y)^2 - \frac{1}{2}(x^2+y^2) \geqslant -\frac{1}{2}(x^2+y^2)$ and $-xy = \frac{1}{2}(x-y)^2 - \frac{1}{2}(x^2+y^2) \geqslant -\frac{1}{2}(x^2+y^2)$. Hence, the desired existence result is guaranteed by theorem 1 in [**11**]. □

3.2. A priori estimates

In order to prove theorem 2.5, we state and prove two key propositions which control L^r- and $L^\infty-$estimates of the solution $(u_\varepsilon, v_\varepsilon)$ of (3.1).

Proposition 3.2. *Assume that* (2.10), *and* (3.2) *hold. Let* $N \geqslant 3$, $q_i \geqslant 2$ *and* $m^- > q_i - \frac{2}{3}$ *for all* $i = 1, .., N$. *Then* $(u_\varepsilon, v_\varepsilon)$ *satisfies the following estimates*

$$\sup_{0<t<T} \|u_\varepsilon(t)\|_{L^r(\Omega)} \leqslant C, \text{ for all } r \in [1, \infty), \tag{3.4}$$

$$\sup_{0<t<T} \|v_\varepsilon(t)\|_{W^{1,\infty}(\Omega)} \leqslant C, \tag{3.5}$$

where C *is a positive constant independent of* ε.

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Proof. By taking $r \in (1, \infty)$, multiplying the first equation in (3.1) by u_ε^{r-1} and integrating by parts, we get

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial t}\|u_\varepsilon(t)\|^r_{L^r(\Omega)} &= \sum_{i=1}^{N}\Big[-\int_\Omega d_1 m_i (u_\varepsilon+\varepsilon)^{m_i-1}\frac{\partial u_\varepsilon}{\partial x_i}(r-1)u_\varepsilon^{r-2}.\frac{\partial u_\varepsilon}{\partial x_i} \\
&\quad + \chi(r-1)\int_\Omega \frac{(u_\varepsilon+\varepsilon)^{q_i-2}u_\varepsilon^{r-1}}{(\gamma+v_\varepsilon)^{q_i-1}}\frac{\partial v_\varepsilon}{\partial x_i}.\frac{\partial u_\varepsilon}{\partial x_i}\Big] \\
&\leqslant \sum_{i=1}^{N}\Big[-\frac{4d_1 m^-(r-1)}{(r+\alpha-1)^2}\left\|\frac{\partial}{\partial x_i}u^{\frac{\alpha+r-1}{2}}\right\|^2_{L^2(\Omega)} \\
&\quad + \frac{(r-1)\chi}{\gamma^{q^- -2}}\int_\Omega (u_\varepsilon+\varepsilon)^{q_i-2}u_\varepsilon^{r-1}\frac{\partial u_\varepsilon}{\partial x_i}.\frac{\partial v_\varepsilon}{\partial x_i}\Big] \\
&= \sum_{i=1}^{N}\Big[-\frac{4d_1 m^-(r-1)}{(r+\alpha-1)^2}\left\|\frac{\partial}{\partial x_i}u^{\frac{\alpha+r-1}{2}}\right\|^2_{L^2(\Omega)} + \frac{(r-1)\chi}{\gamma^{q^- -2}}I_i\Big],
\end{aligned} \tag{3.6}$$

where

$$\alpha = \begin{cases} m^-, & \text{if } (u_\varepsilon+\varepsilon)\geqslant 1, \\ m^+, & \text{if } (u_\varepsilon+\varepsilon)< 1. \end{cases} \tag{3.7}$$

Next, we set

$$F_i(s) = \int_0^s (\tau+\varepsilon)^{q_i-2}\tau^{r-1}\ d\tau,\ \ s\geqslant 0,$$

such that

$$F_i(s) \leqslant 2^{q_i-2}\left[\frac{s^{r+q_i-2}}{r+q_i-2}+\frac{s^r}{r}\right].$$

Therefore, I_i becomes

$$\begin{aligned}
I_i &= \int_\Omega \frac{\partial}{\partial x_i}[F_i(u_\varepsilon)].\frac{\partial v_\varepsilon}{\partial x_i}\ \mathrm{d}x = -\int_\Omega F_i(u_\varepsilon).\frac{\partial^2 v_\varepsilon}{\partial^2 x_i}\ \mathrm{d}x \\
&\leqslant \frac{2^{q^+-2}}{r+q^- -2}\int_\Omega u_\varepsilon^{r+q_i-2}\left|\frac{\partial^2 v_\varepsilon}{\partial^2 x_i}\right|\ \mathrm{d}x + \frac{2^{q^+-2}}{r}\int_\Omega u_\varepsilon^{r}\left|\frac{\partial^2 v_\varepsilon}{\partial^2 x_i}\right|\ \mathrm{d}x \\
&= I_i' + I_i''.
\end{aligned} \tag{3.8}$$

Next, we are going to integrate I_i' over $(0, t)$ for $t \in (0, T)$ such that

$$\begin{aligned}
\sum_{i=1}^{N}\int_0^t I_i'(s)\ \mathrm{d}s &= \frac{2^{q^+-2}}{r+q^- -2}\sum_{i=1}^{N}\int_0^t\int_\Omega u_\varepsilon^{r+q_i-2}\left|\frac{\partial^2 v_\varepsilon}{\partial^2 x_i}\right|\ \mathrm{d}x\mathrm{d}s \\
&\leqslant \frac{2^{q^+-2}C}{r+q^- -2}\left\{\int_0^t\int_\Omega u_\varepsilon^{r+q^+-2}|\Delta v_\varepsilon|\ \mathrm{d}x\mathrm{d}s + \int_0^t\int_\Omega |\Delta v_\varepsilon|\ \mathrm{d}x\mathrm{d}s\right\} \\
&\leqslant \frac{2^{q^+-2}C}{r+q^- -2}\left\{\left(\int_0^t\int_\Omega u_\varepsilon^{r+q^+-1}\ \mathrm{d}x\mathrm{d}s\right)^{\frac{r+q^+-2}{r+q^+-1}}\right.
\end{aligned}$$

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$$
\begin{aligned}
&\times\left(\int_0^t\int_\Omega |\Delta v_\varepsilon|^{r+q^+-1}\ \mathrm{d}x\mathrm{d}s\right)^{\frac{1}{r+q^+-1}}\\
&+\|\Delta v_\varepsilon\|_{L^{r+q^+-1}(0,t;L^{r+q^+-1}(\Omega))}\Bigg\}\\
\leqslant\ & \frac{2^{q^+-2}C}{r+q^--2}\Bigg\{\|\Delta v_0\|_{L^{r+q^+-1}(\Omega)}\left(1-e^{-(r+q^+-1)t}\right)^{\frac{1}{r+q^+-1}}\\
&\times\left(\int_0^t\int_\Omega u_\varepsilon^{r+q^+-1}\ \mathrm{d}x\mathrm{d}s\right)^{\frac{r+q^+-2}{r+q^+-1}}\\
&+C_{<r+q^+-1>}\int_0^t\int_\Omega u_\varepsilon^{r+q^+-1}\ \mathrm{d}x\mathrm{d}s\\
&+\|\Delta v_0\|_{L^{r+q^+-1}(\Omega)}\left(1-e^{-(r+q^+-1)t}\right)^{\frac{1}{r+q^+-1}}\\
&+C_{<r+q^+-1>}\left(\int_0^t\int_\Omega u_\varepsilon^{r+q^+-1}\ \mathrm{d}x\mathrm{d}s\right)^{\frac{1}{r+q^+-1}}\Bigg\}\\
\leqslant\ & \frac{2^{q^+-2}C}{r+q^--2}\left\{\|\Delta v_0\|^{r+q^+-1}_{L^{r+q^+-1}(\Omega)}+1+\int_0^t\int_\Omega u_\varepsilon^{r+q^+-1}\ \mathrm{d}x\mathrm{d}s\right\},
\end{aligned}
\tag{3.9}
$$

where we used Hölder's inequality, (2.9) and Young's inequality.

