BULL. AUSTRAL. MATH. SOC. 16W30, 81R10
VOL. 47 (1993) [353-375]

# QUANTUM DOUBLE CONSTRUCTION FOR GRADED HOPF ALGEBRAS

M.D. GOULD, R.B. ZHANG AND A.J. BRACKEN

A detailed proof of the quantum double construction is given for $\mathbb{Z}_2$-graded Hopf algebras, and an explicit formula for the graded universal $R$-matrix is obtained in a general fashion.

## 1. INTRODUCTION

The quantum double construction [5, 9, 11] states that, given a Hopf algebra $A$ with a bijective antipode (and satisfying certain other mild conditions), there exists a unique quasitriangular Hopf algebra $(D(A), R)$ such that (1) $D(A)$ contains $A$ and its dual $A^o$ as Hopf subalgebras; (2) the universal $R$ matrix is the image of the canonical element of $A \otimes A^o$ under the embedding $A \otimes A^o \to D(A)$ and (3) the linear mapping $A \otimes A^o \to D(A)$ given by $a \otimes b \mapsto ab$ is bijective. The importance of the construction is two-fold. Firstly, it allows one to construct new Hopf algebras which are non-commutative and non-co-commutative from *non-commutative or non-co-commutative* Hopf algebras; and secondly it provides a constructive way of finding solutions of the quantum Yang-Baxter equation which arises in a variety of areas in physics. The universal $R$-matrix for $U_q(sl(m))$ [6], is constructed in [12], and more recently, the universal $R$-matrices for all the quantum groups [6] have been obtained in explicit form by Kirillov and Reshetikhin [7].

Certain category-theoretical arguments indicate that the quantum double construction may be formally generalised to $\mathbb{Z}_2$-graded Hopf algebras, and recent research on quantum supergroups [2, 3, 4, 8, 13, 15], especially the works on $U_q(osp(1/2))$ and $U_q(osp(2/2))$ [4, 8, 13], have provided us with concrete examples. However, a detailed discussion of the double construction in the graded case is still lacking, and in particular an explicit formula for the graded universal $R$-matrix is not known in a general fashion. The aim of the present paper is to give a systematic and detailed description of the double construction of $\mathbb{Z}_2$-graded Hopf algebras, and to construct the corresponding graded universal $R$-matrix explicitly. We hope that our proof will be readily accessible by the mathematical physics community.

Received 29th April, 1992.





https://doi.org/10.1017/S0004972700015197 Published online by Cambridge University Press

The structure of the paper is as follows. In Section 2 we give the basic definitions which will be needed in the remainder of the paper and present some examples of graded Hopf algebras; in Section 3 we discuss the dual of a graded Hopf algebra and prove certain technical results which are of crucial importance for the development of the quantum double construction. In Section 4 we generalise the definition of quasi-triangularity to graded Hopf algebras following [8] and then prove the quantum double construction in detail. Finally in Section 5 we briefly summarise the main results of the paper and also indicate how the quantum double construction applies for quantum supergroups.

## 2. Graded Hopf Algebras

To make the paper self-contained we give some basic definitions in this section. We choose a fixed field $K$. A $\mathbb{Z}_2$-graded vector space $V$ over $K$ is the direct sum of two vector spaces,

$$V = V_0 \oplus V_1.$$

An element $v \in V$ is called *homogeneous* if $v \in V_0 \cup V_1 \subset V$, otherwise it is *inhomogeneous*. To each element $v \in V_i \subset V$, $i = 0$ or 1, we assign the gradation index $[v] = i$ and call $v$ even if $[v] = 0$ and odd if $[v] = 1$. We refer to $V_0$ and $V_1$ as the even and odd subspaces of $V$, respectively. The dual $V^*$ of a $\mathbb{Z}_2$-graded vector space $V$ admits a natural $\mathbb{Z}_2$-grading $V^* = V_0^* \oplus V_1^*$, where $V_i^* \simeq \operatorname{Hom}(V_i, K)$, $i = 0, 1$ with $\operatorname{Hom}(V_i, K)$ representing the morphisms from $V_i$ to $K$. For any $x^* \in V^*$, $v \in V$, we shall denote $x^*(v) \in K$ by $\langle x^*, v\rangle$. Given two $\mathbb{Z}_2$-graded vector spaces $V$ and $W$ over $K$, a linear mapping $f : V \to W$ is said to be *homogeneous of degree* $r \in \mathbb{Z}_2$, if

$$f(V_i) \subseteq W_{i+r \pmod 2}, \qquad i = 0, 1.$$

The mapping $f$ is called a homomorphism of the $\mathbb{Z}_2$-graded vector space $W$ to the $\mathbb{Z}_2$-graded vector space $V$ if $f$ is homogeneous of degree 0. An isomorphism of $\mathbb{Z}_2$-graded vector spaces is a homomorphism which is one-to-one and onto. The tensor product $V \otimes W$ is again a $\mathbb{Z}_2$-graded vector space with the naturally induced grading given by

$$V \otimes W = (V \otimes W)_0 \oplus (V \otimes W)_1,\ (V \otimes W)_i = \bigoplus_{k+l \equiv i \pmod 2} (V_k \otimes W_l).$$

In the remainder of the paper, the twisting map $T : V \otimes W \to W \otimes V$ will frequently appear. It is defined for homogeneous elements $v \in V$, $w \in W$ by

$$T(v \otimes w) = (-)^{[v][w]} w \otimes v$$

and extends to all elements of $V$ and $W$ through linearity.

A $\mathbb{Z}_2$-graded algebra $A$ over $K$ is a $\mathbb{Z}_2$-graded vector space equipped with graded vector space homomorphisms $M : A \otimes A \to A$ and $u : K \to A$ such that the following diagrams

$$
\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "I\otimes M"] \arrow[d, "M\otimes I"'] & A\otimes A \arrow[d, "M"] \\
A\otimes A \arrow[r, "M"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A\otimes A \arrow[dd, "M"] & \\
K\otimes A \arrow[ur, "u\otimes I"] & & A\otimes K \arrow[ul, "I\otimes u"'] \\
 & A \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

are commutative, where $A \to K \otimes A$ and $A \to A \otimes K$ are the natural maps; $M$ and $u$ are called the *multiplication* and *unit* of $A$, respectively. For convenience, we shall write $M(a \otimes b)$ as $ab$, $\forall\, a, b \in A$, whenever this does not cause confusion.

A $\mathbb{Z}_2$-graded algebra $A$ is called *commutative* if the following diagram commutes

$$
\begin{tikzcd}
 & A\otimes A \arrow[dl, "M"'] \arrow[dd, "T"] \\
A & \\
 & A\otimes A \arrow[ul, "M"]
\end{tikzcd}
$$

where $T : A \otimes A \to A \otimes A$ is the twisting map. Let $A$ and $B$ be $\mathbb{Z}_2$-graded algebras with multiplications $M_A$, $M_B$, and units $u_A$, $u_B$, respectively. Then $A \otimes B$ is also a $\mathbb{Z}_2$-graded algebra with unit $u = u_A \otimes u_B$ and multiplication $M$ defined by the composition

$$
(1) \qquad A \otimes B \otimes A \otimes B \xrightarrow{I_A \otimes T \otimes I_B} A \otimes A \otimes B \otimes B \xrightarrow{M_A \otimes M_B} A \otimes B,
$$

where $I_A : A \to A$ and $I_B : B \to B$ are the identity morphisms of $A$ and $B$, respectively, and $T : A \otimes B \to B \otimes A$ is the twisting morphism.

A homomorphism of $\mathbb{Z}_2$-graded algebras $f : A \to B$ is a homomorphism of $\mathbb{Z}_2$-graded vector spaces such that the diagrams

$$
\begin{tikzcd}
A\otimes A \arrow[r, "M_A"] \arrow[d, "f\otimes f"'] & A \arrow[d, "f"] \\
B\otimes B \arrow[r, "M_B"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & B \arrow[dd, "f"] \\
K \arrow[ur, "u_A"] \arrow[dr, "u_B"'] & \\
 & A
\end{tikzcd}
$$

https://doi.org/10.1017/S0004972700015197 Published online by Cambridge University Press

are commutative. A $\mathbb{Z}_2$-graded *co-algebra* $C$ is a $\mathbb{Z}_2$-graded vector space with the $\mathbb{Z}_2$-graded vector space homomorphisms $\Delta : C \to C \otimes C$ and $\varepsilon : C \to K$ such that the following diagrams are commutative

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "I \otimes \Delta"] \\ C \otimes C \arrow[r, "\Delta \otimes I"] & C \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} & C \otimes C \arrow[dl, "I \otimes \varepsilon"'] \arrow[dr, "\varepsilon \otimes I"] & \\ C \otimes K \arrow[dr] & & K \otimes C \arrow[dl] \\ & C \arrow[uu, "\Delta"'] & \end{tikzcd}$$

where $I : C \to C$ is the identity homomorphism, $\Delta$ and $\varepsilon$ are, respectively, called the *co-multiplication* and *co-unit* of $C$. Note that the commutativity of the first diagram is equivalent to $(\Delta \otimes I) \cdot \Delta = (I \otimes \Delta) \cdot \Delta$, called the *co-associativity* of $\Delta$.

