Ergod. Th. & Dynam. Sys. (1985), **5**, 291–299
Printed in Great Britain

On non-additive processes

ULRICH WACKER

Fakultät für Mathematik und Informatik der Universität Passau, Innstr. 27,
D-8390 *Passau, West Germany*

(*Received* 16 *April* 1984)

Abstract. The aim of this paper is to introduce and study the class of boundedly non-additive processes. The main result is the decomposition in theorem (2.1) and theorem (3.1), which says that a boundedly non-additive process is the sum of a non-positive subadditive, a non-negative superadditive and an additive process. By this decomposition we can extend the mean and local ergodic theorems for super-additive processes of M. A. Akcoglu and U. Krengel to boundedly non-additive processes. At the end of this paper some examples are given.

1. *Definitions*

For $a=(a_i)$, $b=(b_i)\in\mathbb{R}^l$, $l\geq 1$ we set:

$a\leq b$ $(a<b)$: $\Leftrightarrow a_i\leq b_i$, $(a_i<b_i)$, for $1\leq i\leq l$;

$[a, b[:=\{u\in\mathbb{R}^l: a\leq u<b\}$, $\prod([a, b[):=\prod_{i=1}^{l}(b_i-a_i)$;

$0=(0,0,\ldots,0)$, $e=(1,1,\ldots,1)$; $\mathbb{R}_+^l:=\{a\in\mathbb{R}^l: a\geq 0\}$;

$I_d:=\{[a, b[: a, b\in\mathbb{N}_0^l\}$, with $\mathbb{N}_0:=\{0,1,2,\ldots\}$; and finally

$I_s:=\{[a, b[: a, b\in\mathbb{R}_+^l\}$.

Let $\tau=(\tau_u)_{u\in I'}(I=\mathbb{N}_0^l$ or $\mathbb{R}_+^l)$ be a measurable semigroup of measure preserving transformations on a measure space (X, Q, P). For a Q-measurable function f we denote the equivalence class of all functions, which are a.s. equal to f, by $\bar{f}$. Let $\mathcal{M}$ be the vector lattice of these equivalence classes, and let $T=(T_u)_{u\in I}$ be the semigroup of linear operators acting on $\mathcal{M}$ by the relation $T_u\bar{f}=\bar{f}(\tau_u\cdot)$.

Let F be a set function defined on I_d (I_s) with values in $\mathcal{M}$. We distinguish the following conditions:

(1.1) $T_uF_J=F_{J+u}$ whenever $J\in I_d$ (I_s) and $u\in\mathbb{N}_0^l$ $(\mathbb{R}_+^l)$;

(1.2) if $J_1,\ldots,J_n$ are disjoint sets in I_d (I_s) and if $J=\bigcup_{i=1}^n J_i$ is also in I_d (I_s) then $F_J\geq\sum_{i=1}^n F_{J_i}$;

(1.3) $\sup\{\prod(J)^{-1}\int F_J\,dP: J\in I_d\ (I_s), \prod(J)>0\}=\gamma=\gamma(F)<+\infty$,

(1.4) $\int F^{-}_{[0,e[}\,dP<+\infty$, where f^+, f^- denote the positive and negative part of an $f\in\mathcal{M}$.

Definition (1.5). If F satisfies (1.1), we will say that F is a *stationary process* with discrete (or continuous) parameter. If F also satisfies (1.2), (1.3), and takes values in L_1, F is called a *superadditive* process. If $-F$ is a superadditive process, F is called a *subadditive* process. F is called an *extended superadditive* process if F satisfies (1.1), (1.2) and (1.4). If $-F$ is an extended superadditive process, F is called an *extended subadditive* process. F is called an *additive* process, if it is a superadditive and a subadditive process.