Next, we are going to simplify the last integral in the right-hand side of (3.9) by using lemma 2.2. As a consequence, by letting $r>r_0=\max\{\alpha-2q^++1, \frac{N}{2}(q^+-\alpha)-q^++1, \frac{2(N-1)}{N}-\alpha\}$, we have the following

$$
\begin{aligned}
\int_0^t\int_\Omega u_\varepsilon^{r+q^+-1}\ \mathrm{d}x\mathrm{d}s &= \int_0^t\int_\Omega u_\varepsilon^{\frac{r+\alpha-1}{2}\frac{2(r+q^+-1)}{r+\alpha-1}}\ \mathrm{d}x\mathrm{d}s = \int_0^t \left\|u_\varepsilon^{\frac{r+\alpha-1}{2}}\right\|^{\frac{2(r+q^+-1)}{r+\alpha-1}}_{L^{\frac{2(r+q^+-1)}{r+\alpha-1}}(\Omega)}\ \mathrm{d}s\\
&\leqslant C\int_0^t\left\{\|u_\varepsilon\|^{(r+q^+-1)(1-N\rho)}_{L^1(\Omega)}\prod_{i=1}^N\left\|\frac{\partial}{\partial x_i}u_\varepsilon^{\frac{r+\alpha-1}{2}}\right\|^{\frac{2\rho(r+q^+-1)}{r+\alpha-1}}_{L^2(\Omega)}\right\}\ \mathrm{d}s,
\end{aligned}
\tag{3.10}
$$

where

$$
\rho=\frac{\frac{r+\alpha-1}{2}\left(\frac{r+q^+-2}{r+q^+-1}\right)}{1+\frac{N}{2}(r+\alpha-2)}.
$$

We can replace the geometric mean on the right-hand side of (3.10) by an arithmetic mean. Indeed, by the inequality between geometric and arithmetic means we get

$$
\begin{aligned}
\prod_{i=1}^N\left\|\frac{\partial}{\partial x_i}u_\varepsilon^{\frac{r+\alpha-1}{2}}\right\|^{\frac{2\rho(r+q^+-1)}{r+\alpha-1}}_{L^2(\Omega)} &= \prod_{i=1}^N\left\|\frac{\partial}{\partial x_i}u_\varepsilon^{\frac{r+\alpha-1}{2}}\right\|^{\frac{1}{N}\left(\frac{N(r+q^+-2)}{1+\frac{N}{2}(r+\alpha-2)}\right)}_{L^2(\Omega)}\\
&\leqslant \frac{1}{N}\sum_{i=1}^N\left\|\frac{\partial}{\partial x_i}u_\varepsilon^{\frac{r+\alpha-1}{2}}\right\|^{\frac{N(r+q^+-2)}{1+\frac{N}{2}(r+\alpha-2)}}_{L^2(\Omega)}
\end{aligned}
\tag{3.11}
$$

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Since, we took $q_i < \frac{2}{N} + m^- \leqslant \frac{2}{N} + \alpha$ for all $i = 1, .., N$. Then we get

$$\rho' = \frac{2N(r + q^+ - 2)}{2 + N(r + \alpha - 2)} < 2. \tag{3.12}$$

Therefore, by using (3.11), (3.12), Young's inequality and the mass conservation law (3.3), we obtain

$$\begin{aligned}\sum_{i=1}^{N} \int_0^t I_i'(s)\ \mathrm{d}s \leqslant \frac{2^{q^+-2}C}{r+q^- - 2}\Bigg[\|\Delta v_0\|^{r+q^+-1}_{L^{r+q^+-1}(\Omega)} + 1 \\ + C(\nu)\int_0^t \|u_0\|_{L^1(\Omega)}^{\frac{2(r+q^+-1)(1-N\rho)}{2-\rho'}} \\ + \nu \sum_{i=1}^{N}\int_0^t \left\| \frac{\partial}{\partial x_i} u_\varepsilon^{\frac{r+\alpha-1}{2}} \right\|^2_{L^2(\Omega)} \ \mathrm{d}s\Bigg].\end{aligned} \tag{3.13}$$

The integral I_i'' can be controlled by the same way as I_i' for all $i = 1, .., N$. Consequently, we omit that $\sum_i^N \int_0^t I_i''\ \mathrm{d}s$ satisfy the same estimation as in (3.13). Therefore, by integrating (3.6) over $(0, t)$ and using the previous estimates, we arrive at

$$\begin{aligned}\|u_\varepsilon(t)\|^r_{L^r(\Omega)} &\leqslant \|u_0\|^r_{L^r(\Omega)} - \sum_{i=1}^{N} \frac{4d_1 m^-(r-1)}{(r+\alpha-1)^2}\int_0^t \left\|\frac{\partial}{\partial x_i} u^{\frac{\alpha+r-1}{2}}\right\|^2_{L^2(\Omega)}\ \mathrm{d}s \\ &\quad + \frac{C2^{q^+-2}\chi(r-1)}{\gamma^{q^--2}(r+q^--2)}\left[\|\Delta v_0\|^{r+q^+-1}_{L^{r+q^+-1}(\Omega)} + \int_0^t \|u_0\|_{L^1(\Omega)}^{\frac{2(r+q^+-1)(1-N\rho)}{2-\rho'}}\ \mathrm{d}s + 1\right] \\ &\quad + \sum_{i=1}^{N} \frac{C2^{q^+-2}\chi(r-1)\nu}{\gamma^{q^--2}(r+q^--2)}\int_0^t \left\|\frac{\partial}{\partial x_i} u^{\frac{\alpha+r-1}{2}}\right\|^2_{L^2(\Omega)}\ \mathrm{d}s \\ &\leqslant \|u_0\|^r_{L^r(\Omega)} + \frac{C2^{q^+-2}\chi(r-1)t}{\gamma^{q^--2}(r+q^--2)}\left[\|\Delta v_0\|^{r+q^+-1}_{L^{r+q^+-1}(\Omega)} + \|u_0\|_{L^1(\Omega)}^{\frac{2(r+q^+-1)(1-N\rho)}{2-\rho'}} + 1\right],\end{aligned} \tag{3.14}$$

