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VISCOSITY SOLUTIONS TO THE INFINITY LAPLACIAN EQUATION WITH SINGULAR NONLINEAR TERMS

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Abstract

In this paper, we study the singular boundary value problem

$$\begin{cases} \Delta_\infty^h u = \lambda f(x,u,Du) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$ is a parameter, $h > 1$ and $\Delta_\infty^h u = |Du|^{h-3}\langle D^2uDu, Du\rangle$ is the highly degenerate and h-homogeneous operator related to the infinity Laplacian. The nonlinear term $f(x,t,p) : \Omega \times (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function and may exhibit singularity at $t \to 0^+$. We establish the comparison principle by the double variables method for the general equation $\Delta_\infty^h u = F(x,u,Du)$ under some conditions on the term $F(x,t,p)$. Then, we establish the existence of viscosity solutions to the singular boundary value problem in a bounded domain based on Perron's method and the comparison principle. Finally, we obtain the existence result in the entire Euclidean space by the approximation procedure. In this procedure, we also establish the local Lipschitz continuity of the viscosity solution.

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1. Introduction

In this paper, we consider the singular boundary value problem

$$\begin{cases} \Delta_\infty^h u = \lambda f(x,u,Du) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-1}$$

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where the domain $\Omega \subseteq \mathbb{R}^n$, $\lambda > 0$ is a parameter and

$$\Delta_\infty^h u := |Du|^{h-3}\langle D^2uDu, Du\rangle = |Du|^{h-3}\sum_{i,j=1}^{n}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i\partial x_j}, \quad h > 1$$

is the h-homogeneous quasilinear operator associated with the infinity Laplacian. The nonlinear term $f(x, t, p) : \Omega \times (0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function and may exhibit singularity at $t \to 0^+$.

When $h = 3$, the operator is the infinity Laplacian

$$\Delta_\infty u := \langle D^2uDu, Du\rangle = \sum_{i,j=1}^{n}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i\partial x_j},$$

which was first introduced by Aronsson in the study of the absolutely minimizing Lipschitz extension (AMLE) [2–5] in the 1960s. The infinity Laplacian is quasilinear and highly degenerate, and the solutions are usually understood in the viscosity sense. See for example Crandall *et al.* [18]. In [24], Jensen proved that the AMLE functions are equivalent to the infinity harmonic functions (viscosity solutions to the homogeneous infinity Laplacian equation $\Delta_\infty u = 0$) and also proved the existence and uniqueness of AMLE. In [16], Crandall *et al.* showed that the infinity harmonic functions can be compared with linear cones that can be regarded as the fundamental solution of the equation $\Delta_\infty u = 0$. For more results on the infinity harmonic functions, one can see [1, 15, 17].

When $h = 1$, the operator is the normalized infinity Laplacian

$$\Delta_\infty^N u := |Du|^{-2}\langle D^2uDu, Du\rangle = |Du|^{-2}\sum_{i,j=1}^{n}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i\partial x_j}.$$

Peres *et al.* [38] discovered a wonderful connection between the Dirichlet problem corresponding to the normalized infinity Laplacian and a random game named 'tug-of-war'. The game is played by two players and the token is initially at any point $x_0 \in \Omega$. When the game starts, the two players take turns to move the token arbitrarily and the length of the movement is not greater than $\varepsilon > 0$. When one of the players moves the token to the point $x \in \partial\Omega$, the game is over and the players are rewarded or punished through the running payoff function f and the final payoff function g. As $\varepsilon \to 0$, Peres *et al.* proved that the continuum value function of the game is the unique viscosity solution of the problem

$$\begin{cases} \Delta_\infty^N u = f(x) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases} \tag{1-2}$$

for the continuous functions f ($|f| > 0$) and g. Lu and Wang [32] established the existence and uniqueness of the solution for the problem (1-2) based on partial differential equation methods. The normalized infinity Laplacian equations related to some 'tug-of-war' game have attracted increasingly more interest. The normalized infinity Laplacian with a transport term

$$\begin{cases} \Delta_\infty^N u + \langle \xi, Du \rangle = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

was first studied by López-Soriano *et al.* [30]. When ξ is a continuous gradient vector field, they obtained the existence and uniqueness of the viscosity solutions. When ξ is Lipschitz continuous but not necessarily a gradient, they established the existence of viscosity solutions using 'tug-of-war' game arguments. In addition, Peres *et al.* [37] introduced a biased 'tug-of-war' and showed the existence and uniqueness of viscosity solutions under the continuous Dirichlet boundary condition for the β-biased infinity Laplacian equation

$$\Delta_\infty^N u + \beta |Du| = 0, \tag{1-3}$$

where $\beta \in \mathbb{R}$ is a fixed constant denoting the bias. If $\beta = 0$, (1-3) reduces to the unbiased case.

In [31], Lu and Wang showed the existence and uniqueness of viscosity solutions for the inhomogeneous Dirichlet problem

$$\begin{cases} \Delta_\infty u = f(x) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

when the nonhomogeneous term f has one sign. Bhattacharya and Mohammed [7] studied the existence and nonexistence of viscosity solutions to the Dirichlet problem

$$\begin{cases} \Delta_\infty u = f(x, u) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where f has sign and monotonicity restrictions. In [8], they presented some structure conditions on f and established the existence results without sign and monotonicity restrictions. Liu and Yang [29] gave existence and uniqueness results of viscosity solutions of the nonhomogeneous problem

$$\begin{cases} \Delta_\infty^h u = f(x) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where $1 \le h \le 3$. In [26], Li and Liu established the existence of viscosity solutions of the Dirichlet problem

$$\begin{cases} \Delta_\infty^h u = f(x, u) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

when the right-hand side $f(x, t)$ is nondecreasing in t and nonincreasing in t. For other studies on the infinity Laplacian operator, one can refer to [27, 33, 34].

Over the years, the singular boundary value problem for the partial differential equation has attracted much attention and the earliest study on the singular boundary value problem was introduced by Fulks and Maybee in [21]. They studied the physical problem about heat conduction in electrically conducting materials. Let $\Omega \subseteq \mathbb{R}^3$ be occupied by an electrical conductor. Then each point $x \in \Omega$ becomes a source of heat when a current is passing through Ω. Suppose that the function $E(x,t)$ describes the local voltage drop in Ω and $\sigma(u)$ is the electrical resistivity, where u is the temperature. According to the resistance heat generation formula, $E^2(x,t)\backslash\sigma(u)$ denotes the rate of generation of heat at any point x and time t. Let c and κ be the specific heat and thermal conductivity of Ω, respectively, which can be taken as constant. Then the temperature u at the point $x \in \Omega$ and time t satisfies

$$cu_t - \kappa\Delta u = \frac{E^2(x,t)}{\sigma(u)}. \tag{1-4}$$

In one simple case, one can take $\sigma(u) = \alpha u$, where α is a positive constant. It is obvious that $\sigma(u) \to 0$ as $u \to 0$. Thus, (1-4) is singular since the right-hand side becomes unbounded at $u = 0$. This physical problem naturally leads to the study of the singular problem related to the parabolic Laplacian equation

$$u_t - a^2\Delta u = F(x,t,u),$$

where $a^2 = \kappa/c$ and $F(x,t,u) = E^2(x,t)/c\sigma(u)$. The singular boundary value problems can be widely applied in non-Newtonian fluids, boundary layer phenomena for viscous fluids and chemical heterogeneous catalysts [19, 20, 22].

It is worth noting that $\Delta_\infty^N u = |Du|^{-2}\Delta_\infty u$ reduces to Δu when the dimension is one. Therefore, it is natural to study the singular boundary value problems related to the infinity Laplacian. Bhattacharya and Mohammed [7] considered the singular boundary value problem

$$\begin{cases} -\Delta_\infty u = b(x)h(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-5}$$

where $b \in C(\Omega) \cap L^\infty(\Omega)$ is a positive function in Ω, and $h \in C(0,\infty)$ is positive and may exhibit singularity at zero, in other words, $h(t) \to \infty$ as $t \to 0^+$. They gave the existence, uniqueness and asymptotic behaviour of the positive viscosity solution of Problem (1-5). In [10], Biset and Mohammed extended the existence results of viscosity solutions to the singular boundary value problem

$$\begin{cases} -\Delta_\infty u = \lambda f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

in a bounded domain Ω as well as in the whole Euclidean space, where $\lambda > 0$ is a parameter. In [26], Li and Liu established the existence, uniqueness and the asymptotic estimate of the viscosity solution to the singular boundary value problem

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$$\begin{cases} -\Delta_\infty^h u = b(x)g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

For other studies of the singular boundary value problems, one can refer to [35, 36, 39].

Inspired by these works, we study the singular boundary value problem (1-1) of the strongly degenerate operator $\Delta_\infty^h u$. For all cases $h > 1$, we prove that Δ_∞^h share a common theory. Let us point out that, unlike the case $h = 1$, the operator Δ_∞^h is quasilinear even in dimension 1. Therefore, we must make subtle analysis. Our main results are as follows.