We shall adopt Sweedler's Sigma notation [14] throughout the paper and write

$$\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}, \quad (\Delta \otimes I) \cdot \Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)} \otimes c_{(3)}, \quad c \in C.$$

In this notation, the co-unit property may be expressed as

$$\sum_{(c)} \varepsilon\big(c_{(1)}\big) c_{(2)} = \sum_{(c)} c_{(1)} \varepsilon\big(c_{(2)}\big) = c.$$

A $\mathbb{Z}_2$-graded co-algebra $C$ is called *co-commutative* if the following diagram is commutative

$$\begin{tikzcd} & C \otimes C \arrow[dd, "T"] \\ C \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & \\ & C \otimes C \end{tikzcd}$$

where $T : C \otimes C \to C \otimes C$ is the twisting map. Let $B$, $C$ be $\mathbb{Z}_2$-graded co-algebras with co-multiplications $\Delta_B$, $\Delta_C$ and co-units $\varepsilon_B$, $\varepsilon_C$, respectively, then $B \otimes C$ is also a $\mathbb{Z}_2$-graded co-algebra with co-multiplication given by the composition

$$B \otimes C \overset{\Delta_B \otimes \Delta_C}{\longrightarrow} B \otimes B \otimes C \otimes C \overset{I_B \otimes T \otimes I_C}{\longrightarrow} B \otimes C \otimes B \otimes C$$

and co-unit given by

$$B \otimes C \overset{\varepsilon_B \otimes \varepsilon_C}{\longrightarrow} K \otimes K \to K.$$

A homomorphism of $\mathbb{Z}_2$-graded co-algebras $f : A \to B$ is a homomorphism of $\mathbb{Z}_2$-graded vector spaces such that the diagrams

$$\begin{tikzcd} A \arrow[r, "\Delta_A"] \arrow[d, "f"'] & A\otimes A \arrow[d, "f\otimes f"] \\ B \arrow[r, "\Delta_B"] & B\otimes B \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "\varepsilon_A"'] \arrow[dd, "f"] \\ K & \\ & B \arrow[ul, "\varepsilon_B"] \end{tikzcd}$$

are commutative.

Let $H$ be a $\mathbb{Z}_2$-graded algebra with multiplication $M$ and unit $u$, and at the same time a $\mathbb{Z}_2$-graded co-algebra with co-multiplication $\Delta$ and co-unit $\varepsilon$. Then $H$ is called a $\mathbb{Z}_2$-graded *bi-algebra* provided that one of the following equivalent conditions are satisfied:

(i) $\Delta$ and $\varepsilon$ are $\mathbb{Z}_2$-graded algebra homomorphisms;
(ii) $M$ and $u$ are $\mathbb{Z}_2$-graded co-algebra homomorphisms.

Further, if $H$ admits a $\mathbb{Z}_2$-graded vector space homomorphism $S : H \to H$ which satisfies the following defining relation

$$M\cdot(I\otimes S)\cdot\Delta = M\cdot(S\otimes I)\cdot\Delta = u\cdot\varepsilon \tag{2}$$

then $H$ is called a $\mathbb{Z}_2$-graded *Hopf algebra*, and $S$ is called the *antipode* of $H$.

The antipode $S$ has the following properties [**1**, **10**, **14**]

LEMMA 1. *Let $H$ be a $\mathbb{Z}_2$-graded Hopf algebra and $S$ its antipode; then*

(1) $S\cdot M = M\cdot T\cdot(S\otimes S)$,
(2) $S\cdot u = u$,
(3) $\varepsilon\cdot S = \varepsilon$,
(4) $T\cdot(S\otimes S)\cdot\Delta = \Delta\cdot S$,
(5) *if $H$ is commutative or co-commutative, then* $S\cdot S = I$,

*where $I : H \to H$ is the identity morphism and $T : H\otimes H \to H\otimes H$ is the twisting morphism.*

A fact which is important for later applications is that if the antipode $S$ of a $\mathbb{Z}_2$-graded Hopf algebra $H$ is bijective, then $H$ is also a $\mathbb{Z}_2$-graded Hopf algebra with the opposite co-multiplication $\Delta' = T\cdot\Delta$ and the opposite antipode $S' = S^{-1}$. We note that all finite dimensional $\mathbb{Z}_2$-graded Hopf algebras possess bijective antipodes.

EXAMPLES: Finally we give some concrete examples of $\mathbb{Z}_2$-graded Hopf algebras. Classical examples of co-commutative graded Hopf algebras are the universal enveloping algebras of graded Lie algebras. Consider for example the universal enveloping algebra $U(g)$ of a simple basic classical Lie superalgebra $g$, which is a $\mathbb{Z}_2$-graded Lie algebra over the complex number field $\mathbb{C}$. Denote by $\Delta$ the familiar diagonal homomorphism $\Delta: U(g) \to U(g) \otimes U(g)$ such that

$$\begin{aligned} &\Delta(a) = a \otimes 1 + 1 \otimes a, \quad &&\forall\, a \in g \\ &\Delta(uv) = \Delta(u)\Delta(v), &&\forall\, u, v \in U(g) \end{aligned} \tag{3}$$

and let the morphism $\varepsilon: U(g) \to K$ be defined by

$$\begin{aligned} &\varepsilon(1) = 1 \\ &\varepsilon(a) = 0, &&\forall\, a \in g \\ &\varepsilon(uv) = \varepsilon(u)\varepsilon(v), &&\forall\, u, v \in U(g), \end{aligned} \tag{4}$$

then $U(g)$ constitutes a $\mathbb{Z}_2$-graded bi-algebra with co-multiplication $\Delta$ and co-unit $\varepsilon$.

Introduce the gradation index $[v]$ for homogeneous elements $v \in U(g)$ such that

$$[v] = \begin{cases} 0 & v \text{ even}, \\ 1 & v \text{ odd}, \end{cases}$$

and define $S: U(g) \to U(g)$ by

$$S(a) = -a, \qquad \forall\, a \in g; \qquad S(uv) = (-)^{[u][v]} S(v) S(u) \tag{5}$$

for homogeneous elements of $U(g)$ which extends to all $U(g)$ by linearity. Then $S$ gives rise to an antipode on $U(g)$, and thus turns it into a $\mathbb{Z}_2$-graded Hopf algebra. Note that $\Delta' = T \cdot \Delta = \Delta$, and $S^2 = I$.

The $\mathbb{Z}_2$-graded Hopf algebra $U(g)$ admits one-parameter deformations leading to non-commutative and non-co-commutative $\mathbb{Z}_2$-graded Hopf algebras, namely, the quantum supergroups [2, 3, 8, 13] $U_q(g)$ which are defined as follows.

Let $\alpha_i$, $i = 1, 2, \dots, r$, $r =$ rank of $g$, be the simple roots of $g$. Denote by $H = \{h_{\alpha_i} \mid i = 1, 2, \dots, r\}$ the vector space spanned by the Cartan generators $h_{\alpha_i}$ of $g$, and by $(\ ,\ )$ the invariant bilinear form on $H^* = \oplus_{i=1}^r \mathbb{C}\alpha_i$. Define the matrix $A = (a_{ij})$ by

$$a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i), \text{ if } (\alpha_i, \alpha_i) \neq 0, \quad \forall\, j$$

and

$$a_{ij} = (\alpha_i, \alpha_j), \text{ if } (\alpha_i, \alpha_i) = 0, \quad \forall\, j.$$