where we took $\nu = \frac{4d_1 m^- \gamma^{q^- -2}(r+q^- -2)}{2^{q^+-2}C\chi(r+\alpha-1)^2}$. Moreover, by letting $r > r_1 = \{r_0, \beta - q^+ + 1\}$ for $\beta >> 1$, we obtain that

$$\begin{aligned}\sup_{0<t<T} \|u_\varepsilon\|_{L^r(\Omega)} \leqslant \Bigg\{ \|u_0\|^{r-1}_{L^\infty(\Omega)}\|u_0\|_{L^1(\Omega)} + \frac{C2^{q^+-2}\chi(r-1)T}{\gamma^{q^--2}(r+q^--2)}\Bigg[\|\Delta v_0\|^{r+q^+-1-\beta}_{L^\infty(\Omega)} \\ + \|\Delta v_0\|^\beta_{L^\beta(\Omega)} + \|u_0\|_{L^1(\Omega)}^{\frac{2(r+q^+-1)(1-N\rho)}{2-\rho'}} + 1\Bigg]\Bigg\}^{\frac{1}{r}} = C,\end{aligned} \tag{3.15}$$

where C is a positive constant independent of ε.

Now, for the case $1 \leqslant r \leqslant r_1$ we have the following

$$\|u_\varepsilon(t)\|_{L^r(\Omega)} \leqslant \|u_0\|_{L^1(\Omega)} + \|u_\varepsilon(t)\|_{L^{r_1}(\Omega)}, \quad \text{for every } t \in (0, T), \tag{3.16}$$

where we used Hölder's inequality, the mass conservation law (3.3) and Young's inequality. Hence, (3.15) and (3.16) give us the desired estimation for every $r \in [1, \infty)$.

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Finally estimation (3.5) is a direct consequence of (2.6), (2.7) and (3.4) with $r = N + 1$. □

We conclude this section with the proof of L^∞-estimates of the approximated solutions.

PROPOSITION 3.3. *Let the same assumptions as those in proposition 3.2 hold. Then, there exists a positive constant C independent of ε such that*

$$\sup_{0<t<T} \|u_\varepsilon(t)\|_{L^\infty(\Omega)} \leqslant C. \tag{3.17}$$

Proof. We begin by multiplying the first equation in (3.1) by u_ε^{r-1} such that

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial t}\|u_\varepsilon\|^r_{L^r(\Omega)} &= \sum_{i=1}^N \Big[-\int_\Omega d_1 m_i (u_\varepsilon+\varepsilon)^{m_i-1}\frac{\partial u_\varepsilon}{\partial x_i}(r-1)u_\varepsilon^{r-2}\frac{\partial u_\varepsilon}{\partial x_i}\ \mathrm{d}x \\ &\quad + \chi\int_\Omega \frac{(u_\varepsilon+\varepsilon)^{q_i-2}}{(\gamma+v_\varepsilon)^{q_i-1}}\frac{\partial v_\varepsilon}{\partial x_i}(r-1)u_\varepsilon^{r-1}\frac{\partial u_\varepsilon}{\partial x_i}\ \mathrm{d}x\Big] \\ &\leqslant \sum_{i=1}^N \Big[-\frac{4d_1 m^-(r-1)}{(r+\alpha-1)^2}\left\|\frac{\partial}{\partial x_i}u_\varepsilon^{\frac{\alpha+r-1}{2}}\right\|^2_{L^2(\Omega)} + \frac{(r-1)\chi}{\gamma^{q^--2}}\left\|\frac{\partial v_\varepsilon}{\partial x_i}\right\|_{L^\infty(\Omega)} \\ &\quad \left(\int_\Omega u_\varepsilon^{q_i+r-3}\frac{\partial u_\varepsilon}{\partial x_i}\ \mathrm{d}x + \int_\Omega u_\varepsilon^{r-1}\frac{\partial u_\varepsilon}{\partial x_i}\ \mathrm{d}x\right)\Big], \end{aligned} \tag{3.18}$$

where α is defined in (3.7). Next, we are going to simplify the last two integrals in the right-hand side of (3.18). Then, for all $i = 1,..,N$ we have

$$\begin{aligned} &\frac{(r-1)\chi}{\gamma^{q^--2}}\left\|\frac{\partial v_\varepsilon}{\partial x_i}\right\|_{L^\infty(\Omega)}\left(\int_\Omega u_\varepsilon^{q_i+r-3}\frac{\partial u_\varepsilon}{\partial x_i}\ \mathrm{d}x + \int_\Omega u_\varepsilon^{r-1}\frac{\partial u_\varepsilon}{\partial x_i}\ \mathrm{d}x\right) \\ &= \frac{(r-1)\chi}{\gamma^{q^--2}}\left\|\frac{\partial v_\varepsilon}{\partial x_i}\right\|_{L^\infty(\Omega)}\frac{2}{r+\alpha-1}\Big[\int_\Omega u_\varepsilon^{\frac{r+2q_i-\alpha-3}{2}}\frac{\partial}{\partial x_i}u_\varepsilon^{\frac{\alpha+r-1}{2}}\ \mathrm{d}x \\ &+ \int_\Omega u_\varepsilon^{\frac{r-\alpha+1}{2}}\frac{\partial}{\partial x_i}u_\varepsilon^{\frac{\alpha+r-1}{2}}\ \mathrm{d}x\Big] \\ &\leqslant 2(r-1)\Big[\frac{2\nu}{(r+\alpha-1)^2}\int_\Omega\left|\frac{\partial}{\partial x_i}u_\varepsilon^{\frac{\alpha+r-1}{2}}\right|^2\ \mathrm{d}x + \frac{C(\nu)\chi^2}{\gamma^{2(q^--2)}}\left\|\frac{\partial v_\varepsilon}{\partial x_i}\right\|^2_{L^\infty(\Omega)} \\ &\left(\int_\Omega u_\varepsilon^{r+2q_i-\alpha-3}\ \mathrm{d}x + \int_\Omega u_\varepsilon^{r-\alpha+1}\ \mathrm{d}x\right)\Big] \\ &= \Big[\frac{2d_1m^-(r-1)}{(r+\alpha-1)^2}\int_\Omega\left|\frac{\partial}{\partial x_i}u_\varepsilon^{\frac{\alpha+r-1}{2}}\right|^2\ \mathrm{d}x + \frac{\chi^2(r-1)}{d_1m^-\gamma^{2(q^--2)}}\left\|\frac{\partial v_\varepsilon}{\partial x_i}\right\|^2_{L^\infty(\Omega)} \\ &\left(\|u_\varepsilon\|^{r+2q_i-\alpha-3}_{L^{r+2q_i-\alpha-3}(\Omega)} + \|u_\varepsilon\|^{r-\alpha+1}_{L^{r-\alpha+1}(\Omega)}\right)\Big], \end{aligned} \tag{3.19}$$

where we used Young's inequality and choose ν accordingly.