We first give the comparison principle for the following general equation:

$$\Delta_\infty^h u = F(x, u, Du). \tag{1-6}$$

THEOREM 1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. Suppose that the function $F(x, t, p) \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ is negative and nondecreasing in t, and the map $\tau \mapsto F(x, t, \tau p)$ is nondecreasing in $[1, \rho)$ for each $(x, t, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, where $\rho > 1$. Assume that $u \in C(\overline{\Omega})$ and $v \in C(\overline{\Omega})$ satisfy*

$$\Delta_\infty^h u \geq F(x, u, Du), \quad x \in \Omega$$

and

$$\Delta_\infty^h v \leq F(x, v, Dv), \quad x \in \Omega$$

in the viscosity sense, respectively. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

The proof of the comparison principle Theorem 1.1 is based on the double variables method in viscosity solution theory. If the nonhomogeneous term $F(x, u, Du)$ is independent of the gradient Du, Li and Liu proved the comparison principle for the equation $\Delta_\infty^h u = F(x, u)$ in [25]. However, due to the strong degeneracy of the operator Δ_∞^h and the dependence of p of the nonlinear term $F(x, t, p)$, we must perturb twice to make Jensen's method useful [24]. In [25], they also established the existence of the boundary blow-up viscosity solution and analysed the boundary asymptotic behaviour of the blow-up solutions based on the comparison principle and Karamata's regular variation theory, which was first introduced by Cîrstea and Rădulescu [11–13].

THEOREM 1.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $\varphi \in C(\partial\Omega)$. Assume that $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function and satisfies*

$$\sup_{\Omega \times I \times \mathbb{R}^n} |F(x, t, p)| < \infty \tag{1-7}$$

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for every compact interval $I \subseteq \mathbb{R}$. Suppose that $u_* \in C(\overline{\Omega})$ is a viscosity subsolution and $u^* \in C(\overline{\Omega})$ is a viscosity supersolution of

$$\begin{cases} \Delta_\infty^h u = F(x, u, Du) & \textit{in } \Omega, \\ u = \varphi & \textit{on } \partial\Omega, \end{cases} \tag{1-8}$$

in Ω. If $u_* \leq u^*$ in Ω, then the problem (1-8) admits a viscosity solution $u \in C(\overline{\Omega})$ such that $u_* \leq u \leq u^*$ in Ω.

Theorem 1.2 gives the existence of the viscosity solution of (1-8) when the nonlinear $F(x, t, p)$ is bounded in Condition (1-7) for the class of operators Δ_∞^h with a parameter $h > 1$. The method is based on the standard Perron idea. Let us point out that, unlike the case $h = 1$, the operator Δ_∞^h is quasilinear even in dimension one. Hence, we must make subtle analysis. During this procedure, the construction of barrier functions is more complicated. This observation is very important for the study of the existence.

Obviously, taking $F(x, u, Du) = \lambda f(x, u, Du)$, one can immediately get the existence result of Problem (1-1) if $f(x, u, Du)$ is bounded, that is $\sup_{\Omega \times I \times \mathbb{R}^n} |f(x, u, Du)| < \infty$. Now we turn to the singular boundary problem (1-1). That is, $f(x, t, p)$ may exhibit the singularity when $t \to 0^+$. We need some basic assumptions on the singular nonlinear term $f(x, t, p)$ to construct a viscosity subsolution and supersolution.

Let $\mathbb{R}^+ := (0, \infty)$ and $f(x, t, p) : \Omega \times \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function that satisfies the following conditions.

(F-1): There are $t_0 \in \mathbb{R}^+$ and two continuous functions $k : (0, t_0) \to \mathbb{R}^+$ and $a : \overline{\Omega} \to \mathbb{R}^+$ such that

$$f(x, t, p) \leq -a(x)k(t) \quad \text{for all } (x, t, p) \in \Omega \times (0, t_0) \times \mathbb{R}^n.$$

(F-2): There are two continuous functions $g : \mathbb{R}^+ \to \mathbb{R}^+$ and $b : \Omega \to \mathbb{R}^+$ such that

$$f(x, t, p) \geq -b(x)g(t) \quad \text{for all } (x, t, p) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^n.$$

Now we define the following notation in connection with the functions k and g:

$$g_\infty := \limsup_{t \to +\infty} \frac{g(t)}{t^h}, \quad k_0 := \liminf_{t \to 0^+} \frac{k(t)}{t^h}. \tag{1-9}$$

We always assume that $0 < k_0 \leq \infty$.

Furthermore, we assume that the function b in Condition **(F-2)** above satisfies the following condition.

(B-w): The following problem:

$$\begin{cases} \Delta_\infty^h w = -b(x) & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{1-10}$$

admits a viscosity solution $w_\Omega \in C(\overline{\Omega})$. When the function $b \in C(\Omega)$ is positive, the existence of the Problem (1-10) has been established by Li and Liu (see [26]).

To investigate the existence of the viscosity solutions to the singular boundary value problem (1-1), one key is to find a viscosity subsolution. Due to the nonlinear term $f(x, t, p)$ on the right-hand side, it is difficult to construct a viscosity subsolution. Here we adopt the principal eigenfunction for the following eigenvalue problem:

$$\begin{cases} \Delta_\infty^h u + \mu a(x) u^h = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-11}$$

where Ω is a bounded domain and $a(x)$ is a positive continuous function in $\overline{\Omega}$. Liu *et al.* [28] established the existence of a positive principal eigenfunction Φ_Ω and gave a characteristic for the principal eigenvalue $\Lambda_1(\Omega)$ of Problem (1-11). Additionally, it is obvious that $\Lambda_1(\Omega_1) \geq \Lambda_1(\Omega_2)$ if $\Omega_1 \subseteq \Omega_2$. Since the principal Dirichlet-eigenvalue $\Lambda_1(\Omega)$ and the corresponding eigenfunction Φ_Ω are both positive, we need the positivity of k_0 as in (1-9) to construct an appropriate viscosity subsolution and then Perron's method guarantees the existence of viscosity solutions to the singular boundary problem (1-1).

THEOREM 1.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $f : \Omega \times \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function. If f satisfies the conditions **(F-1)**, **(F-2)** and b satisfies the condition **(B-w)**, then there exists a constant λ_Ω^* depending on Ω, g_∞ and $\|w_\Omega\|_\infty$ such that Problem (1-1) admits a viscosity solution $u = u_\lambda \in C(\overline{\Omega})$ for any λ with $\Lambda_1(\Omega)k_0^{-1} < \lambda < \lambda_\Omega^*$.*

Theorem 1.3 shows the existence of the viscosity solution of the singular boundary value problem (1-1) when the parameter λ is lying in an appropriate range, that is $\Lambda_1(\Omega)k_0^{-1} < \lambda < \lambda_\Omega^*$. To overcome the difficulty of the singularity of Problem (1-1), we construct the appropriate cut-off function and then use the comparison principle, compactness analysis and Perron's method to establish the existence of the viscosity solution. Due to the high degeneracy and quasilinearity of the operator Δ_∞^h and the singularity of the nonlinear term $f(x, t, p)$, we cannot follow the scheme in [9]. To overcome this difficulty, we propose conditions **(F-1)** and **(F-2)** and then construct a suitable viscosity supersolution of Problem (1-10). However, we invoke the positive eigenfunction of the eigenvalue problem (1-11) to construct an appropriate viscosity subsolution to Problem (1-1). Then by the standard Perron method, we can obtain the existence of the viscosity solution of the singular boundary value problem.

With Theorem 1.3 in hand, we can establish the existence result of the singular problem in the entire Euclidean space by the approximation procedure. Due to the strong degeneracy of the operator, we need the following additional monotone condition **(F-3)** on the nonlinear term $f(x, t, p)$. Note that it would be interesting to consider the singular problem on unbounded domains other than $\mathbb{R}^n$.

(F-3): For any $(x, t, p) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^n$, the function $t \to f(x, t, tp)/a(x)t^h$ is nondecreasing in t, where the function $a(x)$ is the one in **(F-1)**.

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THEOREM 1.4. *Let f satisfy the conditions **(F-1)**, **(F-2)** and **(F-3)** in $\mathbb{R}^n$, where b satisfies the condition **(B-w)**. Then for any λ with $\Lambda_1(B(O,1))k_0^{-1} < \lambda < \lambda^*$, the problem*

$$\begin{cases} \Delta_\infty^h u = \lambda f(x,u,Du) & \text{in } \mathbb{R}^n, \\ u > 0 & \text{in } \mathbb{R}^n, \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases} \tag{1-12}$$

admits a viscosity solution $u = u_\lambda \in C(\mathbb{R}^n)$, where $\Lambda_1(B(O,1))$ is the principal eigenvalue of Problem (1-11) on the unit ball $\Omega = B(O,1)$.