For the next definition we need some further notation. We denote a set by the letter M, iff it is contained in I_d (I_s) and its elements are disjoint sets. For two sets M_1, M_2we write $M_1 \lhd M_2$, if every element in M_1 is the disjoint union of elements of M_2. For $I \in I_d$ we set

$$M_I := \{[u, u+e[: u \in I \cap \mathbb{N}_0^l\}.$$

For $I \in I_d$ (I_s) and $\{I\} \lhd M_1 \lhd M_2$, we define

$$d''(M_1, M_2) := \sum_{I \in M_1} \left| F_I - \sum_{\substack{J \in M_2 \\ J \subset I}} F_J \right|$$

$$d^*(M_1, M_2) := \sum_{I \in M_1} \left(F_I - \sum_{\substack{J \in M_2 \\ J \subset I}} F_J \right)^* \qquad \text{for } * = +, -.$$

For $A = (M_i)_{1 \le i \le m}$, $m \in \mathbb{N}$, with $\{I\} = M_1 \lhd M_2 \lhd \cdots \lhd M_m$, where M_m consists only of finitely many elements, we define:

$$F_I^*(A) := \sum_{i=1}^{m-1} d^*(M_i, M_{i+1}), \qquad * = '', +, -.$$

Let $\bar{A}(I)$ be the set of all such A. In the discrete parameter case we define

$$F_I^* := \max\{F_I^*(A) : A \in \bar{A}(I)\}, \qquad * = '', +, -,$$

and

$$p(F) := \sup\left\{ \prod(I)^{-1} \int F_I'' \, dP : \prod(I) > 0, I \in I_d \right\}.$$

In the continuous parameter case we suppose that for every countable $K \subset \bar{A}(I)$, $\sup\{F_I^*(A) : A \in K\}$, $* = '', +, -$, is in L_1 and their integrals are uniformly bounded. Then $F_I^* := \sup\{F_I^*(A) : A \in \bar{A}(I)\}$, $* = '', +, -$, exists and is in L_1. If this is satisfied for every $I \in I_s$ we define:

$$p(F) := \sup\left\{ \prod(I)^{-1} \int F_I'' \, dP : \prod(I) > 0, I \in I_s \right\}.$$

Definition (1.6). A stationary process F is called *locally boundedly non-additive* if F_I'' is integrable for some I with $\prod(I) > 0$. It is called *boundedly non-additive* if $p(F)$ is finite.

$p(\cdot)$ is a seminorm on the vector space of all stationary, boundedly non-additive processes, with a common semigroup T. If we build the canonical quotient space relative to p, we obtain a Banach space, where two processes are identified iff their

https://doi.org/10.1017/S0143385700002911 Published online by Cambridge University Press

difference is additive. This can be seen easily. The only non-trivial part is the completeness of the quotient space. Let $\{F'_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in the quotient space, and F_n a representative of F'_n. If $\{n_i\}$ increases fast enough we have

$$p(F_{n_i}-F_{n_{i-1}})<2^{-i}, \qquad i\geq 2.$$

Then the processes F^*, $*=+,-$ defined by

$$F^*_I=\sum_{i=2}^{\infty}(F_{n_i,I}-F_{n_{i-1},I})^*+F^*_{n_1,I}, \qquad *=+,-; \qquad I\in I_d\ (I_s),$$

exist and they are non-negative superadditive processes. The equivalence class of F, which is defined by

$$F_I=F^+_I-F^-_I, \qquad I\in I_d\ (I_s),$$

is the limit point of the sequence. Let F be a superadditive process, then we obtain the following relationship between the constants $\gamma(F)$ and $p(F)$.
In the discrete parameter case:

$$p(F)=\gamma(F)-\int F_{[0,e[}\,dP.$$

And in the continuous parameter case, if in addition $\sup\{\prod(I)^{-1}\int|F_I|\,dP: I\in I_s, \prod(I)>0\}$ is finite:

$$p(F)=\gamma(F)-s_F,$$

where s_F is defined by

$$s_F:=\lim_{t\to 0+}\prod([0,t\cdot e[)^{-1}\int F_{[0,te[}\,dP.$$

For the second equality see the proof of lemma (4.7) in [**1**].

2. *The discrete parameter case*

THEOREM (2.1). *Let F be a stationary process with discrete parameter. Then*

$$F=F^+-F^-+G, \tag{2.1.1}$$

where G is defined by

$$G_I:=\sum_{u\in I\cap\mathbb{N}'_0}F_{[u,u+e[}. \tag{2.1.2}$$

Furthermore, F^+ and F^- are non-negative extended superadditive processes. If $p(F)$ is finite, then F^+ and F^- are superadditive processes.