For a nonzero parameter $q \in \mathbb{C}$, we let

$$q_i = \begin{cases} q^{(\alpha_i, \alpha_i)/2} & (\alpha_i, \alpha_i) \neq 0 \\ q & \text{otherwise}, \end{cases}$$

and consider the algebra generated by $\{K_i^{\pm 1} = q^{\pm h_{\alpha_i}}, \widehat{e}_{\alpha_i}, \widehat{f}_{\alpha_i} \mid i = 1, 2, \dots, r\}$ subject to the following constraints

$$[K_i, K_j] = 0, \quad K_i \widehat{e}_{\alpha_j} K_i^{-1} = q_i^{a_{ij}} e_j, \quad K_i \widehat{f}_{\alpha_j} K_i^{-1} = q_i^{-a_{ij}} f_{\alpha_j}$$

$$[\widehat{e}_{\alpha_i}, \widehat{f}_{\alpha_j}\} = \delta_{ij}(K_i - K_i^{-1})/(q_i - q_i^{-1}),$$

$$\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \left[\begin{matrix} 1-a_{ij} \\ \nu \end{matrix}\right\}_{q_i} \widehat{e}_{\alpha_i}^{1-a_{ij}-\nu} \widehat{e}_{\alpha_j} \widehat{e}_{\alpha_i}^{\nu} = 0, \quad i \neq j, \ (\alpha_i, \alpha_i) \neq 0$$

$$\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \left[\begin{matrix} 1-a_{ij} \\ \nu \end{matrix}\right\}_{q_i} \widehat{f}_{\alpha_i}^{1-a_{ij}-\nu} \widehat{f}_{\alpha_j} \widehat{f}_{\alpha_i}^{\nu} = 0, \quad i \neq j, \ (\alpha_i, \alpha_i) \neq 0$$

$$\widehat{e}_{\alpha_i}^2 = \widehat{f}_{\alpha_i}^2 = 0, \quad \text{if} \quad (\alpha_i, \alpha_i) = 0,$$

where the notations are as follows. Define the gradation index

$$[h_i] = 0, \qquad \forall i$$

$$[e_{\alpha_i}] = [f_{\alpha_i}] = \begin{matrix} 0, & \alpha_i \text{ even}, \\ 1, & \alpha_i \text{ odd}, \end{matrix}$$

and for any $u, v$ which are monomials in the simple and Cartan generators

$$[uv] \equiv [u] + [v] \pmod 2.$$

Then

$$\left[\widehat{e}_{\alpha_i}, \widehat{f}_{\alpha_j}\right\} = \widehat{e}_{\alpha_i}\widehat{f}_{\alpha_j} - (-)^{[\widehat{e}_{\alpha_i}][\widehat{f}_{\alpha_j}]}\widehat{f}_{\alpha_j}\widehat{e}_{\alpha_i}, \quad \text{et cetera}$$

$$\left[\begin{matrix} 1-a_{ij} \\ \nu \end{matrix}\right\}_{q_i} = \begin{cases} \left[\begin{matrix} 1-a_{ij} \\ \nu \end{matrix}\right]_{q_i}^{(-)}, & \text{if } \alpha_i \text{ is even}, \\ (-1)^{1/2}\nu\left(\nu - (-1)^{[\widehat{e}_j]}\right)\left[\begin{matrix} 1-a_{ij} \\ \nu \end{matrix}\right]_{q_i}^{(+)}, & \text{if } \alpha_i \text{ is odd}, \end{cases}$$

with

$$\left[\begin{matrix} m \\ n \end{matrix}\right]_q^{(\pm)} = \begin{cases} \frac{(q^m \pm q^{-m})(q^{m-1} \pm q^{-m+1})\cdots(q^{m-n+1} \pm q^{-m+n-1})}{(q \pm q^{-1})(q^2 \pm q^{-2})\cdots(q^n \pm q^{-n})}, & m > n > 0 \\ 1, & n = 0, m. \end{cases}$$

It is a matter of straightforward manipulations to check case by case for all simple basic classical Lie superalgebras that $\Delta : U_q(g) \to U_q(g) \otimes U_q(g)$, with

$$\Delta(\widehat{e}_{\alpha_i}) = \widehat{e}_{\alpha_i} \otimes q^{-h_{\alpha_i}/2} + q^{h_{\alpha_i}/2} \otimes \widehat{e}_{\alpha_i}$$

$$\Delta\left(\widehat{f}_{\alpha_i}\right) = \widehat{f}_{\alpha_i} \otimes q^{-h_{\alpha_i}/2} + q^{h_{\alpha_i}/2} \otimes \widehat{f}_{\alpha_i}$$

$$\Delta(q^{h_{\alpha_i}}) = q^{h_{\alpha_i}} \otimes q^{h_{\alpha_1}} \tag{6}$$

https://doi.org/10.1017/S0004972700015197 Published online by Cambridge University Press

defines an algebra homomorphism, which we call the co-multiplication on $U_q(g)$. The algebra homomorphism $\varepsilon : U_q(g) \to \mathbb{C}$ defined by

$$\varepsilon(\widehat{e}_{\alpha i}) = \varepsilon\left(\widehat{f}_{\alpha_i}\right) = 0, \quad \varepsilon\left(q^{h_{\alpha_i}}\right) = \varepsilon(1) = 1 \tag{7}$$

determines a co-unit for $U_q(g)$; and an antipode $S : U_q(g) \to U_q(g)$ is defined by

$$\begin{aligned} S\left(q^{h_{\alpha_i}/2}\right) &= q^{-h_{\alpha_i}/2}, \quad S\left(q^{-h_{\alpha_i}/2}\right) = q^{h_\alpha/2}, \\ S\left(\widehat{e}_{\alpha_i}\right) &= -q^{-(\alpha_i,\alpha_i)/2}\widehat{e}_{\alpha_i} \\ S\left(\widehat{f}_{\alpha_i}\right) &= -q^{(\alpha_i,\alpha_i)/2}\widehat{f}_{\alpha_i}, \end{aligned} \tag{8.a}$$

which we extend to an algebra anti-homomorphism for all of $U_q(g)$, so that

$$S(uv) = (-1)^{[u][v]}S(v)S(u), \tag{8.b}$$

thus turning $U_q(g)$ into a $\mathbb{Z}_2$-graded Hopf algebra.

In the limit as $q \to 1$, $U_q(g)$ reduces to $U(g)$, the universal enveloping algebra of the simple basic classical Lie superalgebra $g$, and equations (6), (7) and (8) reduce to (3), (4) and (5), respectively.

## 3. The dual of a graded Hopf algebra

In this section we study the dual of a graded Hopf algebra. The results proved here, although technical, will be of crucial importance for the development of the quantum double construction.

Let $A$ be a $\mathbb{Z}_2$-graded algebra over the field $K$, $M$ and $u$ be its multiplication and unit, respectively. Denote by $A^*$ the dual of $A$ regarded as a $\mathbb{Z}_2$-graded vector space, and define

$$A^o = \{a^* \in A^* \mid \text{Ker}(a^*) \text{ contains a co-finite } \mathbb{Z}_2\text{-graded ideal}\}, \tag{9}$$

where a $\mathbb{Z}_2$-graded ideal $I \subset A$ is assumed to be two-sided, that is,

$$aI \subseteq I, \quad Ia \subseteq I, \qquad \forall a \in A$$

and it is co-finite if $\dim(A/I) < +\infty$. If $A^o \neq (0)$, then it has the structure of a $\mathbb{Z}_2$-graded co-algebra with co-multiplication $M^*|_{A^o}$ and co-unit $u^*|_{A^o}$, where $M^* : A^* \to (A \otimes A)^*$ and $u^* : A^* \to K$ are the $\mathbb{Z}_2$-graded vector space homomorphisms induced from $M : A \otimes A \to A$ and $u : K \to A$, respectively [1, 14].

If for any $0 \neq a \in A$ there exists $x^* \in A^o$ such that $\langle x^*, a\rangle \neq 0$, we say that $A^o$ is *dense* in $A^*$. This occurs if and only if for any non-zero $a \in A$ there is a co-finite $\mathbb{Z}_2$-graded ideal $I$ of $A$ which excludes $a$ [1, 14]; and in this case, we call $A$ a *proper algebra*.

For a $\mathbb{Z}_2$-graded Hopf algebra $A$ we denote by $A_a$ its underlying $\mathbb{Z}_2$-graded algebra. Then $A^o = (A_a)^o$ is a $\mathbb{Z}_2$-graded co-algebra because of the above discussion. Since the dual of a $\mathbb{Z}_2$-graded co-algebra is an algebra with the same gradation, we conclude that

**Lemma 2.** *If $A$ is a $\mathbb{Z}_2$-graded Hopf algebra with multiplication $M$, unit $u$, co-multiplication $\Delta$, co-unit $\varepsilon$ and antipode $S$, then $A^o$ is also a $\mathbb{Z}_2$-graded Hopf algebra with multiplication $M^o$, unit $u^o$, co-multiplication $\Delta^o$, co-unit $\varepsilon^o$ and antipode $S^o$ such that*

$$\begin{aligned} &M^o = \Delta^*|_{A^o}, \quad u^o = \varepsilon^*|_{A^o} \\ (10) \qquad &\Delta^o = M^*|_{A^o}, \quad \varepsilon_o = u^*|_{A^o}, \quad S^o = S^*|_{A^o}. \end{aligned}$$

We shall call the graded Hopf algebra $A$ proper if its underlying graded algebra $A_a$ is proper, that is, $A^o$ is dense in $A^*$.