The paper is organized as follows. In Section 2, we give the definition of the viscosity solutions and establish the comparison principle by the double variables method based on viscosity solutions theory. In Section 3, we prove the existence of the viscosity solutions to the boundary value problem by Perron's method if the nonlinear term on the right-hand side is bounded. In Section 4, we establish the existence of the viscosity solution of the singular boundary problem (1-1) in a bounded domain Ω by truncation and Perron's method. In Section 5, we extend the existence result to the entire Euclidean space by the approximation procedure.

2. Comparison principle

In this section, we first give the definition of viscosity solutions to the general equation

$$\Delta_\infty^h u = F(x,u,Du) \quad \text{in } \Omega, \tag{2-1}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous. Then we establish the comparison principle by the perturbation method based on viscosity solutions theory.

Since the operator Δ_∞^h is highly degenerate and singular at the points where the gradient vanishes, one should give an appropriate explanation at these points. Here we adopt the viscosity solutions based on the semicontinuous extension [18, 23]. Note that one can rewrite (2-1) as

$$G_h(D^2u, Du) = F(x,u,Du), \quad x \in \Omega,$$

where $G_h : \mathbb{S} \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}$, $G_h(X,p) := |p|^{h-3}(Xp) \cdot p$ and $\mathbb{S}$ is the set of all $n \times n$ real symmetric matrices. When $h > 1$, we have $\lim_{p\to 0} G_h(X,p) = 0$ for any $X \in \mathbb{S}$. Thus, we can define the following continuous extension of G_h:

$$\overline{G}_h(X,p) = \begin{cases} G_h(X,p) & \text{if } p \neq 0, \\ 0 & \text{if } p = 0. \end{cases}$$

Now we give the definition of viscosity solutions to (2-1).

DEFINITION 2.1. Let $\Omega \subseteq \mathbb{R}^n$ be a domain. An upper semicontinuous (USC) function u defined in Ω is said to be a viscosity subsolution of (2-1) if and only if for every

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$x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $u(x_0) = \varphi(x_0)$ and $u(x) \le \varphi(x)$ for all $x \in \Omega$ near x_0, there holds

$$\overline{G}_h(D^2\varphi(x_0), D\varphi(x_0)) \ge F(x_0, \varphi(x_0), D\varphi(x_0)).$$

Similarly, a lower semicontinuous (LSC) function u defined in Ω is said to be a viscosity supersolution of (2-1) if and only if for every $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $u(x_0) = \varphi(x_0)$ and $u(x) \ge \varphi(x)$ for all $x \in \Omega$ near x_0, there holds

$$\overline{G}_h(D^2\varphi(x_0), D\varphi(x_0)) \le F(x_0, \varphi(x_0), D\varphi(x_0)).$$

If a continuous function $u \in C(\Omega)$ is both a viscosity supersolution and a viscosity subsolution of (2-1), then we say that u is a viscosity solution of (2-1).

Note that one can give the definition of viscosity subsolutions and viscosity supersolutions equivalently by super-jets and sub-jets. We first recall the definition of super-jets and sub-jets, and then give an equivalent definition of viscosity subsolutions and viscosity supersolutions. One can see the details in for example [18].

Definition 2.2. The second-order super-jet of $u \in \mathrm{USC}(\Omega)$ at $x_0 \in \Omega$ is the set

$$\mathcal{J}^{2,+}u(x_0) = \{(D\varphi(x_0), D^2\varphi(x_0)) : \varphi \in C^2(\Omega) \text{ and } u - \varphi \text{ has a local maximum at } x_0\},$$

and the closure of $\mathcal{J}^{2,+}u(x_0)$ is

$$\overline{\mathcal{J}}^{2,+}u(x_0) := \Big\{(p, X) \in \mathbb{R}^n \times \mathbb{S} : \text{there exists } (x_n, p_n, X_n) \in \Omega \times \mathbb{R}^n \times \mathbb{S} \text{ such that } (p_n, X_n) \in \mathcal{J}^{2,+}u(x_n) \text{ and } (x_n, u(x_n), p_n, X_n) \to (x_0, u(x_0), p, X)\Big\}.$$

Similarly, the second-order sub-jet of $u \in \mathrm{LSC}(\Omega)$ at $x_0 \in \Omega$ is the set

$$\mathcal{J}^{2,-}u(x_0) = \{(D\varphi(x_0), D^2\varphi(x_0)) : \varphi \in C^2(\Omega) \text{ and } u - \varphi \text{ has a local minimum at } x_0\},$$

and the closure of $\mathcal{J}^{2,-}u(x_0)$ is

$$\overline{\mathcal{J}}^{2,-}u(x_0) := \Big\{(p, X) \in \mathbb{R}^n \times \mathbb{S} : \text{there exists } (x_n, p_n, X_n) \in \Omega \times \mathbb{R}^n \times \mathbb{S} \text{ such that } (p_n, X_n) \in \mathcal{J}^{2,-}u(x_n) \text{ and } (x_n, u(x_n), p_n, X_n) \to (x_0, u(x_0), p, X)\Big\}.$$

Definition 2.3. We say that $u \in \mathrm{USC}(\Omega)$ is a viscosity subsolution of (2-1) if

$$\overline{G}_h(X, p) \ge F(x_0, u(x_0), p) \quad \text{for all } x_0 \in \Omega \text{ and for all } (p, X) \in \overline{\mathcal{J}}^{2,+}u(x_0).$$

Similarly, we say that $u \in \mathrm{LSC}(\Omega)$ is a viscosity supersolution of (2-1) if

$$\overline{G}_h(X, p) \le F(x_0, u(x_0), p) \quad \text{for all } x_0 \in \Omega \text{ and for all } (p, X) \in \overline{\mathcal{J}}^{2,-}u(x_0).$$

If a continuous function $u \in C(\Omega)$ is both a viscosity supersolution and viscosity subsolution of (2-1), we say that u is a viscosity solution of (2-1).

By the definition of the viscosity subsolution (supersolution), one can easily get the following results.

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REMARK 2.4. (1) If u_1, u_2 are viscosity subsolutions of (2-1), then $\max\{u_1, u_2\}$ is a viscosity subsolution of (2-1) as well.

(2) If u_1, u_2 are viscosity supersolutions of (2-1), then $\min\{u_1, u_2\}$ is a viscosity supersolution of (2-1) as well.

Now we give the local Lipschitz continuity of a viscosity solution to $\Delta_\infty^h u = \alpha$, where α is a constant. For more regularity results of infinity Laplacian equations, one can see [8, 31].

LEMMA 2.5. *Let α be a constant. If $u \in C(\Omega) \bigcap L^\infty(\Omega)$ satisfies $\Delta_\infty^h u \geq \alpha$ in the viscosity sense, then u is locally Lipschitz continuous in Ω. Moreover, for any given $x_0 \in \Omega$, there exists a constant C such that*

$$|u(x) - u(y)| \leq C|x-y| \ \textit{for all } x, y \in B_{\mathrm{dist}(x_0,\partial\Omega)/3}(x_0),$$

where C depends on $x_0, \mathrm{diam}(\Omega), |\alpha|$ and $\|u\|_{L^\infty(\Omega)}$. A similar result holds if u satisfies $\Delta_\infty^h u \leq \alpha$.

PROOF. Let $d(x_0) := \mathrm{dist}(x_0, \partial\Omega)$ for each $x_0 \in \Omega$. Set

$$k(x_0) = \frac{2(M-m)}{d(x_0)} + 1 + |\alpha|\mathrm{diam}(\Omega), \tag{2-2}$$

where $M := \max_\Omega u$ and $m := \min_\Omega u$. For all $y \in B_{d(x_0)/3}(x_0)$, we consider the function

$$\psi(x) := u(y) + k|x-y| - \frac{|\alpha|}{2}|x-y|^2,$$

where $k := k(x_0)$ is defined in (2-2). Note that $\psi \in C^\infty(\mathbb{R}^n - \{y\})$. For $x \neq y$,

$$\Delta_\infty^h \psi(x) = -|\alpha|(k - |\alpha||x-y|)^{h-1}.$$

Since $k \geq 1 + |\alpha|\mathrm{diam}(\Omega)$, we see that $\Delta_\infty^h \psi \leq \alpha$ in $\Omega \setminus \{y\}$. Note that $d(y) \geq 2d(x_0)/3$ for any $y \in B_{d(x_0)/3}(x_0)$. For any $x \in \partial B_{d(y)}(y)$,

$$\begin{aligned}
\psi(x) &= u(y) + kd(y) - \frac{|\alpha|}{2}d^2(y) \\
&\geq m + \frac{d(x_0)}{2}\left(k - \frac{|\alpha|}{2}d(y)\right) \\
&\geq m + \frac{d(x_0)}{2}\left(k - \frac{|\alpha|}{2}\mathrm{diam}(\Omega)\right) \geq M \geq u(x),
\end{aligned}$$

where we use (2-2). Therefore, $u \leq \psi$ on $\partial(B_{d(y)}(y) \setminus \{y\})$. Since $\Delta_\infty^h \psi \leq \alpha$ and $\Delta_\infty^h u \geq \alpha$ in $B_{d(y)}(y) \setminus \{y\}$, by the comparison principle in [25], we have $u \leq \psi$ in $B_{d(y)}(y)$. Thus, for any $y \in B_{d(x_0)/3}(x_0)$ and any $z \in B_{d(y)}(y)$,

$$u(z) \leq u(y) + k|z-y| - \frac{|\alpha|}{2}|z-y|^2. \tag{2-3}$$

For any $p \in B_{d(x_0)/3}(x_0)$, one has $B_{d(x_0)/3}(x_0) \subseteq B_{d(p)}(p)$. According to (2-3), for any $x, y \in B_{d(x_0)/3}(x_0)$,

$$u(y) \le u(x) + k|x-y| - \frac{|\alpha|}{2}|x-y|^2$$

and

$$u(x) \le u(y) + k|x-y| - \frac{|\alpha|}{2}|x-y|^2.$$

That is,

$$|u(x)-u(y)| \le \Big(k - \frac{|\alpha|}{2}|x-y|\Big)|x-y| \le k|x-y| \quad \text{for all } x, y \in B_{d(x_0)/3}(x_0).$$