Proof. For $A\in\bar{A}(I)$ with $A=(M_i)_{1\leq i\leq m}$ and $M_m\neq M_I$ we obtain

$$F^*_I(A)\leq F^*_I(A'), \qquad *='',+,-,$$

with $A'=(M_1,\ldots,M_m,M_I)$. Let $A^*(I)$ be the set of all $A\in\bar{A}(I)$ of the form $A=(M_1,\ldots,M_m,M_I)$. We obtain:

$$F^*_I=\sup\{F^*_I(A): A\in A^*(I)\}, \qquad *='',+,-.$$

https://doi.org/10.1017/S0143385700002911 Published online by Cambridge University Press

So $\bar{A}(I)$ can be replaced by $A^*(I)$. First we show (2.1.1) and

$$F_I^+ + F_I^- = F_I'' \qquad \text{for every } I \in I_d. \tag{2.1.3}$$

For every $A \in A^*(I)$ we obtain

$$F_I^+(A) - F_I^-(A) = F_I - \sum_{J \in M_I} F_J.$$

Let $A_i \in A^*(I)$, $1 \le i \le N$. The last equality implies

$$\begin{aligned} F_I^+(A_1) \vee \cdots \vee F_I^+(A_N) &= \left(\left(F_I - \sum_{J \in M_I} F_J\right) + F_I^-(A_1)\right) \\ &\quad \vee \cdots \vee \left(\left(F_I - \sum_{J \in M_I} F_J\right) + F_I^-(A_N)\right) \\ &= \left(F_I - \sum_{J \in M_I} F_J\right) + F_I^-(A_1) \vee \cdots \vee F_I^-(A_N). \end{aligned}$$

Hence

$$F_I^+(A_1) \vee \cdots \vee F_I^+(A_N) - F_I^-(A_1) \vee \cdots \vee F_I^-(A_N) = F_I - \sum_{J \in M_I} F_J. \tag{2.1.4}$$

In the case $\{A_1, \ldots, A_N\} = A^*(I)$ we obtain (2.1.1). We further obtain

$$\begin{aligned} F_I^+ + F_I^- &= 2F_I^- + \left(F_I - \sum_{J \in M_I} F_J\right) \\ &= \left(2F_I^-(A_1) + \left(F_I - \sum_{J \in M_I} F_J\right)\right) \vee \cdots \vee \left(2F_I^-(A_N) + \left(F_I - \sum_{J \in M_I} F_J\right)\right) \\ &= F_I''(A_1) \vee \cdots \vee F_I''(A_N) = F_I''. \end{aligned}$$

By definition F^+, F^-, and G satisfy (1.1), and we have $F^* \ge 0$ for $* = ''$, $+$, $-$. Next we shall prove (1.2) for F^+. For F^- the proof is the same.

Let I_j, $1 \le j \le n$, in I_d be disjont sets, such that $I = \bigcup_{j=1}^n I_j$ is also in I_d. Take

$$A_j^p := (M_{1,j}^p, \ldots, M_{l_p,j}^p) \in A^*(I_j), \qquad 1 \le p \le N_j,$$

such that $F_{I_j}^+ = F_{I_j}^+(A_j^1) \vee \cdots \vee F_{I_j}^+(A_j^{N_j})$. Put $l = \max l_{j,p}$. For $\bar{p} = (p_1, \ldots, p_n)$, $1 \le p_j \le N_j$, let $M_1^{\bar{p}} = \{I\}$, and for $1 \le k \le l$ let $M_{k+1}^{\bar{p}}$ be the collection of all sets in $\bar{M}_{k,j}^{p_j}$, $1 \le j \le n$, where we set $\bar{M}_{k,j}^{p_j} = M_{k,j}^{p_j}$, if $k \le l_{j,p_j}$ and $\bar{M}_{k,j}^{p_j} = M_{I_j}$ otherwise. Set $A^{\bar{p}} = (M_1^{\bar{p}}, \ldots, M_{l+1}^{\bar{p}})$, then we obtain

$$F_I^+ \ge \max_{e \le \bar{p} \le \bar{N}} F_I^+(A^{\bar{p}}) \ge \sum_{j=1}^n F_{I_j}^+,$$

with $\bar{N} = (N_1, \ldots, N_n)$, and (1.2) is proved.