When $S$ is bijective, $(S^o)^{-1} : A^o \to A^o$ exists and this guarantees that $A^o$ is also a graded Hopf algebra with the opposite co-multiplication $\Delta_o = T \cdot \Delta^o$ and opposite antipode $S_o = (S^{-1})^*|_{A^o} = (S^o)^{-1}$, or more explicitly, $\Delta_o$ and $S_o$ are defined by the following equations

$$(11) \qquad \langle \Delta_o(a^*),\ T(b \otimes c)\rangle = \langle a^*, bc\rangle,$$

$$(12) \qquad \langle S_o(a^*), b\rangle = \langle a^*,\ S^{-1}(b)\rangle, \qquad \forall\, a^* \in A^o,\ b, c \in A,$$

while the multiplication, unit and co-unit remain the same as in Lemma 2. For uniformity of notation we let $M_o = M^o$, $u_o = u^o$ and $\varepsilon_o = \varepsilon^o$.

Now let us consider the tensor products $A^o \otimes A$ and $A \otimes A^o$. They inherit graded Hopf algebra structures from those of $A$ and $A^o$. For example, the natural co-multiplication for $A^o \otimes A$, which we denote by $\widehat{\Delta} : A^o \otimes A \to A^o \otimes A \otimes A^o \otimes A$, is given by

$$(13.a) \qquad \widehat{\Delta} = (I \otimes T \otimes I) \cdot (\Delta_o \otimes \Delta),$$

and the natural co-multiplication for $A \otimes A^o$ is $\widehat{\Delta}' : A \otimes A^o \to A \otimes A^o \otimes A \otimes A^o$ given by

$$(13.b) \qquad \widehat{\Delta}' = (I \otimes T \otimes I) \cdot (\Delta \otimes \Delta_o).$$

(To avoid cumbersome notation we have in (13) used the same symbol $I$ to denote the identity homomorphisms of both $A$ and $A^o$, and $T$ for both the twisting maps $A\otimes A^o \to A^o\otimes A$ and $A^o\otimes A \to A\otimes A^o$. We shall keep this convention in the remainder of the paper.) However, in this section we are mainly interested in $A^o\otimes A$ and $A\otimes A^o$ as graded vector spaces, and we have the following

**PROPOSITION 1.** *Let $A$ be a $\mathbb{Z}_2$-graded Hopf algebra with a bijective antipode $S$. The homomorphism of $\mathbb{Z}_2$-graded vector spaces $\mu: A^o\otimes A \to A\otimes A^o$ given by the composition*

$$A^o\otimes A \xrightarrow{(Str\,\otimes I^{\otimes 2})\cdot(S_o\otimes I^{\otimes 3})\cdot\hat{\Delta}} A^o\otimes A \xrightarrow{T} A\otimes A^o \xrightarrow{(I^{\otimes 2}\otimes\, Str\,)\hat{\Delta}'} A\otimes A^o \tag{14}$$

*defines an isomorphism between the $\mathbb{Z}_2$-graded vector spaces $A^o\otimes A$ and $A\otimes A^o$ with the inverse morphism $\mu^{-1}: A\otimes A^o\to A^o\otimes A$ given by the composition*

$$A\otimes A^o \xrightarrow{(Str\,\otimes I^{\otimes 2})\cdot\hat{\Delta}'} A\otimes A^o \xrightarrow{T} A^o\otimes A \xrightarrow{(I^{\otimes 2}\otimes\, Str)\cdot(I^{\otimes 3}\otimes S^{-1})\cdot\hat{\Delta}} A^o\otimes A, \tag{15}$$

*where $Str$ is defined by*

$$Str\,(a^*\otimes b) = .\ Str\,(T(a^*\otimes b)) = \langle a^*, b\rangle, \quad \forall\, a^*\in A^o,\ b\in A.$$

PROOF: We only need to prove the proposition for homogeneous elements of $A$ and $A^o$, since it can be trivially generalised to the inhomogeneous elements through linearity. Denote by $\nu$ the composite morphism (15). For homogeneous elements we have

$$\begin{aligned}\mu(a^*\otimes b) = \sum_{(a*),(b)} &\langle S_o\left(a^*_{(1)}\right), b_{(1)}\rangle\langle a^*_{(3)}, b_{(3)}\rangle b_{(2)}\otimes a^*_{(2)}\\ &\times(-)^{[a^*][b]+[b_{(1)}][b]+[a^*][b_{(3)}]+[b_{(1)}][b_{(3)}]}, \quad a^*\in A^o,\ b\in A,\end{aligned} \tag{16}$$

$$\begin{aligned}\nu(b\otimes a^*) = \sum_{(a*),(b)} &\langle a^*_{(1)}, b_{(1)}\rangle\langle a^*_{(3)}, S^{-1}\left(b_{(3)}\right)\rangle a^*_{(2)}\otimes b_{(2)}\\ &\times(-)^{[b_{(1)}]+[b_{(3)}]+\left([a^*_{(2)}]+[b_{(1)}]\right)\left([b_{(2)}]+[b_{(3)}]\right)}, \quad a^*\in A^o,\ b\in A.\end{aligned} \tag{17}$$

Therefore

$$\begin{aligned}\nu\cdot\mu(a^*\otimes b) &= \sum_{(a*),(b)} \langle S_o\left(a^*_{(1)}\right), b_{(1)}\rangle\langle a^*_{(2)}, b_{(2)}\rangle\langle a^*_{(4)}, S^{-1}\left(b_{(4)}\right)\rangle\langle a^*_{(5)}, b_{(5)}\rangle a^*_{(3)}\otimes b_{(3)}\\ &\qquad\times(-)^{\xi+[b_{(1)}][b_{(2)}]+\left([b_{(1)}]+[b_{(2)}]\right)\left([b]+[b_{(5)}]\right)}\\ &= \sum_{(a*),(b)} \langle S^o\left(a^*_{(1)}\right)\otimes a^*_{(2)}, b_{(1)}\otimes b_{(2)}\rangle\langle a^*_{(4)}, S^{-1}\left(b_{(4)}\right)\rangle\langle a^*_{(5)}, b_{(5)}\rangle a^*_{(3)}\otimes b_{(3)}\\ &\qquad\times(-)^{\xi+\left([b_{(1)}]+[b_{(2)}]\right)\left([b]+[b_{(5)}]\right)},\end{aligned} \tag{18}$$

where

$$\xi \equiv [a^*][b] + [a^*][b_{(5)}] + [b_{(4)}] + [a^*_{(3)}]\left([b_{(3)}]+[b_{(4)}]\right) \pmod 2.$$

The first bracket on the far right-hand side of (18) can be eliminated by using the defining relation (2) of an antipode, leading to

$$\nu \cdot \mu(a^* \otimes b) = \sum_{(a*),(b)} \langle a^*_{(2)} \otimes a^*_{(3)}, S^{-1}\left(b_{(2)}\right) \otimes b_{(3)} \rangle a^*_{(1)} \otimes b_{(1)}$$
$$\times (-)^{[a^*][b]+[a^*_{(1)}][b_{(1)}]+[b_{(2)}][b_{(3)}]+[a^*][b_{(3)}]+[b_{(2)}][a^*_{(1)}]}.$$

The bracket on the right-hand side of the above equation can again be eliminated in the same way and we arrive at

$$\nu \cdot \mu(a^* \otimes b) = \sum_{(a*),(b)} \varepsilon_o\left(a^*_{(2)}\right)\varepsilon\left(b_{(2)}\right) a^*_{(1)} \otimes b_{(1)} (-)^{[a^*][b]+[a^*_{(1)}][b]+[a^*_{(2)}]}$$

(19) $$= a^* \otimes b, \qquad a^* \in A^o,\ b \in A.$$

In a similar way we can prove that

$$\mu \cdot \nu(b \otimes a^*) = b \otimes a^*, \qquad a^* \in A^o,\ b \in A,$$

thus completing the proof of the proposition. ∎

The isomorphism $\mu : A^o \otimes A \to A \otimes A^o$ actually defines a graded co-algebra isomorphism between $A^o \otimes A$ and $A \otimes A^o$. In particular,

**Lemma 3.**

(20) $$\widehat{\Delta}' \cdot \mu = (\mu \otimes \mu) \cdot \widehat{\Delta}.$$

Proof: Let $a^* \in A^o$, $b \in A$ be homogeneous elements. Then

$$(\mu \otimes \mu) \cdot \widehat{\Delta}(a^* \otimes b) = \sum_{(a*),(b)} \mu\left(a^*_{(1)} \otimes b_{(1)}\right) \otimes \mu\left(a^*_{(2)} \otimes b_{(1)}\right) (-)^{[b_{(1)}][a^*_{(2)}]}$$
$$= \sum_{(a*),(b)} \langle S_o\left(a^*_{(1)}\right), b_{(1)} \rangle \langle a^*_{(3)}, b_{(3)} \rangle \langle S_o\left(a^*_{(4)}\right), b_{(4)} \rangle \langle a^*_{(6)}, b_{(6)} \rangle$$
$$\times\, b_{(2)} \otimes a^*_{(2)} \otimes b_{(5)} \otimes a^*_{(5)}$$
$$\times (-)^{[b_{(3)}][b_{(4)}]+\left([b_{(3)}]+[b_{(4)}]\right)\left([b_{(1)}]+[b_{(2)}]+[a^*_{(5)}]+[b_{(6)}]\right)+\nu}$$

with $\nu \equiv \left([b_{(1)}] + [b_{(2)}] + [b_{(5)}]\right)\left([a^*_{(5)}] + [b_{(6)}]\right) + [a^*_{(2)}]\left([b_{(1)}] + [b_{(2)}]\right) \pmod 2$.