Thus, for a given $x_0 \in \Omega$,

$$|u(x)-u(y)| \le C|x-y| \quad \text{for all } x, y \in B_{d(x_0)/3}(x_0),$$

where C depends on x_0, $\operatorname{diam}(\Omega)$, $|\alpha|$ and $\|u\|_{L^\infty(\Omega)}$. □

Now we recall the maximum principle of infinity harmonic functions [6, 14].

LEMMA 2.6. *If $u \in C(\overline{\Omega})$ satisfies $\Delta_\infty u \ge 0$ in the viscosity sense, then u attains its maximum only on the boundary $\partial\Omega$ unless u is a constant.*

PROOF OF THEOREM 1.1. Define

$$v_\varepsilon = v(1+\varepsilon) - \varepsilon \inf_{\overline{\Omega}} v, \quad 0 < \varepsilon < \rho - 1.$$

Since $F(x, t, p)$ is negative, nondecreasing in t and $\tau \to F(x, t, \tau p)$ is nondecreasing in $[1, \rho)$,

$$\begin{aligned} \Delta_\infty^h v_\varepsilon &= (1+\varepsilon)^h \Delta_\infty^h v \\ &\le (1+\varepsilon)^h F(x, v, Dv) \\ &\le (1+\varepsilon)^h F(x, v_\varepsilon, Dv_\varepsilon) \\ &< F(x, v_\varepsilon, Dv_\varepsilon) \end{aligned}$$

in the viscosity sense. That is, v_ε is a viscosity supersolution of (1-6).

Next we want to show $u \le v_\varepsilon$ in Ω. Suppose in contrast that $u > v_\varepsilon$ at some point $x_0 \in \Omega$ and

$$M = \sup_{\Omega}(u - v_\varepsilon) = u(x_0) - v_\varepsilon(x_0) > 0.$$

According to [18], we double the variables

$$w_j(x, y) := u(x) - v_\varepsilon(y) - \frac{j}{4}|x-y|^4, \quad (x, y) \in \Omega \times \Omega,\ j = 1, 2, \dots.$$

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Suppose that w_j attains its maximum at $(x_j, y_j) \in \overline{\Omega} \times \overline{\Omega}$. According to [18, Proposition 3.7],

$$\lim_{j\to\infty} M_j = \lim_{j\to\infty}\Big(u(x_j) - v_\varepsilon(y_j) - \frac{j|x_j - y_j|^4}{4}\Big) = M$$

and

$$\lim_{j\to\infty} \frac{j|x_j - y_j|^4}{4} = 0.$$

It is clear that $x_j \to x_0, y_j \to x_0$ as $j \to \infty$. Since $M > 0 \geq \sup_{\partial\Omega}(u - v_\varepsilon)$, there exists an open set Ω_0 such that x_0, x_j and $y_j \in \Omega_0 \subseteq \Omega$ as $j \to \infty$.

Let

$$\varphi(x) = \frac{j|x - y_j|^4}{4}, \quad \psi(y) = -\frac{j|x_j - y|^4}{4}.$$

It is obvious that the function $u - \varphi$ has a local maximum at x_j and $v_\varepsilon - \psi$ has a local minimum at y_j.

We consider the following two cases: either $x_j = y_j$ or $x_j \neq y_j$ for j large enough.

Case 1: If $x_j = y_j$, we have $D\psi(y_j) = 0$ and $D^2\psi(y_j) = 0$. Since v_ε is a viscosity supersolution,

$$F(y_j, \psi(y_j), D\psi(y_j)) = F(y_j, v_\varepsilon(y_j), D\psi(y_j)) \geq 0,$$

which contradicts $F < 0$.

Case 2: If $x_j \neq y_j$, we use jets and the maximum principle for semicontinuous functions [18, Theorem 3.2]. There exist $n \times n$ symmetric matrices X_j and Y_j such that $X_j \leq Y_j$ and

$$(p_j, X_j) \in \overline{\mathcal{J}}^{2,+} u(x_j), \qquad (p_j, Y_j) \in \overline{\mathcal{J}}^{2,-} v_\varepsilon(y_j),$$

where $p_j = j|x_j - y_j|^2(x_j - y_j)$. By the definitions of the viscosity subsolution and supersolution, since $\Delta_\infty^h u \geq F(x, u, Du)$ and $\Delta_\infty^h v_\varepsilon \leq (1+\varepsilon)^h F(x, v_\varepsilon, Dv_\varepsilon)$ in the viscosity sense,

$$\begin{aligned} 0 &\leq |p_j|^{h-3}\langle X_j p_j, p_j\rangle - F(x_j, u(x_j), p_j) \\ &\leq |p_j|^{h-3}\langle Y_j p_j, p_j\rangle - F(x_j, u(x_j), p_j) \\ &\leq (1+\varepsilon)^h F(y_j, v_\varepsilon(y_j), p_j) - F(x_j, u(x_j), p_j). \end{aligned} \tag{2-4}$$

Since w_j attains its maximum at $(x_j, y_j) \in \overline{\Omega} \times \overline{\Omega}$,

$$u(x) - v_\varepsilon(y) - \frac{j}{4}|x - y|^4 \leq u(x_j) - v_\varepsilon(y_j) - \frac{j|x_j - y_j|^4}{4} \quad \text{for all } x, y \in \overline{\Omega}. \tag{2-5}$$

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Since v_ε is a viscosity supersolution, we see that v_ε is locally Lipschitz continuous by Lemma 2.5. Taking $x = y = x_j$ in (2-5),

$$\frac{j|x_j - y_j|^4}{4} \le v_\varepsilon(x_j) - v_\varepsilon(y_j) \le L|x_j - y_j|,$$

where L is the Lipschitz constant of v_ε. Thus,

$$\frac{j|x_j - y_j|^3}{4} \le L.$$

Upon taking a subsequence if necessary, we can assume that $p_j \to p$. Taking the limit in (2-4),

$$(1+\varepsilon)^h F(x_0, v_\varepsilon(x_0), p) - F(x_0, u(x_0), p) \ge 0.$$

Thus,

$$F(x_0, v_\varepsilon(x_0), p) > (1+\varepsilon)^h F(x_0, v_\varepsilon(x_0), p) \ge F(x_0, u(x_0), p). \tag{2-6}$$

Since $F(x, t, p)$ is nondecreasing in t and $u(x_0) > v_\varepsilon(x_0)$,

$$F(x_0, u(x_0), p) \ge F(x_0, v_\varepsilon(x_0), p),$$

which contradicts (2-6).

Thus, we have $u \le v_\varepsilon$ in Ω. Letting $\varepsilon \to 0$, we obtain $u \le v$ in Ω. □

Remark 2.7. If $F(x, t, p) > 0$, $F(x, t, p)$ is nondecreasing in t and $\tau \to F(x, t, \tau p)$ is nonincreasing in $(\rho, 1]$ for each $(x, t, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, where $0 < \rho < 1$, then the comparison principle still holds if we perturb the supersolution v to $v_\varepsilon = v(1-\varepsilon) + \varepsilon \sup_{\overline{\Omega}} v$, where $0 < \varepsilon < 1 - \rho$.

3. Existence when nonlinear term $\sup_{\Omega\times I\times\mathbb{R}^n} |F(x, t, p)| < \infty$

In this section, we want to establish the existence of viscosity solutions of the boundary value problem (1-8) when the nonlinear term $F(x, t, p)$ is bounded, that is, $\sup_{\Omega\times I\times\mathbb{R}^n} |F(x, t, p)| < \infty$. The method is based on the standard Perron idea, but we must make subtle analysis due to the strong degeneracy of the operator Δ_∞^h.