(2.1.3) implies $\gamma(F^+) + \gamma(F^-) = p(F)$, and this proves the last statement in theorem (2.1). □

The decomposition (2.1.1) is minimal in the sense that for every decomposition $F = H^1 - H^2 + H$, with non-negative superadditive processes H_1, H_2 and additive H, $F_I^+ \le H_I^1$ and $F_I^- \le H_I^2$, for $I \in I_d$, is satisfied. For equivalence classes mod P, $\lim_{n\to\infty} f_n = f$ a.s. means that $\lim_{n\to\infty} \tilde{f}_n = \tilde{f}$ a.s. is satisfied if $\tilde{f}_n$, $\tilde{f}$ are representatives of f_n, f. Theorem (2.1) and theorem (2.5) in [1] yield:

THEOREM (2.2). *Let F be a stationary process with discrete parameter, which satisfies $F_{[0,e[} \in L_1$ and $p(F) < +\infty$. Set $J_n := [0, n \cdot e[$, $n \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} \prod (J_n)^{-1} F_{J_n}$$

exists a.s.

The theorem remains valid if $(J_n)_{n\in\mathbb{N}}$ is a regular family of sets with $\lim_{n\to\infty} J_n = \mathbb{P}_0^I$, as defined in [1]. If (X, Q, P) is a finite measure space, then we can replace the condition $p(F) < +\infty$ by the existence of the time constant of one of the processes F^+ and F^- in (2.1.1). E.g. we can suppose $\gamma(F^-) < +\infty$. Then, by theorem (2.5) in [1], $\lim_{n\to\infty} \prod (J_n)^{-1} F^-_{J_n}$ and $\lim_{n\to\infty} \prod (J_n)^{-1} G_{J_n}$ exist a.s. and are in L_1. By a truncation argument like that in [4, p.188] the existence of $\lim_{n\to\infty} \prod (J_n)^{-1} F^+_{J_n}$ follows from theorem (2.5) in [1]. Together these prove the last statement. Let F be a stationary process with discrete parameter on a probability space (X, Q, P) with $F_{[n,k[} \in L_1$ for $[n, k[\in I_d$. Y. Derriennic [2] proved that $(1/N)F_{[0,N[}$ converges a.s. and in L_1, if the following two conditions are satisfied:

(a) there is a sequence $(h_k)_{k\in\mathbb{N}} \subset L_1$, $h_k \geq 0$, with $\sup_{k\geq 1} \|h_k\|_1 < +\infty$ and

$$F_{[0,n+k[} - F_{[0,n[} - F_{[n,n+k[} \leq T^n h_k \qquad (T = T_1),$$

for every n and k,

(b) $\inf_{N\geq 1} (1/N) \int F_{[0,N[}\, dP > -\infty.$

From this result the Shannon-McMillan-Breiman theorem follows at once (see [2]).

If we replace $\sup_{k\geq 1} \|h_k\|_1 < +\infty$ by $\sup_{k\geq 1} h_k \in L_1$ in (a), then F is boundedly non-additive. The question arises as to whether all processes which satisfy (a) and (b) are boundedly non-additive.

3. *The continuous parameter case*

THEOREM (3.1). *Let F be a stationary process with continuous parameter which satifies $p(F) < +\infty$. Then*

$$F = F^+ - F^- + G, \tag{3.1.1}$$

where F^+ and F^- are non-negative superadditive processes and G is additive.