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We will deal only with the norm $\|u_\varepsilon\|^{r+2q_i-\alpha-3}_{L^{r+2q_i-\alpha-3}(\Omega)}$ in the right-hand side of (3.19), because the last norm can be controlled by the same way. Furthermore, we are going to study the following two possible cases.

Case 1: $q_i > 3 - \frac{2}{N}$, $\forall i = 1, .., N$.

Let l be a natural number which is chosen later. Therefore, by applying lemma 2.2, we obtain

$$\begin{aligned}
\|u_\varepsilon\|^{r+2q_i-\alpha-3}_{L^{r+2q_i-\alpha-3}(\Omega)} &= \left\|u_\varepsilon^{\frac{r+\alpha-1}{2}}\right\|^{\frac{2(r+2q_i-\alpha-3)}{r+\alpha-1}}_{L^{\frac{2(r+2q_i-\alpha-3)}{r+\alpha-1}}(\Omega)} \\
&\leqslant \sigma^{\frac{1}{\beta_i}} \left\|u_\varepsilon^{\frac{r+\alpha-1}{2}}\right\|^{\frac{2(r+2q_i-\alpha-3)}{r+\alpha-1}(1-N\rho_i)}_{L^{\frac{2r}{l(r+\alpha-1)}}(\Omega)} \prod_{j=1}^{N} \left\|\frac{\partial}{\partial x_j} u_\varepsilon^{\frac{r+\alpha-1}{2}}\right\|^{\frac{2(r+2q_i-\alpha-3)\rho_i}{r+\alpha-1}}_{L^2(\Omega)} \\
&\leqslant \frac{\sigma^{\frac{1}{\beta_i}}}{N} \|u_\varepsilon\|^{(r+2q_i-\alpha-3)(1-N\rho_i)}_{L^{\frac{r}{l}}(\Omega)} \sum_{j=1}^{N} \left\|\frac{\partial}{\partial x_j} u_\varepsilon^{\frac{r+\alpha-1}{2}}\right\|^{\frac{2N\rho_i(r+2q_i-\alpha-3)}{r+\alpha-1}}_{L^2(\Omega)},
\end{aligned} \tag{3.20}$$

for $r > r_0 = \max\{3\alpha - 4q_i + 5, -\frac{N}{2}(\alpha-1) + \frac{(N-2)}{2}(2q_i - \alpha - 3), \alpha - 2q_i + 3\}, l > 1$,

$$\rho_i = \frac{\frac{r+\alpha-1}{2}\left(\frac{l(r+2q_i-\alpha-3)-r}{r(r+2q_i-\alpha-3)}\right)}{1 + \frac{N}{2r}(l(r+\alpha-1)-r)}, \quad \text{and } \beta_i = \frac{(2+N)\left(\frac{2(r+2q_i-\alpha-3)}{r+\alpha-1} - \frac{2r}{l(r+\alpha-1)}\right)}{\left(\frac{2(r+2q_i-\alpha-3)}{r+\alpha-1} - 1\right)\left(\frac{2r(2-N)}{l(r+\alpha-1)} + 2N\right)}.$$

By simple computation, we find that $\frac{2N\rho_i(r+2q_i-\alpha-3)}{r+\alpha-1} < 2$ and $\frac{1}{\beta_i} \leqslant 6$ for every $r > r_0$ and $i = 1, .., N$. Therefore, by Young's inequality we get

$$\begin{aligned}
&\frac{\chi^2(r-1)}{d_1 m^- \gamma^{2(q^- -2)}} \left\|\frac{\partial v_\varepsilon}{\partial x_i}\right\|^2_{L^\infty(\Omega)} \|u_\varepsilon\|^{r+2q_i-\alpha-3}_{L^{r+2q_i-\alpha-3}(\Omega)} \\
&\quad \leqslant \frac{\chi(r-1)}{d_1 m^- \gamma^{2(q^- -2)}} \left\|\frac{\partial v_\varepsilon}{\partial x_i}\right\|^2_{L^\infty(\Omega)} \frac{C}{N} \|u_\varepsilon\|^{(r+2q_i-\alpha-3)(1-N\rho_i)}_{L^{\frac{r}{l}}(\Omega)} \\
&\sum_{j=1}^{N} \left\|\frac{\partial}{\partial x_j} u_\varepsilon^{\frac{r+\alpha-1}{2}}\right\|^{\frac{2N\rho_i(r+2q_i-\alpha-3)}{r+\alpha-1}}_{L^2(\Omega)} \\
&\quad \leqslant \frac{1}{N} \sum_{j=1}^{N} (r-1)\nu \left\|\frac{\partial}{\partial x_j} u_\varepsilon^{\frac{r+\alpha-1}{2}}\right\|^2_{L^2(\Omega)} + C(\nu)(r-1) \\
&\left(C \frac{\chi^2}{d_1 m^- \gamma^{2(q^- -2)}} \left\|\frac{\partial v_\varepsilon}{\partial x_i}\right\|^2_{L^\infty(\Omega)}\right)^{\xi_{i,1}} \|u_\varepsilon\|^{(r+2q_i-\alpha-3)(1-N\rho_i)\xi_{i,1}}_{L^{\frac{r}{l}}(\Omega)},
\end{aligned} \tag{3.21}$$

where

$$\xi_{1,i} = \frac{r+\alpha-1}{(r+\alpha-1) - N\rho_i(r+2q_i-\alpha-3)}.$$

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Next, by taking $\nu = \frac{d_1 m^-}{(r+\alpha-1)^2}$, then $C(\nu) = \dfrac{1}{q_i(\nu p_i)^{\frac{q_i}{p_i}}}$, where

$$q_i = \frac{r+\alpha-1}{(r+\alpha-1) - N\rho_i(r+2q_i-\alpha-3)}, \text{ and } p = \frac{r+\alpha-1}{N\rho_i(r+2q_i-\alpha-3)}.$$

Then, (3.21) becomes

$$\begin{aligned}&\frac{\chi^2(r-1)}{d_1 m^- \gamma^{2(q^- -2)}} \left\| \frac{\partial v_\varepsilon}{\partial x_i} \right\|^2_{L^\infty(\Omega)} \|u_\varepsilon\|^{r+2q_i-\alpha-3}_{L^{r+2q_i-\alpha-3}(\Omega)} \\ &\quad \leqslant \frac{1}{N} \sum_{j=1}^{N} \frac{(r-1)d_1 m^-}{(r+\alpha-1)^2} \left\| \frac{\partial}{\partial x_j} u_\varepsilon^{\frac{r+\alpha-1}{2}} \right\|^2_{L^2(\Omega)} + C(r-1) r^{2\xi_{i,2}} \\ &\left(C \frac{\chi^2}{d_1 m^- \gamma^{2(q^- -2)}} \left\| \frac{\partial v_\varepsilon}{\partial x_i} \right\|^2_{L^\infty(\Omega)} \right)^{\xi_{i,1}} \|u_\varepsilon\|^{(r+2q_i-\alpha-3)(1-N\rho_i)\xi_{i,1}}_{L^{\frac{r}{l}}(\Omega)},\end{aligned} \tag{3.22}$$

where

$$\xi_{i,2} = \frac{N\rho_i(r+2q_i-\alpha-3)}{(r+\alpha-1) - N\rho_i(r+2q_i-\alpha-3)}.$$