Proof of Theorem 1.2. Let

$$\mathbb{T} := \{\alpha \in C(\overline{\Omega}) : \Delta_\infty^h \alpha \ge F(x, \alpha, D\alpha) \text{ in } \Omega, \alpha \le \varphi \text{ on } \partial\Omega \text{ and } \alpha \le u^* \text{ in } \Omega\}.$$

Here $\mathbb{T}$ is nonempty since $u_* \in \mathbb{T}$.

Let

$$u(x) := \sup_{\alpha\in\mathbb{T}} \alpha(x), \qquad x \in \overline{\Omega}. \tag{3-1}$$

Setting $v_* := \inf_\Omega u_*$ and $v^* := \sup_\Omega u^*$,

$$v_* \le u_* \le u \le u^* \le v^* \quad \text{in } \overline{\Omega}.$$

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Step 1. We show that u is a viscosity subsolution. Let $\psi \in C^2(\Omega)$ and $u-\psi$ have a local maximum at $x_0 \in \Omega$. Then there exists some small ball $B_\rho(x_0) \subseteq \Omega$ such that $u(x)-\psi(x) \le u(x_0)-\psi(x_0)$ for any $x \in B_\rho(x_0)$. We want to show

$$\Delta_\infty^h \psi(x_0) \ge F(x_0, u(x_0), D\psi(x_0)).$$

According to (3-1), $u(x_0) = \sup_{\alpha\in\mathbb{T}} \alpha(x_0)$. We fix $0<\delta<\rho^{2(h+1)}$ and pick a sequence $\{\alpha_k\}$ in $\mathbb{T}$ such that $u(x_0)-\alpha_k(x_0) < \delta/k$ for each positive integer k. Clearly,

$$\alpha_k(x)-\psi(x) \le u(x)-\psi(x) \le u(x_0)-\psi(x_0) \le \alpha_k(x_0)-\psi(x_0)+\frac{\delta}{k}, \quad x \in B_\rho(x_0). \tag{3-2}$$

That is,

$$\alpha_k(x)-\psi(x)-\frac{\delta}{k} \le \alpha_k(x_0)-\psi(x_0), \quad x \in B_\rho(x_0).$$

Therefore,

$$\alpha_k(x)-[\psi(x)+|x-x_0|^{2(h+1)}] < \alpha_k(x)-\psi(x)-\frac{\delta}{k} \le \alpha_k(x_0)-\psi(x_0)$$

for all $x \in B_\rho(x_0)\backslash\overline{B}_{(\delta/k)^{1/[2(h+1)]}}(x_0)$. This implies that $\alpha_k(x)-[\psi(x)+|x-x_0|^{2(h+1)}]$ attains its maximum at some point $x_k \in \overline{B}_{(\delta/k)^{1/[2(h+1)]}}(x_0)$. In particular,

$$\alpha_k(x_0)-\psi(x_0) \le \alpha_k(x_k)-[\psi(x_k)+|x_k-x_0|^{2(h+1)}]. \tag{3-3}$$

Let $\psi_0(x) := \psi(x)+|x-x_0|^{2(h+1)}$. By a direct calculation,

$$\Delta_\infty^h \psi_0(x_k) = \Delta_\infty^h \psi(x_k) + O((\delta/k)^{h/(h+1)}) \ge F(x_k, \alpha_k(x_k), D\psi_0(x_k)), \tag{3-4}$$

where we use $\alpha_k \in \mathbb{T}$. Combining (3-2) and (3-3),

$$\alpha_k(x_0)-\psi(x_0) \le \alpha_k(x_k)-[\psi(x_k)+|x_k-x_0|^{2(h+1)}] \le u(x_0)-\psi(x_0)-|x_k-x_0|^{2(h+1)},$$

which implies $\lim_{k\to\infty}\alpha_k(x_k) = u(x_0)$. Letting $k\to\infty$ in (3-4),

$$\Delta_\infty^h \psi(x_0) \ge F(x_0, u(x_0), D\psi(x_0)).$$

Therefore, u is a viscosity subsolution.

Step 2. We want to show $u \in C(\overline{\Omega})$ and $u=\varphi$ on $\partial\Omega$. Take two constants $C_* < 0 < C^*$ such that

$$C_* \le \inf\{F(x,t,p) : (x,t,p) \in \Omega\times[v_*, v^*]\times\mathbb{R}^n\}$$

and

$$C^* \ge \sup\{F(x,t,p) : (x,t,p) \in \Omega\times[v_*, v^*]\times\mathbb{R}^n\}.$$

Since u is a viscosity subsolution of $\Delta_\infty^h u = F(x,u,Du)$ and $F(x,u,Du)$ is bounded, u is locally Lipschitz in Ω by Lemma 2.5. Now we proceed to show that u is continuous on $\partial\Omega$ and $u=\varphi$ on $\partial\Omega$. By [26], there exist $\alpha_*, \beta^* \in C(\overline{\Omega})$ such that

$$\Delta_\infty^h \alpha_* = C^* \text{ in } \Omega \quad \text{and} \quad \alpha_* = \varphi \text{ on } \partial\Omega,$$

$$\Delta_\infty^h \beta^* = C_* \text{ in } \Omega \quad \text{and} \quad \beta^* = \varphi \text{ on } \partial\Omega.$$

Since $\Delta_\infty^h u^* \le C^*$ and $\alpha_* \le u^*$ on $\partial\Omega$, we have $\alpha_* \le u^*$ in Ω by the comparison principle (see Remark 2.7). Similarly, we have $u_* \le \beta^*$ in Ω. Let

$$\hat{\alpha} := \max\{\alpha_*, u_*\} \quad \text{and} \quad \hat{\beta} := \min\{\beta^*, u^*\}.$$

Then $\hat{\alpha} \in C(\overline{\Omega}), \hat{\beta} \in C(\overline{\Omega})$, and

$$v_* \le u_* \le \hat{\alpha}, \hat{\beta} \le u^* \le v^* \quad \text{in } \Omega \text{ and } \hat{\alpha} = \hat{\beta} = \varphi \text{ on } \partial\Omega.$$

Note that α_* and β^* are a viscosity subsolution and a viscosity supersolution of (1-8), respectively. By Remark 2.4, we conclude that $\hat{\alpha}$ and $\hat{\beta}$ are a viscosity subsolution and a viscosity supersolution of (1-8), respectively. Thus, $\hat{\alpha} \in \mathbb{T}$ and $\hat{\alpha} \le u$ in $\overline{\Omega}$. For any $z \in \mathbb{T}$, we have $z \le u^*$ in $\overline{\Omega}$. Additionally, by the comparison principle, we also have $z \le \beta^*$ in $\overline{\Omega}$. Therefore, $z \le \hat{\beta}$ in $\overline{\Omega}$. This implies $u \le \hat{\beta}$ in $\overline{\Omega}$. Then we have $\hat{\alpha} \le u \le \hat{\beta}$ in $\overline{\Omega}$.

Since $\hat{\alpha}, \hat{\beta} \in C(\overline{\Omega})$ and $\hat{\alpha} = \hat{\beta} = \varphi$ on $\partial\Omega$, it follows that $u \in C(\overline{\Omega})$ and $u = \varphi$ on $\partial\Omega$.

Step 3. We show that u is a viscosity supersolution. Suppose that it does not hold. Then there exists a point $x_0 \in \Omega$ and a function $\psi \in C^2(\Omega)$ such that $u - \psi$ has a local minimum at x_0, but

$$\Delta_\infty^h \psi(x_0) > F(x_0, u(x_0), D\psi(x_0)). \tag{3-5}$$

Suppose that $u(x_0) = u^*(x_0)$. Then for any x near x_0,

$$u^*(x) - \psi(x) \ge u(x) - \psi(x) \ge u(x_0) - \psi(x_0) = u^*(x_0) - \psi(x_0).$$

Since u^* is a viscosity supersolution,

$$\Delta_\infty^h \psi(x_0) \le F(x_0, u^*(x_0), D\psi(x_0)) = F(x_0, u(x_0), D\psi(x_0)),$$

which is contrary to (3-5).