Proof. Fix $I \in I_s$. As F is boundedly non-additive, there is a countable set $K = \{A_1, \ldots\} \subset \bar{A}(I)$ with

$$F_I^* = \sup\{F_I^*(A): A \in K\}, \qquad * = '', +, -.$$

Put $f_N^* := F_I^*(A_1) \vee \cdots \vee F_I^*(A_N)$. We obtain $f_N^* \uparrow F_I^*$ and $f''_N = f^+_N + f^-_N$. Hence

$$F''_I = F^+_I + F^-_I, \tag{3.1.2}$$

Put $g_N := F_I - f^+_N + f^-_N$, and let G be defined by $G_I := \lim_{N\to\infty} g_N$. Now we will show, that F^*, G and $-G$ satisfy (1.2). Together with (3.1.2) this implies that the F^* are superadditive processes. Let $I_1, \ldots, I_m$ be disjoint sets in I_s such that $\bigcup_{i=1}^m I_i = I$. Take $K_i = \{A_1^i, A_2^i, \ldots\} \subset \bar{A}(I_i)$ with

$$F_I^* = \sup\{F_I^*(A): A \in K_i\}, \qquad 1 \leq i \leq m.$$

Put $f_N^{*,i} = F_{I_i}^*(A_1^i) \vee \cdots \vee F_{I_i}^*(A_N^i)$. By the same argument as in the proof of theorem (2.1), we obtain $B_1^N, \ldots B_M^N \in \bar{A}(I)$ with

$$F_I^*(B_1^N) \vee \cdots \vee F_I^*(B_M^N) \geq \sum_{i=1}^m f_N^{*,i}.$$

So for every $N \in \mathbb{N}$, $F_I^* \geq \sum_{i=1}^m f_N^{*,i}$ is satisfied, and (1.2) is proved for F^*, $* = '', +, -$. It remains to prove

$$G_I = \sum_{i=1}^m G_{I_i}. \tag{3.1.3}$$

Let $A_j = (M_1^j, \ldots, M_{K_j}^j)$ and $A_j^i = (M_{1,i}^j, \ldots, M_{K_{j,i},i}^j)$, $1 \leq i \leq m$, $j \in \mathbb{N}$. For $1 \leq i \leq m$ let M_i^N be the collection of all intersections of sets in $M_{K_{j,i},i}^j$ and in $M_{K_j}^j$, $1 \leq j \leq N$. Let M_N be the collection of all these intersections. We set

$$Z_N^i = \sum_{J \in M_i^N} F_J \quad \text{and} \quad Z_N = \sum_{J \in M_N} F_J.$$

Because $\sum_{i=1}^m Z_N^i = Z_N$ is satisfied, (3.1.3) follows from

$$Z_N^i \to G_{I_i} \quad \text{and} \quad Z_N \to G_I. \tag{3.1.4}$$

By $M_i^N \lhd M_{K_{j,i},i}^j$ and $M_N \lhd M_{K_j}^j$ for $1 \leq j \leq N$, we obtain

$$A_{j,N}^i := (M_{1,i}^j, \ldots, M_{K_{j,i},i}^j, M_i^N) \in \bar{A}(I_i)$$

and

$$A_{j,N} := (M_1^j, \ldots, M_{K_j}^j, M_N) \in \bar{A}(I).$$

Put $h_N^{*,i} = F_{I_i}^*(A_{1,N}^i) \vee \cdots \vee F_{I_i}^*(A_{N,N}^i)$ and $h_N^* = F_I^*(A_{1,N}) \vee \cdots \vee F_I^*(A_{N,N})$, $* = +, -$. We obtain $F_{I_i}^* \geq h_N^{*,i} \geq f_N^{*,i}$ and $F_I^* \geq h_N^* \geq f_N^*$. This implies $Z_N^i = F_{I_i} - h_N^{+,i} + h_N^{-,i} \to G_{I_i}$ and $Z_N := F_I - h_N^+ + h_N^- \to G_I$, and (3.1.4) is proved. □

We will call a process F *bounded*, if

$$b(F) := \sup\left\{\prod (I)^{-1} \int |F_I| \, dP : I \in I_s, \prod (I) > 0\right\}$$

is finite.

THEOREM (3.2). *Let F be a bounded and locally boundedly non-additive process. Set $J_t = [0, t \cdot e[$, $t \in \mathbb{Q}$. Then*

$$\lim_{t \to 0+} \prod (J_t)^{-1} F_{J_t}$$

exists a.s.