Next, for $l > 1$, $r > r_0$ and for every $i = 1, ..N$, we have

$$N\rho_i \longrightarrow \frac{\frac{1}{2}(l-1)}{\frac{1}{2}(l-1) + \frac{1}{N}} \text{ as } r \longrightarrow \infty. \tag{3.23}$$

Consequently, we obtain that

$$\frac{\frac{1}{2}(l-1) - \frac{1}{2N}}{\frac{1}{2}(l-1) + \frac{1}{N}} \leqslant N\rho_i \leqslant \frac{\frac{1}{2}(l-1) + \frac{1}{2N}}{\frac{1}{2}(l-1) + \frac{1}{N}}, \; \forall r > r_0 \text{ and every } i = 1, .., N. \tag{3.24}$$

Then, by (3.24) we get the following estimations

$$\xi_{i,1} \leqslant Nl + 2, \text{ and } \xi_{i,2} \leqslant Nl \text{ for all } r > r_0.$$

Therefore, (3.22) becomes

$$\begin{aligned}&\frac{\chi^2(r-1)}{d_1 m^- \gamma^{2(q^- -2)}} \left\| \frac{\partial v_\varepsilon}{\partial x_i} \right\|^2_{L^\infty(\Omega)} \|u_\varepsilon\|^{r+2q_i-\alpha-3}_{L^{r+2q_i-\alpha-3}(\Omega)} \\ &\quad \leqslant \frac{1}{N} \sum_{j=1}^{N} \frac{(r-1)d_1 m^-}{(r+\alpha-1)^2} \left\| \frac{\partial}{\partial x_j} u_\varepsilon^{\frac{r+\alpha-1}{2}} \right\|^2_{L^2(\Omega)} + C r^C \|u_\varepsilon\|^{(r+2q_i-\alpha-3)(1-N\rho_i)\xi_{i,1}}_{L^{\frac{r}{l}}(\Omega)}.\end{aligned} \tag{3.25}$$

Next, we are going to simplify the last term in the right-hand side of (3.25). For this reason, we choose l to verify

$$l > \max\left\{1, \frac{2\left((q + \frac{2}{N} - 3)(\frac{1}{N} - \frac{1}{2}) + (\alpha-1)(1-\frac{1}{N})\right)}{\alpha - q_i - \frac{2}{N} + 2}\right\}, \text{ for all } i = 1, .., N,$$

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such that

$$\frac{q_i + \frac{2}{N} - 3}{\alpha - 1} < \frac{\frac{1}{2}(l-1) - \frac{1}{2N}}{\frac{1}{2}(l-1) + \frac{1}{N}} \leqslant N\rho_i, \quad \text{for all } i = 1, .., N.$$

Therefore, by taking

$$r > r_1 = \max\left\{ r_0, \frac{(N\rho_i + 1)(\alpha - 1)(2q_i - \alpha - 3)}{(2q_i - \alpha - 3) - N\rho_i(\alpha - 1)} \right\}, \quad \text{for all } i = 1, .., N,$$

we get that

$$\xi_{i,3} = \frac{r}{(r + 2q_i - \alpha - 3)(1 - N\rho_i)\xi_{i,1}} \geqslant 1, \text{ for all } i = 1, .., N.$$

By simple computation, we get also $\xi_{i,3} \leqslant Nl + 2$ for all $i = 1, .., N$. To this end, we apply Young's inequality on the last term in the right-hand side of (3.25) such that

$$\begin{aligned} &\frac{\chi^2(r-1)}{d_1 m^- \gamma^{2(q^- - 2)}} \left\| \frac{\partial v_\varepsilon}{\partial x_i} \right\|^2_{L^\infty(\Omega)} \|u_\varepsilon\|^{r+2q_i-\alpha-3}_{L^{r+2q_i-\alpha-3}(\Omega)} \\ &\quad \leqslant \frac{1}{N} \sum_{j=1}^{N} \frac{(r-1)d_1 m^-}{(r+\alpha-1)^2} \left\| \frac{\partial}{\partial x_j} u_\varepsilon^{\frac{r+\alpha-1}{2}} \right\|^2_{L^2(\Omega)} + 1 + Cr^C \|u_\varepsilon\|^r_{L^{\frac{r}{l}}(\Omega)}. \end{aligned} \tag{3.26}$$

By the same method, we get also that

$$\begin{aligned} &\frac{\chi^2(r-1)}{d_1 m^- \gamma^{2(q^- - 2)}} \left\| \frac{\partial v_\varepsilon}{\partial x_i} \right\|^2_{L^\infty(\Omega)} \|u_\varepsilon\|^{r-\alpha+1}_{L^{r-\alpha+1}(\Omega)} \\ &\quad \leqslant \frac{1}{N} \sum_{j=1}^{N} \frac{(r-1)d_1 m^-}{(r+\alpha-1)^2} \left\| \frac{\partial}{\partial x_j} u_\varepsilon^{\frac{r+\alpha-1}{2}} \right\|^2_{L^2(\Omega)} + 1 + Cr^C \|u_\varepsilon\|^r_{L^{\frac{r}{l}}(\Omega)}, \end{aligned} \tag{3.27}$$

for every $r > r_1$. Then, by putting (3.26) and (3.27) into (3.18) we obtain

$$\frac{1}{r}\frac{\partial}{\partial t} \|u_\varepsilon\|^r_{L^r(\Omega)} \leqslant 2 + Cr^C \|u_\varepsilon\|^2_{L^{\frac{r}{l}}(\Omega)}. \tag{3.28}$$

Integrating (3.28) from 0 to t, we obtain

$$\sup_{0<t<T} \|u_\varepsilon\|^r_{L^r(\Omega)} \leqslant \|u_0\|^r_{L^r(\Omega)} + 2rT + CTr^C \sup_{0<t<T} \|u_\varepsilon\|^r_{L^{\frac{r}{l}}(\Omega)}. \tag{3.29}$$

Since

$$\|u_0\|_{L^r(\Omega)} \leqslant \|u_0\|^{\frac{r-1}{r}}_{L^\infty(\Omega)} \|u_0\|^{\frac{1}{r}}_{L^1(\Omega)} \leqslant C'. \tag{3.30}$$

Then,

$$\sup_{0<t<T} \|u_\varepsilon\|^r_{L^r(\Omega)} \leqslant C(T)^{\frac{1}{r}} r^{\frac{C}{r}} \max\{C', \sup_{0<t<T} \|u_\varepsilon(t)\|_{L^{\frac{r}{l}}(\Omega)}\}, \text{ for any } r > r_1. \tag{3.31}$$