The second and third brackets on the right-hand side of the above equation can be combined and eliminated together leading to

$$(\mu \otimes \mu) \cdot \widehat{\Delta}(a^* \otimes b) = \sum_{(a*),(b)} \langle S_o\left(a^*_{(1)}\right), b_{(1)} \rangle \langle a^*_{(4)}, b_{(4)} \rangle b_{(2)} \otimes a^*_{(2)} \otimes b_{(3)} \otimes a^*_{(3)}$$
$$\times (-)^{\left([b_{(1)}]+[b_{(2)}]+[b_{(3)}]\right)\left([a^*_{(3)}]+[b_{(4)}]\right)+[a^*_{(2)}]\left([b_{(1)}]+[b_{(2)}]\right)}$$
$$= \widehat{\Delta} \cdot \mu(a^* \otimes b).$$

https://doi.org/10.1017/S0004972700015197 Published online by Cambridge University Press

The above calculations can be generalised straightforwardly to inhomogeneous elements of $A$ and $A^o$ through linearity, thus proving the lemma. ∎

Another technical result which will be useful later is

**LEMMA 4.**

$$\mu \cdot (S_o \otimes S) = (S \otimes S_o) \cdot T \cdot \mu^{-1} \cdot T. \tag{21}$$

PROOF: Let

$$\mu_1 = (\mathrm{Str} \otimes I^{\otimes 2}) \cdot (S_o \otimes I^{\otimes 3}) \cdot \widehat{\Delta},$$
$$\mu_2 = (I^{\otimes 2} \otimes \mathrm{Str}) \cdot \widehat{\Delta}'.$$

Then for homogeneous elements $a^* \in A^o$, $b \in A$, we have

$$\begin{aligned} T \cdot \mu_1 \cdot (S_o \otimes S)(a^* \otimes b) &= T \cdot (\mathrm{Str} \otimes I^{\otimes 2}) \cdot (S_o \otimes I^{\otimes 3}) \cdot (I \otimes T \otimes I) \\ &\quad \times (S_o \otimes S_o \otimes S \otimes S) \cdot (T \cdot \Delta_o \otimes T \cdot \Delta)(a^* \otimes b) \\ &= (S \otimes S_o) \sum_{(a*),(b)} \langle a^*_{(2)}, S^{-1}\left(b_{(2)}\right) \rangle b_{(1)} \otimes a^*_{(1)} (-)^{[a^*][b_{(1)}]} \end{aligned}$$

where the fourth relation of Lemma 1 has been used. Using this relation again we can show that

$$\begin{aligned} \mu \cdot (S_o \otimes S)(a^* \otimes b) &= \mu_2 \cdot T \cdot \mu_1 (S_o \otimes S)(a^* \otimes b) \\ &= (S \otimes S_o) \sum_{(a*),(b)} \langle a^*_{(1)}, b_{(1)} \rangle \langle a^*_{(3)}, S^{-1}\left(b_{(3)}\right) \rangle b_{(2)} \otimes a^*_{(2)} (-)^{\left([a^*]+[b_{(1)}]\right)\left([b_{(1)}]+[b_{(2)}]\right)}. \end{aligned}$$

Comparing the far right-hand side of the above equation with equation (17) we immediately see that

$$\mu(S_o \otimes S)(a^* \otimes b) = (S \otimes S_o) \cdot T \cdot \mu^{-1} \cdot T(a^* \otimes b), \qquad a^* \in A^o,\ b \in A,$$

and this holds for inhomogeneous elements of $A^o$ and $A$ as well, due to linearity of the homomorphisms. ∎

We want to emphasise that Proposition 1 and Lemmas 3 and 4 hold only for graded Hopf algebras with bijective antipodes. In the next section we shall show that every proper graded Hopf algebra with a bijective antipode can be embedded in a quasitriangular graded Hopf algebra.

## 4. The quantum double construction

In this section we generalise Drinfeld's quantum double construction for ordinary Hopf algebras to the graded case. But before doing so we first introduce the notion of a *quasitriangular* graded Hopf algebra.

Following [8], we call a $\mathbb{Z}_2$-graded Hopf algebra $A$ quasitriangular if there exists an invertible element $R \in (A \otimes A)_o \subset A \otimes A$ such that

$$R\Delta(a) = \Delta'(a)R, \quad \forall\, a \in A \tag{22}$$

and

$$(\Delta \otimes I)R = R_{13}R_{23}, \quad (I \otimes \Delta)R = R_{13}R_{12}, \tag{23}$$

where $\Delta' = T \cdot \Delta$ and we have adopted the standard notation that

$$\begin{aligned} R_{12} &= \sum_i a_i \otimes b_i \otimes 1_A, \\ R_{13} &= \sum_i a_i \otimes 1_A \otimes b_i, \\ R_{23} &= \sum_i 1_A \otimes a_i \otimes b_i, \end{aligned} \tag{24}$$

where

$$R = \sum_i a_i \otimes b_i, \quad a_i, b_i \in A.$$

If $R$ further satisfies the property that $RT(R) = 1_A \otimes 1_A$, then $A$ is said to be *triangular*.

We shall refer to $R$ as the graded universal $R$-matrix. It has been proved in [4, 8] that

THEOREM 1. *The universal $R$-matrix satisfies the graded Yang-Baxter equation*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{25}$$

PROOF: For completeness we repeat the proof here. It follows from the first equation of (23) and the fact that $R \in (A \otimes A)_o$, the even component of the graded tensor product $A \otimes A$, that

$$(T \cdot \Delta \otimes I)R = R_{23}R_{13}.$$

Using (22) we have

$$R_{12}(\Delta \otimes I)R = [(T \cdot \Delta \otimes I)R]R_{12};$$

which immediately leads to the graded Yang-Baxter equation (25).

https://doi.org/10.1017/S0004972700015197 Published online by Cambridge University Press

We want to emphasise that throughout the paper, the product rule for elements belonging to the tensor product of two $\mathbb{Z}_2$-graded algebras is defined by the composition (1) and generalises to higher rank tensor products iteratively. Therefore, the graded Yang-Baxter equation (25) is essentially different from the ordinary Yang-Baxter equation although they look the same ostensibly.

Now we turn to the quantum double construction for graded Hopf algebras. We shall prove the construction in four steps. Let us first of all define a formal product

$$ba^*, \tag{26}$$

such that for elements $a^* \in A^o$, $b \in A$

$$(b_1 + b_2)a^* = b_1 a^* + b_2 a^*, \quad b(a_1^* + a_2^*) = ba_1^* + ba_2^*$$
$$\forall\, b, b_1, b_2 \in A, \quad a^*, a_1^*, a_2^* \in A^o.$$

Denote the linear span of the elements of the form (26) by $D(A)$. Then $D(A)$ is a $\mathbb{Z}_2$-graded vector space with its gradation naturally induced from $A$ and $A^o$ such that for homogeneous elements $b \in A$, $a^* \in A^o$, $[ba^*] \equiv [b] + [a^*] \pmod 2$.

Consider the $\mathbb{Z}_2$-graded vector space homomorphism $\psi : A \otimes A^o \to D(A)$ defined by

$$\psi(b \otimes a^*) = ba^*, \quad b \in A,\ a^* \in A^o. \tag{27}$$

It is quite obvious that $\psi$ defines a $\mathbb{Z}_2$-graded vector space isomorphism between $A \otimes A^o$ and $D(A)$. Now we introduce on $D(A)$ the multiplication $\overline{M} : D(A) \otimes D(A) \to D(A)$ defined by the composition

(28)

$$D(A) \otimes D(A) \xrightarrow{\psi^{-1} \otimes \psi^{-1}} A \otimes A^o \otimes A \otimes A^o \xrightarrow{I \otimes \mu \otimes I} A \otimes A \otimes A^o \otimes A^o \xrightarrow{M_A \otimes M_{A^o}} A \otimes A^o \xrightarrow{\psi} D(A)$$

where $\mu : A^o \otimes A \to A \otimes A^o$ is defined by (14) and $M_A$ and $M_{A^o}$ are respectively the multiplications of $A$ and $A^o$. More explicitly, given homogeneous elements $ba^*$, $dc^* \in D(A)$, where $b$, $d \in A$ and $a^*$, $c^* \in A^o$ are also homogeneous, we have

$$\overline{M}(ba^* \otimes dc^*) = \sum_{(a*),(b)} \langle S_o\left(a^*_{(1)}\right), d_{(1)} \rangle \langle a^*_{(3)}, d_{(3)} \rangle \left(bd_{(2)}\right) \left(a^*_{(2)} c\right)$$
$$\times (-)^{[a^*][d] + [d_{(1)}][d] + [a^*][d_{(3)}] + [d_{(1)}][d_{(3)}]}, \tag{29}$$

where $bd_{(3)}$ and $a^*_{(3)} c^*$ are the ordinary products in $A$ and $A^o$, respectively. It can be easily proved using (29) that $\overline{M}$ is associative, that is, the following diagram is commutative

$$\begin{tikzcd}
D(A)\otimes D(A)\otimes D(A) \arrow[r, "I\otimes\overline{M}"] \arrow[d, "\overline{M}\otimes I"'] & D(A)\otimes D(A) \arrow[d, "\overline{M}"] \\
D(A)\otimes D(A) \arrow[r, "\overline{M}"] & D(A).
\end{tikzcd}$$