Now suppose that $u(x_0) < u^*(x_0)$. Let $d(x_0) := \mathrm{dist}(x_0, \partial\Omega)$. For any $x \in B_\rho(x_0)$,

$$u(x) - \psi(x) \ge u(x_0) - \psi(x_0).$$

Define $\phi(x) := \psi(x) + (u(x_0) - \psi(x_0))$. For any $x \in B_\rho(x_0)$,

$$\Delta_\infty^h \phi(x_0) = \Delta_\infty^h \psi(x_0) > F(x_0, \phi(x_0), D\phi(x_0)).$$

Since $F(x, t, p)$ is continuous, one can take $0 < \varepsilon_0 < \min\{1, \rho, (d(x_0)/2)^{4(h+1)}\}$ small enough such that

$$\Delta_\infty^h \phi(x_0) > F(x_0, \phi(x_0) + \varepsilon, D\phi(x_0)) \ \text{ for all } 0 < \varepsilon \le \varepsilon_0. \tag{3-6}$$

For $0 < \varepsilon \le \varepsilon_0$, define $\phi_\varepsilon(x) := \phi(x) - \sqrt{\varepsilon}|x - x_0|^{2(h+1)} + \varepsilon$. By a direct computation,

$$\Delta_\infty^h \phi_\varepsilon(x) = \Delta_\infty^h \phi(x) + O(\sqrt{\varepsilon}|x - x_0|^{2h}), \quad \text{as } x \to x_0.$$

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Thus, by (3-6),

$$\Delta_\infty^h \phi_\varepsilon(x_0) = \Delta_\infty^h \phi(x_0) > F(x_0, \phi(x_0) + \varepsilon, D\phi(x_0)) = F(x_0, \phi_\varepsilon(x_0), D\phi_\varepsilon(x_0)).$$

We claim that there exists a small ε_1, with $0 < \varepsilon_1 < \varepsilon_0$, such that

$$\Delta_\infty^h \phi_{\varepsilon_1}(x) > F(x, \phi_{\varepsilon_1}(x), D\phi_{\varepsilon_1}(x)) \quad \text{for all } x \in B_{\varepsilon_1^{1/[4(h+1)]}}(x_0).$$

Suppose that our claim does not hold. Then for each small $\varepsilon > 0$, there exists an $x_\varepsilon \in B_{\varepsilon^{1/[4(h+1)]}}(x_0)$ such that $\Delta_\infty^h \phi_\varepsilon(x_\varepsilon) \le F(x_\varepsilon, \phi_\varepsilon(x_\varepsilon), D\phi_\varepsilon(x_\varepsilon))$. Since $x_\varepsilon \to x_0$ as $\varepsilon \to 0$, we observe that

$$\lim_{\varepsilon\to 0} \Delta_\infty^h \phi_\varepsilon(x_\varepsilon) = \Delta_\infty^h \phi(x_0) \quad \text{and} \quad \lim_{\varepsilon\to 0} F(x_\varepsilon, \phi_\varepsilon(x_\varepsilon), D\phi_\varepsilon(x_\varepsilon)) = F(x_0, \phi(x_0), D\phi(x_0)).$$

We conclude that $\Delta_\infty^h \phi(x_0) \le F(x_0, \phi(x_0), D\phi(x_0))$, which is a contradiction and the claim is proved.

Since $\phi(x_0) = u(x_0) < u^*(x_0)$, we can suppose that ε_1 is small enough and for all $x \in B_{\varepsilon_1^{1/[4(h+1)]}}(x_0)$, $\phi(x) \le \phi_{\varepsilon_1}(x) \le u^*(x)$. Since $u(x_0) < u(x_0) + \varepsilon_1 = \phi(x_0) + \varepsilon_1 = \phi_{\varepsilon_1}(x_0)$, there exists $0 < s_1 < \varepsilon_1^{1/[4(h+1)]}$ such that $u(x) < \phi_{\varepsilon_1}(x)$ for all $x \in B_{s_1}(x_0)$. We note that

$$u(x) \ge \phi(x) \quad \text{for all } x \in B_\rho(x_0),$$

and

$$u(x) - \phi_{\varepsilon_1}(x) = u(x) - \phi(x) + \sqrt{\varepsilon_1}|x - x_0|^{2(h+1)} - \varepsilon_1 > 0$$
$$\text{for all } x \in B_\rho(x_0) \backslash \overline{B}_{\varepsilon_1^{1/[4(h+1)]}}(x_0).$$

To sum up,

$$\begin{cases} (1)\ \Delta_\infty^h \phi_{\varepsilon_1}(x) > F(x, \phi_{\varepsilon_1}(x), D\phi_{\varepsilon_1}(x)) & \text{for all } x \in B_{\varepsilon_1^{1/[4(h+1)]}}(x_0), \\ (2)\ \phi_{\varepsilon_1}(x) < u^*(x) & \text{for all } x \in B_{\varepsilon_1^{1/[4(h+1)]}}(x_0), \\ (3)\ u(x) < \phi_{\varepsilon_1}(x) & \text{for all } x \in B_{s_1}(x_0), \\ (4)\ u(x) > \phi_{\varepsilon_1}(x) & \text{for all } x \in B_\rho(x_0) \backslash \overline{B}_{\varepsilon_1^{1/[4(h+1)]}}(x_0). \end{cases} \tag{3-7}$$

Now define

$$\hat{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \backslash \overline{B}_{\varepsilon_1^{1/[4(h+1)]}}(x_0), \\ \sup\{\phi_{\varepsilon_1}(x), u(x)\} & \text{if } x \in B_{\varepsilon_1^{1/[4(h+1)]}}(x_0). \end{cases}$$

It is obvious that $\hat{u} \in C(\overline{\Omega})$ and $u_* \le u \le \hat{u} \le u^*$ in Ω. Next we want to show that $\hat{u} \in \mathbb{T}$. Take $\hat{\psi} \in C^2(\Omega)$ such that $\hat{u} - \hat{\psi}$ has a local maximum at some point $y \in \Omega$, that is, $\hat{u}(x) - \hat{\psi}(x) \le \hat{u}(y) - \hat{\psi}(y)$ for x in some ball $B_\delta(y)$. It is clear that $\hat{u}(y) = u(y)$ or $\hat{u}(y) = \phi_{\varepsilon_1}(y)$. If $\hat{u}(y) = u(y)$, we note that $u \le \hat{u}$ in Ω. For all $x \in B_\delta(y)$,

$$u(x) - \hat{\psi}(x) \le \hat{u}(x) - \hat{\psi}(x) \le \hat{u}(y) - \hat{\psi}(y) = u(y) - \hat{\psi}(y).$$

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Thus, $u - \hat{\psi}$ has a local maximum at y. Since u is a viscosity subsolution, we have $\Delta_\infty^h \hat{\psi}(y) \geq F(y, u(y), D\hat{\psi}(y)) = F(y, \hat{u}(y), D\hat{\psi}(y))$. If $\hat{u}(y) = \phi_{\varepsilon_1}(y)$, we can see that $u(y) < \phi_{\varepsilon_1}(y)$. According to (3-7)(4), we have $y \in B_{\varepsilon_1^{1/[4(h+1)]}}(x_0)$. Then note that $\phi_{\varepsilon_1} \leq \hat{u}$ and for any $x \in B_{\varepsilon_1^{1/[4(h+1)]}}(x_0) \bigcap B_\delta(y)$,

$$\phi_{\varepsilon_1}(x) - \hat{\psi}(x) \leq \hat{u}(x) - \hat{\psi}(x) \leq \hat{u}(y) - \hat{\psi}(y) = \phi_{\varepsilon_1}(y) - \hat{\psi}(y).$$

Thus, $\phi_{\varepsilon_1} - \hat{\psi}$ has a local maximum at y. This implies that $\Delta_\infty^h \phi_{\varepsilon_1}(y) \leq \Delta_\infty^h \hat{\psi}(y)$. Together with (3-7)(1), one has $\Delta_\infty^h \hat{\psi}(y) \geq F(y, \phi_{\varepsilon_1}(y), D\phi_{\varepsilon_1}(y)) = F(y, \hat{u}(y), D\hat{\psi}(y))$. In any case, we show that $\hat{u}$ is a viscosity subsolution, that is, $\hat{u} \in \mathbb{T}$. By (3-7)(3), we see that $\hat{u} = \phi_{\varepsilon_1} > u$ in $B_{s_1}(x_0)$, which contradicts the definition of u. Therefore, u is a viscosity supersolution.

We have completed the proof that u is a viscosity solution to Problem (1-8) in Ω. ◻

4. Singular boundary value problem in a bounded domain

In this section, we are devoted to the existence of the viscosity solution of the singular boundary value problem. The key is to deal with the singularity of the term $F(x, u, Du)$. We choose an appropriate cut-off function and combine the truncation, compactness method and Theorem 1.2 to deal with the difficulty. We consider the following problem:

$$\begin{cases} \Delta_\infty^h u = F(x, u, Du) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{4-1}$$

where $\Omega \subseteq \mathbb{R}^n$, $F : \Omega \times \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function and $\varphi \in C(\partial\Omega)$ is nonnegative. Note that $\varphi \equiv 0$ is possible and that $F(x, t, p)$ may exhibit singularity at $t = 0$. The corresponding theorem is as follows.