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We are now in a position to derive the claimed $L^\infty-$estimate. Therefore, we set

$$\Lambda_p = \max\{C', \sup_{0<t<T} \|u_\varepsilon(t)\|_{L^{l^p}(\Omega)}\}, \text{ for any } p \geqslant 1. \tag{3.32}$$

Thereafter, we take $r = l^p$ in (3.31) which leads to

$$\begin{aligned}\Lambda_p &= C(T)^{\frac{1}{l^p}} l^{\frac{Cp}{2^p(\frac{l}{2})^p}} \max\{C', \sup_{0<t<T} \|u_\varepsilon(t)\|_{L^{l^{p-1}}(\Omega)}\} \\ &\leqslant C(T)^{\frac{1}{l^p}} l^{\frac{C}{(\frac{l}{2})^p}} \Lambda_{p-1},\end{aligned} \tag{3.33}$$

since $p \leqslant 2^p$ for $p \geqslant 1$. By induction, we get

$$\Lambda_p \leqslant C(T)^{\sum_{k=1}^p l^{-k}} l^{C\sum_{k=1}^p \left(\frac{l}{2}\right)^{-k}} \Lambda_0.$$

Then, by using the mass conservation law (3.3), taking $l > 2$ and letting $p \longrightarrow \infty$, we arrive at

$$\sup_{0<t<T} \|u_\varepsilon(t)\|_{L^\infty(\Omega)} \leqslant C(T) l^c \Lambda_0 = C'', \tag{3.34}$$

where C'' is a positive constant independent of ε.

Case 2: $2 \leqslant q_i \leqslant 3 - \frac{2}{N}$, $\forall i = 1, .., N$.
Knowing that $q_i < \alpha + \frac{2}{N}$, then $2q_i < \alpha + 3$ for every $i = 1, .., N$. Therefore,

$$\|u_\varepsilon\|^{r-\alpha+2q_i-3}_{L^{r-\alpha+2q_i-3}(\Omega)} \leqslant \|u_0\|_{L^1(\Omega)} + \|u_\varepsilon\|^r_{L^r(\Omega)}.$$

Then, (3.18) becomes

$$\begin{aligned}\frac{1}{r}\frac{\partial}{\partial t}\|u_\varepsilon\|^r_{L^r(\Omega)} \leqslant \sum_{i=1}^N \Bigg[&-\frac{2d_1 m^-(r-1)}{(r+\alpha-1)^2} \left\| \frac{\partial}{\partial x_i} u_\varepsilon^{\frac{\alpha+r-1}{2}} \right\|^2_{L^2(\Omega)} \\ &+ \frac{(r-1)\chi^2}{d_1 m^- \gamma^{2(q^--2)}} \left\| \frac{\partial v_\varepsilon}{\partial x_i} \right\|^2_{L^\infty(\Omega)} \left(2\|u_0\|_{L^1(\Omega)} + 2\|u_\varepsilon\|^r_{L^r(\Omega)}\right)\Bigg].\end{aligned} \tag{3.35}$$

Thereafter, by applying lemma 2.2 once again we obtain

$$\begin{aligned}\|u_\varepsilon\|^r_{L^r(\Omega)} &= \left\| u_\varepsilon^{\frac{r+\alpha-1}{2}} \right\|^{\frac{2r}{r+\alpha-1}}_{L^{\frac{2r}{r+\alpha-1}}(\Omega)} \\ &\leqslant \frac{\sigma^{\frac{1}{\beta}}}{N} \|u_\varepsilon\|^{r(1-N\rho)}_{\frac{r}{4}} \sum_{j=1}^N \left\| \frac{\partial}{\partial x_j} u_\varepsilon^{\frac{\alpha+r-1}{2}} \right\|^{\frac{2Nr\rho}{r+\alpha-1}}_{L^2(\Omega)},\end{aligned} \tag{3.36}$$

where

$$\rho = \frac{3(r+\alpha-1)}{2rN\left(\frac{1}{N} - \frac{1}{2} + \frac{2(r+\alpha-1)}{r}\right)}, \text{ and } \beta = \frac{3(r+\alpha-1)}{r(2+3N)+4N(\alpha-1)} < 1.$$

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It is easy to verify that $\frac{2Nr\rho}{r+\alpha-1} < 2$ and $\frac{1}{\beta} \leqslant 6$ for suitable $r > 0$. Then, by using the same method we used to get (3.27), we obtain

$$\begin{aligned} &\frac{2(r-1)\chi^2}{d_1 m^- \gamma^{2(q^- -2)}} \left\| \frac{\partial v_\varepsilon}{\partial x_i} \right\|^2_{L^\infty(\Omega)} \|u_\varepsilon\|^r_{L^r(\Omega)} \\ &\quad \leqslant \frac{2}{N} \sum_{j=1}^{N} \frac{(r-1)d_1 m^-}{(r+\alpha-1)^2} \left\| \frac{\partial}{\partial x_j} u_\varepsilon^{\frac{r+\alpha-1}{2}} \right\|^2_{L^2(\Omega)} + 2Cr^C \|u_\varepsilon\|^r_{L^{\frac{r}{4}}(\Omega)} + 2, \end{aligned} \tag{3.37}$$

for suitable $r > 0$. Hence, by putting (3.37) into (3.35) and applying similar arguments of the case $q > 3 - \frac{2}{N}$ we get the desired $L^\infty-$estimate. □

We complete this section by discussing some uniform estimates (with respect to ε) of u_ε and v_ε.

LEMMA 3.4. *For $q_i \geqslant 2$ and $m^- > q_i - \frac{2}{N}$ for all $i = 1, ..N$, there exists a constant C such that*

$$\sum_{i=1}^{N} \int_0^T \int_\Omega \left| \frac{\partial}{\partial x_i} (u_\varepsilon + \varepsilon)^{\frac{m_i+1}{2}} \right|^2 \, \mathrm{d}x\mathrm{d}t \leqslant C, \tag{3.38}$$

$$\sum_{i=1}^{N} \int_0^T \int_\Omega \left| \frac{\partial}{\partial x_i} (u_\varepsilon + \varepsilon)^{m_i} \right|^2 \, \mathrm{d}x\mathrm{d}t \leqslant C, \tag{3.39}$$

$$\int_0^T \left\| \partial_t u_\varepsilon^\beta \right\|_{(W^{1,N+1}(\Omega))'} \, dt \leqslant C, \tag{3.40}$$

and,

$$\int_0^T \left\| \partial_t v_\varepsilon \right\|_{(W^{1,N+1}(\Omega))'} \, dt \leqslant C, \tag{3.41}$$

for each $\varepsilon \in (0,1)$ and β a big enough positive constant.