Let $u: K \to A$ and $\varepsilon: A \to K$ be the unit and co-unit of the $\mathbb{Z}_2$-graded Hopf algebra $A$ respectively. Then $\varepsilon \in A^o$ is the identity element of $A^o$. Define the homomorphism $\overline{u}: K \to D(A)$ by

$$\overline{u}(k) = u(k)\varepsilon, \quad k \in K. \tag{30}$$

It immediately follows from the relations

$$\mu(a^* \otimes 1_A) = 1_A \otimes a^*,\ \mu(\varepsilon \otimes b) = b \otimes \varepsilon, \quad \forall a^* \in A^o,\ b \in A,$$

where $\mu$ is defined by (14) and the definition of $\overline{M}$ that the following diagram

$$\begin{tikzcd}
& D(A)\otimes D(A) \arrow[dd, "\overline{M}"] & \\
K\otimes D(A) \arrow[ur, "\overline{u}\otimes I"] & & D(A)\otimes K \arrow[ul, "I\otimes\overline{u}"'] \\
& D(A) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

is commutative. Therefore, we have

**Proposition 2.** *$D(A)$ constitutes a $\mathbb{Z}_2$-graded algebra with multiplication $\overline{M}$ and unit $\overline{u}$ defined by (28) and (30), respectively.*

For computational purposes, it is very useful to extend the definition of the formal product (26) by defining

$$a^*b = \psi \cdot \mu(a^* \otimes b), \quad \forall a^* \in A^o,\ b \in A \tag{31}$$

so that we also have $a^*b \in D(A)$, $\forall a^* \in A^o$, $b \in A$. Now the multiplication on $D(A)$ can be interpreted as

$$\overline{M}(ba^* \otimes dc^*) = b(a^*d)c^*, \quad ba^*,\ dc^* \in D(A) \tag{32}$$

where $a^*d$ is expressed as a sum of terms of the form $\alpha_i\beta_j^*$, $\alpha_i \in A$, $\beta_j^* \in A^o$ through (31), and the multiplication of $\alpha_i\beta_j^*$ by $b$ from the left and $c^*$ from the right is taken to be

$$b(\alpha_i\beta_j^*)c^* = (b\alpha_i)(\beta_j^*c^*).$$

https://doi.org/10.1017/S0004972700015197 Published online by Cambridge University Press

This interpretation of $\overline{M}$ is fully consistent with the original definition (28), and the associativity of $\overline{M}$ now becomes an obvious fact.

Let us further define the $\mathbb{Z}_2$-graded vector space homomorphism $\overline{\Delta}: D(A) \to D(A) \otimes D(A)$, by the composition

$$D(A) \xrightarrow{\psi^{-1}} A \otimes A^o \xrightarrow{\widehat{\Delta}'} A \otimes A^o \otimes A \otimes A^o \xrightarrow{\psi \otimes \psi} D(A) \otimes D(A), \tag{33}$$

where $\widehat{\Delta}'$ is defined by (13), and the homomorphism $\overline{\epsilon}: D(A) \to K$ by

$$D(A) \xrightarrow{\psi^{-1}} A \otimes A^o \xrightarrow{\epsilon \otimes \epsilon_o} K \otimes K \to K \tag{34}$$

where $\epsilon_o = u^*|_{A^o}$ is the co-unit of $A^o$. Then

PROPOSITION 3. *The $\mathbb{Z}_2$-graded algebra $D(A)$ together with $\overline{\Delta}$ and $\overline{\epsilon}$ constitutes a $\mathbb{Z}_2$-graded bi-algebra.*

PROOF: Note that

$$\left(\overline{\Delta} \otimes I\right) \cdot \overline{\Delta} = (\psi \otimes \psi \otimes \psi) \cdot \left(\widehat{\Delta}' \otimes I_{A \otimes A^o}\right) \cdot \widehat{\Delta}' \cdot \psi^{-1},$$
$$\left(I \otimes \widehat{\Delta}\right) \cdot \overline{\Delta} = (\psi \otimes \psi \otimes \psi) \cdot \left(I_{A \otimes A^o} \otimes \widehat{\Delta}'\right) \cdot \widehat{\Delta}' \cdot \psi^{-1},$$

where $I: D(A) \to D(A)$ and $I_{A \otimes A^o}: A \otimes A^o \to A \otimes A^o$ are the identity morphisms of $D(A)$ and $A \otimes A^o$, respectively. Since $\widehat{\Delta}'$ is co-associative, we see that

$$\left(\overline{\Delta} \otimes I\right) \cdot \overline{\Delta} = \left(I \otimes \overline{\Delta}\right) \cdot \overline{\Delta}.$$

We may express

$$\overline{\Delta}(ba^*) = \Delta(b)\Delta_o(a^*), \quad \forall\, ba^* \in D(A) \tag{35}$$

where $\Delta(b)\Delta_o(a^*)$ is defined in the usual way, that is

$$\Delta(b)\Delta_o(a^*) = (\psi \otimes \psi)(I_A \otimes T \otimes I_{A^o}) \cdot (\Delta(b) \otimes \Delta_o(a^*)).$$

With (35) we immediately see that

$$(I \otimes \overline{\epsilon}) \cdot \overline{\Delta}(ba^*) = ba^* \otimes \overline{u}(1), \quad (\overline{\epsilon} \otimes I) \cdot \overline{\Delta}(ba^*) = \overline{u}(1) \otimes ba^*, \quad \forall\, ba^* \in D(A),$$

thus $D(A)$ together with $\overline{\Delta}$ and $\overline{\epsilon}$ forms a $\mathbb{Z}_2$-graded co-algebra. What remains to be shown is the fact that $\overline{\Delta}$ and $\overline{\epsilon}$ are also graded algebra homomorphisms. Consider $\overline{\Delta}$ first. We want to prove that

$$\overline{\Delta}(ba^*dc^*) = \overline{\Delta}(ba^*)\overline{\Delta}(dc^*), \quad ba^*,\ dc^* \in D(A). \tag{36}$$

In view of (35) and (32), this is equivalent to showing that

(37) $$\overline{\Delta}(a^*b) = \Delta_o(a^*)\Delta(b), \quad a^* \in A^o,\ b \in A.$$

Now
$$\begin{aligned}\overline{\Delta}(a^*b) &= \overline{\Delta}\cdot\psi\cdot\mu(a^*\otimes b)\\ &= (\psi\otimes\psi)\cdot\widehat{\Delta}'\cdot\mu(a^*\otimes b),\end{aligned}$$

from which we obtain, using Lemma 3,

$$\begin{aligned}\overline{\Delta}(a^*b) &= (\psi\otimes\psi)\cdot(\mu\otimes\mu)\widehat{\Delta}(a^*\otimes b)\\ &= \Delta_o(a^*)\Delta(b).\end{aligned}$$

Similarly, in order to show that $\overline{\epsilon}$ is a $\mathbb{Z}_2$-graded algebra homomorphism, it suffices to demonstrate that

(38) $$\overline{\epsilon}(a^*b) = \varepsilon_o(a^*)\varepsilon(b), \quad a^* \in A^o,\ b \in A.$$

Assume both $a^* \in A^o$ and $b \in A$ are homogeneous, then

$$\begin{aligned}\overline{\epsilon}(a^*b) &= \sum_{(a*),(b)} \langle S_o\left(a^*_{(1)}\right), b_{(1)}\rangle\langle a^*_{(3)}, b_{(3)}\rangle\overline{\epsilon}\left(b_{(2)}a^*_{(2)}\right)\\ &\quad\times(-)^{[a^*][b]+[b_{(1)}][b]+[a^*][b_{(3)}]+[b_{(1)}][b_{(3)}]}\\ &= \sum_{(a*),(b)} \langle S_o\left(a^*_{(1)}\right), b_{(1)}\rangle\langle a^*_{(2)}, b_{(2)}\rangle(-)^{[b_{(1)}][b_{(2)}]}\\ &= \varepsilon_o(a^*)\varepsilon(b).\end{aligned}$$