Sketch of the proof. We can assume $\int F''_{[0,r\cdot e[} \, dP < +\infty$, for an $r > 0$. Let $J = [0, a[\subset [0, r \cdot e[$ with $a = (a_i) > 0$. Let b_i be the largest integer $n \leq r/a_i$ and let $c = (b_i \cdot a_i)$. We obtain

$$\left(\prod_{i=1}^l b_i\right) \int F''_J \, dP \leq \int F''_{[0,c[} \, dP \leq \int F''_{[0,r\cdot e[} \, dP.$$

This and $[0, r \cdot e[\subset [0, 2c[$ imply

$$\prod (J)^{-1} \int F''_J \, dP \leq \prod ([0, c[)^{-1} \int F''_{[0,r\cdot e[} \, dP \leq 2^l \prod ([0, r \cdot e[)^{-1} \int F''_{[0,r\cdot e[} \, dP.$$

So

$$\sup\left\{\prod (J)^{-1}\int F''_J\,dP : J\in I_s, \prod (J)>0, J\subset[0, r\cdot e[\right\}$$
$$\leq 2^l \prod ([0, r\cdot e[)^{-1}\int F''_{[0,r\cdot e[}\,dP<+\infty$$

is proved, and we obtain the decomposition (3.1.1) for all $I\subset[0, r\cdot e[$. By additivity and stationarity we extend G to all $I\in I_s$. We obtain a bounded additive process, for which the existence of

$$\lim_{\substack{t\to 0+\\ t\in\mathbb{Q}}} \prod (J_t)^{-1} G_{J_t} \quad \text{a.s.}$$

is proved in [**1**]. It remains to prove $\lim_{t\to 0+, t\in\mathbb{Q}} \prod (J_t)^{-1}F^*_{J_t}=0$ a.s. for $*=+,-$. This can be done in almost the same manner as in the proofs of theorem (4.2), lemma (4.7) and lemma (4.8) in [**1**]. □

THEOREM (3.3). *Let F be a boundedly non-additive process with continuous parameter for which*

$$\sup\{|F_{[a,b[}| : [a, b[\subset[0, e[\text{ and the coordinates of } a \text{ and } b \text{ are rational}\}$$

is integrable. Set $J_t=[0, t\cdot e[$, $t\in\mathbb{Q}$. *Then*

$$\lim_{\substack{t\to\infty\\ t\in\mathbb{Q}}} \prod (J_t)^{-1} F_{J_t}$$

exists a.s.

The theorem can be proved by a reduction to a discrete case following the proof of theorem (2.5) in [**1**]. Both theorems remain valid if $\{J_t\}_{t\in\mathbb{Q}}$ is a regular family of sets. One can define stationary processes indexed by more general sets than intervals, and it seems that the results carry over to that setting.

5. *Examples*

In this section we give some examples of non-additive processes, which appear in percolation on a lattice. These processes are given by a family of r.v.'s $F=(F_I, I\in I_d)$, where $(F_{I_1},\ldots, F_{I_n})$ has the same distribution as $(F_{I_1+u},\ldots F_{I_n+u})$, for all $I_1,\ldots, I_n\in I_d$, $u\in\mathbb{N}_0^d$. As in the 1-parameter case (see [**3**]) we can pass to an equivalent process $\tilde{F}$ with sample space $\mathbb{R}^{I_d}$, which is a stationary process, defined as in definition (1.5).

Let the graph L be given by a lattice of dimension $d\geq 2$, where the set E of sites is $\mathbb{Z}^d$. Two points in $\mathbb{Z}^d$ are neighbours if their Euclidean distance is 1. The bonds connect any two neighbours. The set of bonds will be denoted by K. All bonds are unoriented. Further let $\{U_l, l\in K\}$ be a set of non-negative i.i.d. r.v.'s with a finite mean see [**5**].