Proof. We multiply the first equation of (3.1) by u_ε and integrate over $\Omega \times (0,T)$ such that

$$\begin{aligned} &\sum_{i=1}^{N} \int_0^T \int_\Omega \frac{4d_1 m_i}{(m_i+1)^2} \left| \frac{\partial}{\partial x_i} (u_\varepsilon + \varepsilon)^{\frac{m_i+1}{2}} \right|^2 \, \mathrm{d}x\mathrm{d}t \leqslant \frac{1}{2} \|u_0\|^2_{L^2(\Omega)} \\ &\quad + \sum_{i=1}^{N} \frac{\chi}{\gamma^{q^- -2}} \int_0^T \int_\Omega (u_\varepsilon + \varepsilon)^{q_i-2} u_\varepsilon \frac{\partial v_\varepsilon}{\partial x_i} . \frac{\partial u_\varepsilon}{\partial x_i} \, \mathrm{d}x\mathrm{d}t \\ &\quad \leqslant \frac{1}{2} \|u_0\|^2_{L^2(\Omega)} + C(T) \|\Delta v_\varepsilon\|_{L^\lambda(0,T;L^\lambda(\Omega))}, \end{aligned}$$

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where we used the same method we introduced to get (3.8), applying proposition 3.17 and for $\lambda >> 1$. Therefore, by (2.9) we get (3.8). Moreover, we note that

$$\sum_{i=1}^{N}\int_0^T\int_\Omega\left|\frac{\partial}{\partial x_i}(u_\varepsilon+\varepsilon)^{m_i}\right|^2 \ \mathrm{d}x\mathrm{d}t$$
$$\leqslant C\sum_{i=1}^{N}\left(\|u_\varepsilon\|_{L^\infty(\Omega)}+\varepsilon\right)^{m_i-1}\int_0^T\int_\Omega\left|\frac{\partial}{\partial x_i}(u_\varepsilon+\varepsilon)^{\frac{m_i+1}{2}}\right|^2 \ \mathrm{d}x\mathrm{d}t.$$

Then, (3.39) follows from (3.38).

Next, taking $\varphi\in C^\infty(\Omega_T)$, multiplying the first equation of (3.1) by $\beta u_\varepsilon^{\beta-1}\varphi$, and integrating by parts, we obtain

$$\begin{aligned}\left|\int_\Omega\beta u_\varepsilon^{\beta-1}\varphi\partial_t u_\varepsilon \ \mathrm{d}x\right| &= \left|\int_\Omega\partial_t u_\varepsilon^\beta\varphi \ \mathrm{d}x\right|\\
&\leqslant \sum_{i=1}^{N}\Bigg\{\left|\int_\Omega d_1 m_i\beta(\beta-1)(u_\varepsilon+\varepsilon)^{m_i-1}u_\varepsilon^{\beta-2}\varphi\left|\frac{\partial u_\varepsilon}{\partial x_i}\right|^2 \ \mathrm{d}x\right|\\
&\quad+\left|\int_\Omega d_1 m_i\beta(u_\varepsilon+\varepsilon)^{m_i-1}u_\varepsilon^{\beta-1}\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial\varphi}{\partial x_i} \ \mathrm{d}x\right|\\
&\quad+\left|\int_\Omega\frac{\beta(\beta-1)\chi(u_\varepsilon+\varepsilon)^{q_i-2}u_\varepsilon^{\beta-1}}{(\gamma+v_\varepsilon)^{q_i-1}}\frac{\partial v_\varepsilon}{\partial x_i}\varphi\frac{\partial u_\varepsilon}{\partial x_i} \ \mathrm{d}x\right|\\
&\quad+\left|\int_\Omega\frac{\beta\chi(u_\varepsilon+\varepsilon)^{q_i-2}u_\varepsilon^{\beta}}{(\gamma+v_\varepsilon)^{q_i-1}}\frac{\partial v_\varepsilon}{\partial x_i}\frac{\partial\varphi}{\partial x_i} \ \mathrm{d}x\right|\Bigg\}\\
&\leqslant C\sum_{i=1}^{N}\Bigg\{(\|u_\varepsilon\|_{L^\infty(\Omega)}+\varepsilon)^{\beta-m_i-1}\int_\Omega|\varphi|\left|\frac{\partial}{\partial x_i}(u_\varepsilon+\varepsilon)^{m_i}\right|^2 \ \mathrm{d}x\\
&\quad+\|u_\varepsilon\|_{L^\infty(\Omega)}^{\beta-1}\int_\Omega\left|\frac{\partial\varphi}{\partial x_i}\right|\left|\frac{\partial}{\partial x_i}(u_\varepsilon+\varepsilon)^{m_i}\right| \ \mathrm{d}x\\
&\quad+(\|u_\varepsilon\|_{L^\infty(\Omega)}+\varepsilon)^{q_i+\beta-m_i-2}\left|\frac{\partial v_\varepsilon}{\partial x_i}\right|\int_\Omega\left|\frac{\partial}{\partial x_i}(u_\varepsilon+\varepsilon)^{m_i}\right||\varphi| \ \mathrm{d}x\\
&\quad+(\|u_\varepsilon\|_{L^\infty(\Omega)}+\varepsilon)\left\|\frac{\partial v_\varepsilon}{\partial x_i}\right\|_{L^\infty(\Omega)}\int_\Omega\left|\frac{\partial\varphi}{\partial x_i}\right| \ \mathrm{d}x\Bigg\}\\
&\leqslant C\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial}{\partial x_i}(u_\varepsilon+\varepsilon)^{m_i}\right|^2 \ \mathrm{d}x\left(\|\varphi\|_{L^\infty(\Omega)}+\left(\left\|\frac{\partial\varphi}{\partial x_i}\right\|_{L^\infty(\Omega)}\right)\right.\\
&\leqslant C\|\varphi\|_{(W^{1,N+1}(\Omega))'},\end{aligned}$$

where we used proposition 3.3, the embedding of $W^{1,N+1}(\Omega)$ into $L^\infty(\Omega)$, and (3.38). Thus, we get (3.40). Also, by the same method and using (3.5) and (3.17) we get (3.41). □

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4. Proof of theorem 2.5

The goal of this section is to prove theorem 2.5. In the proof, we need the strong convergence of u_ε and v_ε. Then, from (3.18), (3.19), proposition 3.3 and integrating over $(0,T)$, we get

$$\begin{aligned}\sum_{i=1}^{N}\frac{2d_1 m^-(r-1)}{(r+\alpha-1)^2}\left\|\frac{\partial}{\partial x_i}u_\varepsilon^{\frac{r+\alpha-1}{2}}\right\|^2_{L^2(\Omega_T)} &\leqslant \frac{1}{r}\|u_0\|^r_{L^\infty(\Omega)}\\ &+\sum_{i=1}^{N}\frac{\chi^2(r-1)}{d_1 m^-\gamma^{2(q^--2)}}\left\|\frac{\partial v_\varepsilon}{\partial x_i}\right\|^2_{L^\infty(\Omega_T)}\left\{\|u_\varepsilon\|^{r+2q_i-\alpha-3}_{L^\infty(\Omega_T)}+\|u_\varepsilon\|^{r-\alpha-1}_{L^\infty(\Omega_T)}\right\},\end{aligned} \tag{4.1}$$