Thus by linearity $\overline{\epsilon} : D(A) \to K$ indeed defines a $\mathbb{Z}_2$-graded algebra homomorphism, and this completes the proof of the proposition. $\square$

Finally we define $\overline{S} : D(A) \to D(A)$ by the composition

(39) $$D(A) \xrightarrow{\psi^{-1}} A\otimes A^o \xrightarrow{T(S\otimes S_o)} A^o\otimes A \xrightarrow{\psi\cdot\mu} D(A),$$

and show that

PROPOSITION 4. *$\overline{S}$ defines an antipode for the $\mathbb{Z}_2$-graded bi-algebra $D(A)$, thus turning it into a $\mathbb{Z}_2$-graded Hopf algebra.*

PROOF: Let $ba^* \in D(A)$ with both $a^* \in A^o$ and $b \in A$ homogeneous. Then

$$\begin{aligned}\overline{M}\cdot(\overline{S}\otimes I)\cdot\overline{\Delta}(ba^*) &= \sum_{(a*),(b)} \overline{S}\left(b_{(1)}a^*_{(1)}\right)b_{(2)}a^*_{(2)}(-)^{[a^*_{(1)}][b_{(2)}]}\\ &= \sum_{(a*),(b)} S_o\left(a^*_{(1)}\right)S\left(b_{(1)}\right)b_{(2)}a^*_{(2)}(-)^{[a^*_{(1)}][b]}\\ &= \overline{u}\cdot\overline{\epsilon}(ba^*).\end{aligned}$$

Generalising the above calculations to inhomogeneous elements linearly we see that

$$\overline{M}\cdot(\overline{S}\otimes I)\cdot\overline{\Delta}=\overline{u}\cdot\overline{\epsilon} \tag{40}$$

and in a similar way we can show that

$$\overline{M}\cdot(I\otimes\overline{S})\cdot\overline{\Delta}=\overline{u}\cdot\overline{\epsilon}. \tag{41}$$

Equations (40) and (41) are necessary and sufficient to guarantee that $\overline{S}$ indeed gives rise to an antipode for $D(A)$. ∎

It is instructive to show that $\overline{S}$ defines a $\mathbb{Z}_2$-graded algebra anti-homomorphism, that is, for homogeneous element $d_1,\ d_2\in D(A)$,

$$\overline{S}(d_1d_2)=(-)^{[d_1][d_2]}\overline{S}(d_2)\overline{S}(d_1). \tag{42}$$

To do this, it sufficies to demonstrate that

$$\overline{S}(a^*b)=(-)^{[a^*][b]}S(b)S_o(a^*), \tag{43}$$

where $a^*\in A^o$ and $b\in A$ are homogeneous. Now

$$\begin{aligned}\overline{S}(a^*b)&=\psi\cdot\mu\cdot T(S\otimes S_o)\cdot\psi^{-1}\cdot\psi\cdot\mu(a^*\otimes b)\\&=\psi\cdot\mu(S_o\otimes S)\cdot T\cdot\mu(a^*\otimes b).\end{aligned}$$

Using Lemma 4, that is, the relation

$$\mu\cdot(S_o\otimes S)=(S\otimes S_o)\cdot T\cdot\mu^{-1}\cdot T$$

we immediately see that

$$\begin{aligned}\overline{S}(a^*b)&=\psi(S\otimes S_o)\cdot T(a^*\otimes b)\\&=(-)^{[a^*][b]}S(b)S_o(a^*),\end{aligned}$$

thus completing the proof.

Another important fact is that there exist natural embeddings of $A$ and $A^o$ in $D(A)$. Define $\overline{A},\ \overline{A^o}\subset D(A)$ by

$$\overline{A}=\{\overline{a}=a\varepsilon\mid a\in A\},\quad \overline{A^o}=\{\overline{a}^*=1_Aa^*\mid a^*\in A^o\}, \tag{44}$$

then $\overline{A}$ and $\overline{A^o}$ are $\mathbb{Z}_2$-graded Hopf subalgebras of $D(A)$, as can be easily seen. Also, $\overline{A}$ and $\overline{A^o}$ are respectively isomorphic to $A$ and $A^o$ via the isomorphisms

$$\varphi:\ A\to\overline{A},\quad \varphi_o:\ A^o\to\overline{A^o}$$

defined by

$$\varphi(a)=\overline{a},\ \varphi_o(a^*)=\overline{a^*},\quad \forall\, a\in A,\ a^*\in A^o. \tag{45}$$

We call $D(A)$ together with $\mathbb{Z}_2$-graded vector space homomorphisms

$$\begin{aligned}
&\overline{M} : D(A) \otimes D(A) \to D(A), \quad \overline{u} : K \to D(A),\\
&\overline{\Delta} : D(A) \to D(A) \otimes D(A), \quad \overline{\epsilon} : D(A) \to K,\\
&\overline{S} : D(A) \to D(A),
\end{aligned}$$

the double of the $\mathbb{Z}_2$-graded Hopf algebra $A$. Summarising the results obtained above, we have

**THEOREM 2.** *The double of a $\mathbb{Z}_2$-graded Hopf algebra $A$ with a nontrivial $A^o$ is a $\mathbb{Z}_2$-graded Hopf algebra containing $A$ and $A^o$ as $\mathbb{Z}_2$-graded Hopf subalgebras.*

From now on, we shall assume that the $\mathbb{Z}_2$-graded Hopf algebra $A$ is proper, that is, $A^o$ is nontrivial and dense in $A^*$. Construct a homogeneous basis $\{a_s \mid s = 1, 2, \dots\}$ for $A$ and the corresponding dual basis $\{a_s^* \mid s = 1, 2, \dots\}$ for $A^o$ such that

$$\langle a_s^*, a_t \rangle = \delta_{s,t} \quad \forall\, s, t.$$

Define

$$R = \sum_s a_s \otimes a_s^* \in A \otimes A^o, \tag{46}$$

then we have the following

**THEOREM 3.** *Let $A$ be a proper $\mathbb{Z}_2$-graded Hopf algebra. Then*

$$\overline{R} = (\varphi \otimes \varphi_o) R \in D(A) \otimes D(A) \tag{47}$$

*defines a universal $R$-matrix for the double $D(A)$ and thus turns it into a quasitriangular $\mathbb{Z}_2$-graded Hopf algebra.*

PROOF: To prove the theorem we have to show that $\overline{R}$ satisfies all the properties of a universal $R$-matrix. Let us firstly demonstrate that $\overline{R}$ is invertable with

$$\overline{R}^{-1} = (\overline{S} \otimes I)\overline{R}.$$

We have

$$\begin{aligned}
(\overline{S} \otimes I)\overline{R} \cdot \overline{R} &= \sum_{p,\,t} \overline{S}(\overline{a}_t)\overline{a}_p \otimes \overline{a}_t^* \overline{a}_p^* (-)^{[\overline{a}_p][\overline{a}_t^*]}\\
&= \sum_t \overline{M} \cdot (\overline{S} \otimes I) \cdot \overline{\Delta}(\overline{a}_t) \otimes \overline{a}_t^* = 1_D \otimes 1_D,
\end{aligned}$$

where $1_D = u(1)\varepsilon$ is the identity element of $D(A)$. In exactly the same way we can prove that

$$\overline{R} \cdot (\overline{S} \otimes I)\overline{R} = 1_D \otimes 1_D,$$

https://doi.org/10.1017/S0004972700015197 Published online by Cambridge University Press

thus, $(\overline{S}\otimes I)\overline{R}$ is the double-sided inverse of $\overline{R}$.