Example (4.1). We suppose $d=2$. For a fixed $0\leq n\leq+\infty$ let $E_n:=\mathbb{Z}\times\{k\in\mathbb{Z}: |k|<n+1\}$. Let the graph L_n be given by the set of sites E_n and the set of all bonds of L whose end points are in E_n. For $l<n+1$ we define

$$M^l_m:=\{(m, p): |p|\leq l\}.$$

For a path w consisting of a connected string $w_1,\ldots, w_m$ of bonds, put $U_w=\sum_{i=1}^m U_{w_i}$. Let $W^l_{m_1,m_2}$ be the set of all paths connecting $M^l_{m_1}$ to $M^l_{m_2}$. Put

$$X^l_{m_1,m_2}(x):=\inf\{U_w(x): w\in W^l_{m_1,m_2}\}, \qquad m_1<m_2.$$

For $l=0$, $X=(X^l_{m_1,m_2})$ is a subadditive process. For $l=n$, X is a superadditive process. For $0<l<n$, $X^{l''}_{0,m}$ is bounded by

$$2\sum_{j=0}^{m}\sum_{i=1-l}^{l} U_{((j,i-1),(j,i))}+\sum_{i=1}^{m} X^l_{i-1,i}-X^l_{0,m},$$

$m\in\mathbb{N}$, so in this case X is a boundedly nonadditive process.

Example (4.2). We suppose $d\geq 2$ and $\|U_l\|_\infty=: C<+\infty$, $(l\in K)$. For $I\in I_d$ let L_I be the subgraph of L, whose set of sites is $E_I=\mathbb{Z}^d\cap I$, and whose set of bonds K_I contains exactly all bonds of L whose end points are in E_I. Let SL_I be the set of all connected subgraphs of L_I, having the set of sites E_I, and which contain no circuit. We define pointwise:

$$F_I(x):=\inf\left\{\sum_{l\in K'} U_l(x): K' \text{ is the set of bonds of a graph } L'\in SL_I\right\},\qquad I\in I_d.$$

We will now show that F is a boundedly non-additive process. As F is non-negative, this statement follows by (2.1.1) and (2.1.3) from: $F_I^+(x)\leq C\prod(I)$, uniformly in x. To prove this fix $I\in I_d$ and $x\in X$. Let $A=(M_i)_{1\leq i\leq m}\in\bar{A}(I)$. Suppose

$$\sum_{\substack{J'\subset J\\ J'\in M_{j+1}}} F_{J'}(x)\leq F_J(x),$$

for a j with $1\leq j\leq m-1$ and a $J\in M_j$. Let, for $I'\in I_d$ and $x\in X$, $L(I',x)\in SL_{I'}$ be a graph with set of sites $E(I',x)$ and set of bonds $K(I',x)$, which satisfies

$$F_{I'}(x)=\sum_{l\in K(I';x)} U_l(x).$$

We complete the graphs $L(J',x)$, $J'\in M_{j+1}$, $J'\subset J$, to an element of SL_J. The set of bonds of this graph may be denoted by K'. By the definition of F, we obtain

$$F_J(x)\leq\sum_{l\in K'} U_l(x).$$

So we obtain:

$$\Big(F_J(x)-\sum_{\substack{J'\subset J\\ J\in M_{j+1}}} F_{J'}(x)\Big)^+\leq C\cdot \operatorname{card}\Big(K'\setminus \bigcup_{\substack{J'\subset J\\ J'\in M_{j+1}}} K(J',x)\Big).$$

The number of new bonds we use for all completions like those described above is smaller than $\prod(I)$ and they all are different; so $F_I^+(x)\leq C\prod(I)$ is proved.

Acknowledgements. This paper contains a part of my dissertation written under the direction of Prof. U. Krengel; I want to thank Prof. U. Krengel for his guidance and encouragement.

REFERENCES

[1] M. A. Akcoglu & U. Krengel. Ergodic theorems for superadditive processes. *J. Reine Ang. Math.* **323** (1981), 53–67.

[2] Y. Derriennic. Un theoreme ergodique presque sous-additif. *Ann. Prob.* **11** (1983), 669–677.

[3] A. del Junco. On the decomposition of a subadditive stochastic process. *Ann. Prob.* **5** (1981), 298–302.

[**4**] J. F. C. Kingman. Subadditive processes. In *Lecture Notes in Mathematics* **539**, 167–223. Springer-Verlag: Berlin–Heidelberg–New York, 1976.
[**5**] R. T. Smythe & J. C. Wierman. *First-Passage Percolation on the Square Lattice.* Lecture Notes in Mathematics **671**, Springer-Verlag: Berlin–Heidelberg–New York, 1978.