THEOREM 4.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, $F : \Omega \times \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function and $\varphi \in C(\partial\Omega)$ be nonnegative. If $\underline{u} \in C(\overline{\Omega})$ and $\overline{u} \in C(\overline{\Omega})$ are a viscosity subsolution and a viscosity supersolution of (4-1) in Ω, respectively, such that $0 < \underline{u} \leq \overline{u}$ in Ω and $\underline{u} = \varphi = \overline{u}$ on $\partial\Omega$, then Problem (4-1) admits a viscosity solution $u \in C(\overline{\Omega})$ such that $\underline{u} \leq u \leq \overline{u}$ in $\overline{\Omega}$.*

PROOF. Define the cut-off function $\hat{F}(x, t, p) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by

$$\hat{F}(x, t, p) = \begin{cases} F(x, t, p) & \text{for } t \geq \underline{u}(x), \\ F(x, \underline{u}(x), p) & \text{for } t < \underline{u}(x). \end{cases} \tag{4-2}$$

Since $\underline{u} > 0$ in Ω, we have $\sup_{\Omega\times I\times\mathbb{R}^n} |\hat{F}(x,t,p)| < \infty$ for any compact interval $I \subseteq \mathbb{R}$. Let $\{\Omega_j\}$ be a sequence of sub-domains of Ω such that

$$\Omega_j \Subset \Omega_{j+1} \quad \text{and} \quad \Omega = \bigcup_{k=1}^{\infty} \Omega_k, \quad (j = 1, 2, \ldots).$$

For each positive integer j, we consider the following Dirichlet problem:

$$\begin{cases} \Delta_\infty^h u = \hat{F}(x,u,Du) & \text{in } \Omega_j, \\ u = \underline{u} & \text{on } \partial\Omega_j. \end{cases} \tag{4-3}$$

Since $\overline{u}$ is a viscosity supersolution of (4-1) and $0 < \underline{u} \le \overline{u}$ in Ω,

$$\Delta_\infty^h \overline{u} \le F(x,\overline{u},D\overline{u}) = \hat{F}(x,\overline{u},D\overline{u}) \quad \text{in } \Omega_j \quad \text{and} \quad \overline{u} \ge \underline{u} \quad \text{on } \partial\Omega_j$$

in the viscosity sense, where we use (4-2). That is, $\overline{u}$ is a viscosity supersolution of (4-3) in Ω_j for each positive integer j. Similarly, $\underline{u}$ is a viscosity subsolution of (4-3) in Ω_j. Since $\underline{u} \le \overline{u}$ in Ω, according to Theorem 1.2, we can find a viscosity solution $\tilde{u}_j \in C(\overline{\Omega}_j)$ of (4-3) such that $\underline{u} \le \tilde{u}_j \le \overline{u}$ in $\overline{\Omega}_j$. We extend u_j to $\overline{\Omega}$ by defining

$$u_j = \begin{cases} \tilde{u}_j & \text{in } \Omega_j, \\ \underline{u} & \text{on } \overline{\Omega} \setminus \Omega_j. \end{cases}$$

Then we obtain a sequence $\{u_j\}$ in $C(\overline{\Omega})$ with $\underline{u} \le u_j \le \overline{u}$ in $\overline{\Omega}$ for all j. In particular, $\{u_j\}$ is uniformly bounded in $\overline{\Omega}$. We also note that $\{u_j\}$ is locally Lipschitz continuous in Ω from Lemma 2.5. Thus, $\{u_j\}$ is equicontinuous. Therefore, there exists a subsequence of $\{u_j\}$ that converges locally uniformly to some $u \in C(\Omega)$. Since $\underline{u} \le u \le \overline{u}$ in $\overline{\Omega}$ and $\underline{u} = u = \overline{u}$ on $\partial\Omega$, we have $u \in C(\overline{\Omega})$.

By (4-2) and $u \ge \underline{u} > 0$ in Ω, we have $\hat{F}(x,u,Du) = F(x,u,Du)$ in Ω. If we want to show that u is a viscosity solution of $\Delta_\infty^h u = F(x,u,Du)$, we only need to show that u is a viscosity solution of $\Delta_\infty^h u = \hat{F}(x,u,Du)$ in Ω. First, we show $\Delta_\infty^h u \le \hat{F}(x,u,Du)$ in the viscosity sense.

Suppose that $\psi \in C^2(\Omega)$ and $u - \psi$ has a local minimum at some $x_0 \in \Omega$. Then,

$$u(x) - \psi(x) \ge u(x_0) - \psi(x_0),$$

where $x \in B_r(x_0) \Subset \Omega_l$ for some small $r > 0$ and some positive integer l. Fix $\varepsilon > 0$ small enough and let $x_j \in \overline{B}_r(x_0)$ be a point of minimum of

$$u_j(x) - \Big(\psi(x) - \frac{\varepsilon}{2}|x - x_0|^2\Big), \quad j \ge l.$$

Then,

$$u_j(x_j) - \Big(\psi(x_j) - \frac{\varepsilon}{2}|x_j - x_0|^2\Big) \le u_j(x_0) - \psi(x_0). \tag{4-4}$$

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Since $x_j \in \overline{B}_r(x_0)$, we assume that $x_j \to \hat{x}$ for some $\hat{x} \in \overline{B}_r(x_0)$. Taking the limit in (4-4) as $j \to \infty$,

$$u(\hat{x}) - \Big(\psi(\hat{x}) - \frac{\varepsilon}{2}|\hat{x} - x_0|^2\Big) \le u(x_0) - \psi(x_0).$$

Therefore,

$$\frac{\varepsilon}{2}|\hat{x} - x_0|^2 \le u(x_0) - \psi(x_0) - (u(\hat{x}) - \psi(\hat{x})) \le 0.$$

This yields $\hat{x} = x_0$. Since u_j is a viscosity solution of (4-3) and x_j is a point of minimum of $u_j(x) - (\psi(x) - \varepsilon/2|x - x_0|^2)$ in $B_r(x_0)$, we take $\phi_\varepsilon(x) := \psi(x) - \varepsilon/2|x - x_0|^2$ as a test function. Then,

$$\Delta_\infty^h \phi_\varepsilon(x_j) \le \hat{F}(x_j, u_j(x_j), D\phi_\varepsilon(x_j)).$$

Recalling that $u_j \to u$ uniformly in $B_r(x_0)$ and taking the limit as $j \to \infty$,

$$\Delta_\infty^h \phi_\varepsilon(x_0) \le \hat{F}(x_0, u(x_0), D\phi_\varepsilon(x_0)).$$

Letting $\varepsilon \to 0$,

$$\Delta_\infty^h \psi(x_0) \le \hat{F}(x_0, u(x_0), D\psi(x_0)),$$

that is, u is a viscosity supersolution of (4-3). Thus, u is a viscosity supersolution of (4-1).

Similarly, one can prove that u satisfies $\Delta_\infty^h u \ge \hat{F}(x, u, Du)$ in the viscosity sense. We leave it to the reader. □

By Theorem 4.1, we can get the following corollary immediately.

COROLLARY 4.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $\lambda > 0$ be a parameter. Suppose that $f(x, t, p) : \Omega \times \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function and may exhibit singularity at $t = 0$. If $\underline{u} \in C(\overline{\Omega})$ and $\overline{u} \in C(\overline{\Omega})$ are a viscosity subsolution and a viscosity supersolution of*

$$\Delta_\infty^h u = \lambda f(x, u, Du) \quad \text{in } \Omega, \tag{4-5}$$

respectively, such that $0 < \underline{u} \le \overline{u}$ in Ω and $\underline{u} = 0 = \overline{u}$ on $\partial\Omega$, then (4-5) admits a viscosity solution $u \in C(\overline{\Omega})$ such that $\underline{u} \le u \le \overline{u}$ in $\overline{\Omega}$.

If we take $F(x, u, Du) = \lambda f(x, u, Du)$, where $\lambda > 0$ and f satisfies conditions **(F-1)**, **(F-2)**, then by Corollary 4.2, the existence of a viscosity solution reduces to finding a viscosity supersolution and subsolution.

LEMMA 4.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $f : \Omega \times \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function. If f satisfies Condition **(F-2)** and b satisfies **(B-w)**, then there exists a constant λ_Ω^* depending on Ω, g_∞ and $\|w_\Omega\|_\infty$ such that Problem (1-1) admits a viscosity supersolution $v_{\Omega,\lambda} \in C(\overline{\Omega})$ for any λ with $0 < \lambda < \lambda_\Omega^*$. Moreover, $v_{\Omega,\lambda}$ satisfies*

$$\Delta_\infty^h v_{\Omega,\lambda} \le -\lambda b(x)\Gamma(v_{\Omega,\lambda}) \quad \text{in } \Omega. \tag{4-6}$$

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PROOF. Define a continuous function $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$

$$G(t,\tau) := \begin{cases} t^h \sup\limits_{t\le s\le\tau} g(s)s^{-h} & 0 < t \le \tau, \\ t^h g(\tau)\tau^{-h} & 0 < \tau \le t. \end{cases}$$

It is easy to see that $G(t,\tau)t^{-h}$ is nonincreasing in t. Let

$$\Gamma(t) = \frac{2}{t}\int_{t/2}^{t} \frac{G(s,t)}{s^h}\,ds, \quad t > 0. \tag{4-7}$$