Secondly we show that $\overline{R}$ satisfies equation (22). Since $A^o$ is dense in $A^*$, we have the following completeness relations

$$\sum_s \langle a_s^*,\, b\rangle a_s = b, \quad \sum_s \langle a^*,\, a_s\rangle a_s^* = a^*, \quad \forall\, a^* \in A^o,\, b \in A.$$

These relations allow us to write $\overline{R\Delta}(\overline{b})$, $\overline{b}\in\overline{A}\subset D(A)$ as

$$\overline{R\Delta}(\overline{b}) = \sum_p \overline{a}_p \otimes \sum_{(a_p^*),(b)} \langle a_{p(1)}^*,\, b_{(1)}\rangle a_{p(2)}^* b_{(2)} (-)^{[a_p^*][b_{(1)}]+[b_{(1)}]}, \tag{48}$$

where $a_{p(1)}^*$ and $a_{p(2)}^*$ are the components of $\Delta_o(a_p^*) = \sum_{(a_p^*)} a_{p(1)}^* \otimes a_{p(2)}^*$ and

$$a_{p(2)}^* b_{(2)} = \psi\cdot\mu\left(a_{p(2)}^*\otimes b_{(2)}\right) \in D(A). \tag{49}$$

Using (49) we obtain

$$\begin{aligned}
&\sum_{(a_{(p)}^*),(b)} \langle a_{p(1)}^*, b_{(1)}\rangle a_{p(2)}^* b_{(2)} (-)^{([a_p^*]+1)[b_{(1)}]}\\
&= \sum_{(a_p^*),(b)} \langle a_{p(1)}^*, b_{(1)}\rangle \langle S^o\left(a_{p(2)}^*\right), b_{(2)}\rangle \langle a_{p(4)}^*, b_{(4)}\rangle b_{(3)} a_{p(3)}^*\\
&\quad \times (-)^{\left([a_{p(3)}^*]+[a_{p(4)}^*]\right)\left([b_{(2)}]+[b_{(3)}]\right)+([a_p^*]+1)[b_{(1)}]}\\
&= \sum_{(a_p^*),(b)} \langle a_{p(2)}^*, b_{(2)}\rangle b_{(1)} a_{p(1)}^* (-)^{[a_p^*][b_{(1)}]}\\
&= \sum_{(b),t} \langle a_p^*, b_{(2)} a_t\rangle b_{(1)} a_t^* (-)^{[a_p^*][b_{(1)}]},
\end{aligned}$$

and inserting this into (48) we arrive at

$$\begin{aligned}
\overline{R\Delta}(\overline{b}) &= \sum_{t,(b)} b_{(2)} a_t \otimes b_{(1)} a_t^* (-)^{[b_{(1)}]\left([b_{(2)}]+[a_t]\right)}\\
&= T\cdot\overline{\Delta}(\overline{b})\cdot\overline{R}.
\end{aligned} \tag{50}$$

In a similar way we can show that

$$\overline{R\Delta}(\overline{a}^*) = T\cdot\overline{\Delta}(\overline{a}^*)\cdot\overline{R}, \quad \forall\, \overline{a^*}\in\overline{A}^o. \tag{51}$$

Since $\overline{\Delta}$ and $T \cdot \overline{\Delta}$ are $\mathbb{Z}_2$-graded algebra homomorphisms, equation (50) and (51) ensure that

$$\overline{R} \cdot \overline{\Delta}(d) = T \cdot \overline{\Delta}(d) \cdot \overline{R} \quad \forall\, d \in D(A). \tag{52}$$

Finally we consider

$$\begin{aligned}(\overline{\Delta} \otimes I)\overline{R} &= \sum_t \overline{\Delta}(\overline{a}_t) \otimes \overline{a}_t^* \\ &= \sum_{t,u,v} \langle a_u^* \otimes a_v^*,\ \Delta(a_t)\rangle \overline{a}_u \otimes \overline{a}_v \otimes \overline{a}_t^* (-)^{[a_u][a_v]} \\ &= \sum_{u,v} \overline{a_u} \otimes \overline{a}_v \otimes \overline{a}_u^* \overline{a}_v^* (-)^{[a_u][a_v]}.\end{aligned}$$

Thus

$$(\overline{\Delta} \otimes I)\overline{R} = \overline{R}_{13}\overline{R}_{23}, \tag{53}$$

and in the same way we can show that

$$(I \otimes \overline{\Delta})\overline{R} = \overline{R}_{13}\overline{R}_{12}. \tag{54}$$

Therefore $\overline{R}$ indeed gives rise to a universal $R$-matrix, and it follows from Theorem 1 that $\overline{R}$ satisfies the graded Yang-Baxter equation. ∎

## 5. Conclusion

We have proved in detail the double construction for $\mathbb{Z}_2$-graded Hopf algebras which states that for every proper $\mathbb{Z}_2$-graded Hopf algebra $A$ there exists a quasitriangular $\mathbb{Z}_2$-graded Hopf algebra $D(A)$ which contains $A$ and $A^o$ as $\mathbb{Z}_2$-graded Hopf subalgebras and its universal $R$-matrix is given by equation (47).

To apply the construction to the quantum supergroups in a direct way, we can follow a similar procedure to that set up in [12]. Let $U_q(g)$ be a quantum supergroup and denote by $U_q(b_+)$ the $\mathbb{Z}_2$-graded Hopf subalgebra of $U_q(g)$ generated by $E_{\alpha_i} = q^{h_i/2}\widehat{e}_{\alpha_i}$, $h_{\alpha_i}$, $i = 1, 2, \ldots, r$. Find a basis $\{E_t \mid t = 1, 2, \ldots\}$ for $U_q(b_+)$ such that $E_{\alpha_i}$, $h_{\alpha_i}$, $\forall\, i$ are basis elements, then construct $(U_q(b_+))^o$ and a basis $\{D_t \mid t = 1, 2, \ldots\}$ for $(U_q(b_+))^o$ which is dual to $\{E_t \mid t = 1, 2, \ldots\}$. Denote by $\sigma_i$, $\rho_i \in \{D_t \mid t = 1, 2, \ldots\}$ the elements dual to $E_{\alpha_i}$ and $h_{\alpha_i}$, respectively. The quantum double $D(U_q(b_+))$ is a quasitriangular $\mathbb{Z}_2$-graded Hopf algebra with the universal $R$-matrix

$$R = \sum_t E_t \otimes D_t,$$

which can be identified with $U_q(g)$ itself via the isomorphism

$$\begin{aligned}&\sigma_i \mapsto \lambda_i \widehat{f}_{\alpha_i} q^{-h_{\alpha_i}/2} \\ &\rho_i \mapsto \chi_i h_{\alpha_i}\end{aligned} \tag{55}$$

https://doi.org/10.1017/S0004972700015197 Published online by Cambridge University Press

where $\lambda_i$ and $\chi_i$ are constants. These constants can be uniquely determined by considering the commutation relations

$$E_{\alpha_i}\rho_j - \rho_j E_{\alpha_i}, \; E_{\alpha_i}\sigma_j - \sigma_j E_{\alpha_i}, \quad \text{et cetera}$$

in $D(U_q(b_+))$, where $\rho_j E_{\alpha_i}$ et cetera are evaluated according to the rule prescribed in (31).

However, it must be pointed out that the existence of the isomorphism (55) is entirely due to the peculiar structure of $U_q(g)$. In the limit as $q \to 1$, this isomorphism breaks down.

The universal $R$-matrices for $U_q(osp(1/2))$ [8] and $U_q(osp(2/2))$ [**8, 13**] have been constructed explicitly. Also the techniques developed by Kirillov and Reshetikhin [7] can be generalised to quantum supergroups to yield explicit expressions for the universal $R$-matrices; results will be published in a separate publication.

## References


[1] E. Abe, *Hopf algebras* (Cambridge University Press, 1980).

[2] A.J. Bracken, M.D. Gould and R.B. Zhang, 'Quantum supergroups and solutions of the Yang-Baxter equation', *Modern Phys. Lett. A* **5** (1990), 831–840.

[3] M. Chaichian and P.P. Kulish, 'Quantum Lie superalgebras and $q$-oscillators', *Phys. Lett. B* **234** (1990), 72–80.

[4] T. Deguchi, A. Fujii, and K. Ito, 'Quantum superalgebra $U_q$ osp$(2,2)$', *Phys. Lett. B* **238** (1990), 242–246.

[5] V.G. Drinfeld, 'Quantum groups', *Proc. I.C.M. Berkeley* **1** (1986), 798–820.

[6] M. Jimbo, 'A $q$-difference analogue of $U(g)$ and the Yang-Baxter equation', *Lett. Math. Phys* **10** (1985), 63–69.

[7] A.M. Kirillov and N.Yu. Reshetikhim, '$q$-Weyl group and a multiplicative formula for universal $R$-matrices', (preprint).

[8] P.P. Kulish and N.Yu. Reshetikhin, 'Universal $R$-matrix of the quantum superalgebra osp$(2|1)$', *Lett.Math.Phys.* **18** (1989), 143–149.

[9] S. Majid, 'Physics for algebraists', *J. Algebra* **130** (1990), 17–64.

[10] J.W. Milner and J.C. More, 'On the structure of Hopf algebras', *Ann. Math.* **81** (1965), 211–264.

[11] N. Yu. Reshetikhin, 'Quantized universal enveloping algebras, the Yang-Baxter equation and invariants of links' (L.O.M.I., Leningrad).

[12] M. Rosso, 'An analogue of PBW theorem and the universal $R$-matrix for $U_h sl(n+1)$', *Comm. Math. Phys.* **124** (1990), 307–318.

[13] H. Saleur, 'Quantum osp$(1/2)$ and solutions of the graded Yang-Baxter equation', *Nuclear Phys. B* **336** (1990), 363–372.

[14] M.E. Sweedler, *Hopf algebras* (Benjamin, New York, 1969).

[15] R.B. Zhang, M.D. Gould and A.J. Bracken, 'Solutions of the graded classical Yang-Baxter equation and integrable models', *J. Phys. A* **24** (1991), 1185–1197.

Department of Mathematics
The University of Queensland
Queensland 4072
Australia