for suitable r. Therefore, by taking $r=2\beta-\alpha+1$ in (4.1) and using (3.5) and proposition 3.3 we get that $u_\varepsilon^\beta\in L^2(0,T;H^1(\Omega))$ while $\partial_t u_\varepsilon^\beta$ is bounded in $L^1(0,T;(W^{1,N+1}(\Omega))')$ by lemma 3.4. Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ and $L^2(\Omega)$ is continuously embedded in $(W^{1,N+1}(\Omega))'$, it follows from corollary 4 in [**22**] that u_ε^β is compact in $L^2(0,T;L^2(\Omega))$. Since $u_\varepsilon\longmapsto u_\varepsilon^{\frac{1}{\beta}}$ is Hölder continuous with exponent $\frac{1}{\beta}$, we get that u_ε is compact in $L^{2\beta}(0,T;L^{2\beta}(\Omega))$. Thus, there exist a function $u\in L^{2\beta}(0,T;L^{2\beta}(\Omega))$ and a subsequence $(\varepsilon_n)_{n\geqslant 1}$ such that

$$u_{\varepsilon_n}\longrightarrow u \text{ Strongly in } L^{2\beta}(0,T;L^{2\beta}(\Omega)). \tag{4.2}$$

This gives

$$u_{\varepsilon_n}\longrightarrow u \text{ a.e. in } \Omega_T. \tag{4.3}$$

On the other hand, by proposition 3.3 we get that

$$\sup_{0<t<T}\|u_\varepsilon\|_{L^\infty(\Omega)}\leqslant M. \tag{4.4}$$

As a consequence, we get that

$$\int_{\Omega_T}|u_\varepsilon|^p\ \mathrm{d}x\mathrm{d}t\leqslant C(T)M^p,\ \text{ for any } 1<p<\infty. \tag{4.5}$$

Therefore, by using Lebesgue dominated convergence theorem, (4.3) and (4.5), we obtain

$$u_\varepsilon\longrightarrow u \text{ Strongly in } L^p(0,T;L^p(\Omega)) \text{ for any } 1<p<\infty. \tag{4.6}$$

By using the following inequality

$$|X^{m_i}-Y^{m_i}|\leqslant m_i^2\max\{|X|^{2(m_i-1)},|Y|^{2(m_i-1)}\}|X-Y|^2,\quad \forall i=1,..,N, \tag{4.7}$$

we get

$$\int_{\Omega_T}|u_{\varepsilon_n}^{m_i}-u^{m_i}|^2\ \mathrm{d}x\mathrm{d}t\leqslant C\int_{\Omega_T}|u_{\varepsilon_n}-u|^2\ \mathrm{d}x\mathrm{d}t\longrightarrow 0, \tag{4.8}$$

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where we used (4.6) for $p = 2$. Then, we get that

$$u_{\varepsilon_n}^{m_i} \longrightarrow u^{m_i} \text{ Strongly in } L^2(0,T;L^2(\Omega)). \tag{4.9}$$

Since $\frac{\partial u^{m_i}}{\partial x_i}$ is bounded in $L^2(0,T;L^2(\Omega))$ by (3.39), and using (4.9) we arrive at

$$\frac{\partial}{\partial x_i}(u_{\varepsilon_n}+\varepsilon_n)^{m_i} \rightharpoonup \frac{\partial u^{m_i}}{\partial x_i} \text{ Weakly in } L^2(0,T;L^2(\Omega)), \tag{4.10}$$

for any $i = 1, .., N$. Thereafter, by using (3.5), (3.41) and the same method we used to get (4.6), we obtain

$$v_{\varepsilon_n} \longrightarrow v \text{ Strongly in } L^p(0,T;L^p(\Omega)), \text{ for any } 1 < p < \infty, \tag{4.11}$$

and

$$\frac{\partial v_{\varepsilon_n}}{\partial x_i} \rightharpoonup \frac{\partial v}{\partial x_i} \text{ Weakly in } L^2(0,T;L^2(\Omega)). \tag{4.12}$$

Using (4.6), (4.11), (4.7) for $q_i - 1$ instead of m_i, and since $q_i \geqslant 2$ and $\gamma \geqslant 1$ we get that

$$\frac{(u_{\varepsilon_n}+\varepsilon_n)^{q_i-2}u_{\varepsilon_n}}{(\gamma+v_{\varepsilon_n})^{q_i-1}} \longrightarrow \left(\frac{u}{\gamma+v}\right)^{q_i-1} \text{ Strongly in } L^2(0,T;L^2(\Omega)). \tag{4.13}$$

Integrating (3.1) with respect to x and t, we see that $(u_{\varepsilon_n}, v_{\varepsilon_n})$ satisfies

$$\begin{aligned}
&\sum_{i=1}^{N}\int_0^T\int_\Omega\left\{d_1\frac{\partial}{\partial x_i}(u_{\varepsilon_n}+\varepsilon_n)^{m_i}.\frac{\partial\varphi}{\partial x_i} - \frac{(u_{\varepsilon_n}+\varepsilon_n)^{q_i-2}u_{\varepsilon_n}}{(\gamma+v_{\varepsilon_n})^{q_i-1}}\frac{\partial v_{\varepsilon_n}}{\partial x_i}.\frac{\partial\varphi}{\partial x_i} - u_{\varepsilon_n}\varphi_t\right\}\,\mathrm{d}x\mathrm{d}t\\
&\quad = \int_\Omega u_0(x)\varphi(x,0)\,\mathrm{d}x,\\
&\int_0^T\int_\Omega\left\{\nabla v_{\varepsilon_n}.\nabla\varphi + v_{\varepsilon_n}\varphi - u_{\varepsilon_n}\varphi - v_{\varepsilon_n}\varphi_t\right\}\,\mathrm{d}x\mathrm{d}t = \int_\Omega v_0(x)\varphi(x,0)\,\mathrm{d}x,
\end{aligned}$$

for any continuously differentiable function φ with compact support in $\Omega \times [0, T)$. Wherefore, by using (4.6), (4.9), (4.10), (4.11), (4.12), (4.13) and by the standard convergence argument we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\int_0^T\int_\Omega\left\{d_1\frac{\partial u^{m_i}}{\partial x_i}.\frac{\partial\varphi}{\partial x_i} - \left(\frac{u}{\gamma+v}\right)^{q_i-1}\frac{\partial v}{\partial x_i}.\frac{\partial\varphi}{\partial x_i} - u\varphi_t\right\}\,\mathrm{d}x\mathrm{d}t\\
&\quad = \int_\Omega u_0(x)\varphi(x,0)\,\mathrm{d}x,\\
&\int_0^T\int_\Omega\left\{\nabla v.\nabla\varphi + v\varphi - u\varphi - v\varphi_t\right\}\,\mathrm{d}x\mathrm{d}t = \int_\Omega v_0(x)\varphi(x,0)\,\mathrm{d}x,
\end{aligned}$$

where $q_i \geqslant 2$ and $m^- > q_i - \frac{2}{N}$ for any $i = 1, .., N$. Hence, we conclude the proof of theorem 2.5.

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