Noting that Γ is a C^1 function,

$$\begin{aligned} \Gamma'(t) &= -\frac{2}{t^2}\int_{t/2}^{t} \frac{G(s,t)}{s^h}\,ds + \frac{2}{t}\Big(\frac{G(t,t)}{t^h} - \frac{1}{2}\frac{G(t/2,t)}{(t/2)^h}\Big) \\ &\le -\frac{1}{t}\frac{G(t,t)}{t^h} + \frac{2}{t}\frac{G(t,t)}{t^h} - \frac{1}{t}\frac{G(t/2,t)}{(t/2)^h} \\ &= \frac{1}{t}\Big(\frac{G(t,t)}{t^h} - \frac{G(t/2,t)}{(t/2)^h}\Big) \\ &\le 0. \end{aligned}$$

Hence, Γ is a nonincreasing function in $\mathbb{R}^+$. It is obvious that

$$\frac{g(t)}{t^h} = \frac{G(t,t)}{t^h} \le \Gamma(t) \le \frac{G(\frac{t}{2},t)}{(\frac{t}{2})^h} = \sup_{(t/2)\le s\le t} \frac{g(s)}{s^h} \le \sup_{(t/2)\le s<\infty} \frac{g(s)}{s^h}. \tag{4-8}$$

Therefore,

$$\lim_{t\to\infty} \Gamma(t) = g_\infty.$$

Now we suppose that $0 < g_\infty < \infty$ and, in this case, we define

$$\lambda_\Omega^* := \frac{1}{g_\infty \|w_\Omega\|_\infty^h}. \tag{4-9}$$

For $0 < \lambda < \lambda_\Omega^*$,

$$\lim_{t\to\infty} \Gamma(t) = \frac{1}{\lambda_\Omega^* \|w_\Omega\|_\infty^h} < \frac{1}{\lambda \|w_\Omega\|_\infty^h}.$$

Consider the function

$$\Psi(t) := \int_0^t \frac{1}{\sqrt[h]{\Gamma(s)}}\,ds, \quad t > 0.$$

We have

$$\lim_{t\to\infty} \frac{\Psi(t)}{t} = \lim_{t\to\infty} \Psi'(t) = \lim_{t\to\infty} \frac{1}{\sqrt[h]{\Gamma(t)}} > \sqrt[h]{\lambda}\|w_\Omega\|_\infty.$$

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Thus, we can find $\theta \geq 1$ large enough such that

$$\Psi(\theta) \geq \theta \sqrt[h]{\lambda} \|w_\Omega\|_\infty. \tag{4-10}$$

In the other case, we suppose $g_\infty = \infty$. One can verify that

$$\alpha_* := \sup_{t \geq 1} \frac{\Psi(t)}{t} < \infty.$$

Set

$$\lambda_\Omega^* := \left(\frac{\alpha_*}{\|w_\Omega\|_\infty} \right)^h. \tag{4-11}$$

Then for $0 < \lambda < \lambda_\Omega^*$,

$$\sqrt[h]{\lambda} \|w_\Omega\|_\infty < \sqrt[h]{\lambda_\Omega^*} \|w_\Omega\|_\infty = \alpha_*.$$

Therefore, there exists $\gamma_\infty \geq 1$, depending on λ, such that

$$\sqrt[h]{\lambda} \|w_\Omega\|_\infty < \frac{\Psi(\gamma_\infty)}{\gamma_\infty}. \tag{4-12}$$

Hence, by (4-10) and (4-12), there exists $\theta \geq 1$ such that

$$\Psi(\theta) > \theta \sqrt[h]{\lambda} \|w_\Omega\|_\infty. \tag{4-13}$$

Let Φ be the inverse of Ψ, that is, Φ satisfies

$$\int_0^{\Phi(t)} \frac{1}{\sqrt[h]{\Gamma(s)}}\, ds = t, \quad t \geq 0.$$

Define

$$v_{\Omega,\lambda}(x) := \Phi(\theta \sqrt[h]{\lambda} w_\Omega(x)), \quad x \in \Omega. \tag{4-14}$$

It is clear that $v_{\Omega,\lambda} > 0$ in Ω and $v_{\Omega,\lambda} = 0$ on $\partial\Omega$.

Next we want to show that $v_{\Omega,\lambda}$ is a viscosity supersolution of Problem (1-1). Let $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $v_{\Omega,\lambda} - \varphi$ has a local minimum at $x_0 \in \Omega$. Without loss of generality, we can suppose that x_0 is a global minimum point of $v_{\Omega,\lambda} - \varphi$ in Ω and $v_{\Omega,\lambda}(x_0) = \varphi(x_0)$. Setting

$$\eta(x) := \frac{1}{\theta \sqrt[h]{\lambda}} \Psi(\varphi) \in C^2(\Omega),$$

we have $w_\Omega(x_0) = \eta(x_0)$ and $w_\Omega - \eta$ has a minimum at x_0. Since w_Ω is a viscosity supersolution of Problem (1-10),

$$\Delta_\infty^h \eta(x_0) \leq -b(x_0).$$

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Since $\Psi'(\varphi) > 0$ and $\Psi''(\varphi) \geq 0$, one can easily check that

$$\begin{aligned}
\Delta_\infty^h \eta &= \Big(\frac{1}{\theta \sqrt[h]{\lambda}}\Big)^h (\Psi''(\varphi)[\Psi'(\varphi)]^{h-1}|D\varphi|^{h+1} + [\Psi'(\varphi)]^h \Delta_\infty^h \varphi) \\
&\geq \Big(\frac{1}{\theta \sqrt[h]{\lambda}}\Big)^h [\Psi'(\varphi)]^h \Delta_\infty^h \varphi \\
&= \Big(\frac{1}{\theta \sqrt[h]{\lambda}}\Big)^h \frac{1}{\Gamma(\varphi)} \Delta_\infty^h \varphi.
\end{aligned}$$

Then we have the following inequalities at x_0

$$\begin{aligned}
\Delta_\infty^h \varphi &\leq (\theta \sqrt[h]{\lambda})^h \Gamma(\varphi) \Delta_\infty^h \eta \\
&\leq -b(\theta \sqrt[h]{\lambda})^h \Gamma(\varphi) \\
&\leq -b(\theta \sqrt[h]{\lambda})^h \Big(\frac{g(\varphi)}{\varphi^h}\Big) \\
&= -\lambda b \Big(\frac{\theta}{\varphi}\Big)^h g(\varphi) \\
&\leq -\lambda b g(\varphi) \\
&\leq \lambda f(x_0, \varphi, D\varphi),
\end{aligned} \tag{4-15}$$

where we use (4-8) and $\varphi(x_0) = v_{\Omega,\lambda}(x_0) = \Phi(\theta \sqrt[h]{\lambda} w_\Omega(x_0)) \leq \Phi(\theta \sqrt[h]{\lambda} \|w_\Omega\|_\infty) \leq \theta$. This shows that $v_{\Omega,\lambda}$ is a viscosity supersolution of Problem (1-1).

Finally, since $\theta \geq 1$, it follows from (4-15) that $\Delta_\infty^h \varphi \leq -\lambda b(x)\Gamma(\varphi)$. Therefore,

$$\Delta_\infty^h v_{\Omega,\lambda} \leq -\lambda b(x) \Gamma(v_{\Omega,\lambda}). \qquad \square$$

REMARK 4.4. If $g_\infty = 0$, then Problem (1-1) admits a positive supersolution for any $0 < \lambda < \infty$, provided that the hypotheses in Lemma 4.3 hold.

Now we proceed to construct a positive viscosity subsolution to Problem (1-1).

LEMMA 4.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $f : \Omega \times \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying Condition **(F-1)**. Then for any $\lambda > \Lambda_1(\Omega) k_0^{-1}$, Problem (1-1) admits a positive viscosity subsolution $\phi_{\Omega,\lambda} \in C(\overline{\Omega})$.*

PROOF. Fix λ such that $\lambda > \Lambda_1(\Omega) k_0^{-1}$. First, if the constant $0 < k_0 < \infty$, where k_0 is defined in (1-9), then there exists a $\delta := \delta(\lambda)$ with $0 < \delta < t_0$ such that

$$k(t) \geq \Big(\frac{\Lambda_1(\Omega) k_0^{-1}}{\lambda}\Big) k_0 t^h = \frac{\Lambda_1(\Omega)}{\lambda} t^h \quad \text{for all } 0 < t \leq \delta.$$

However, if $k_0 = \infty$, then for any $\lambda > 0$, there exists $0 < \delta < t_0$ such that

$$k(t) \geq \frac{\Lambda_1(\Omega)}{\lambda} t^h \quad \text{for all } 0 < t \leq \delta.$$

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In any case, let $\phi_{\Omega,\lambda}$ be a positive eigenfunction of Problem (1-11) corresponding to the principal eigenvalue $\Lambda_1(\Omega)$ such that $0 < \phi_{\Omega,\lambda} \le \min\{1, \delta(\lambda)\}$ in Ω. Then under the assumption on λ and condition **(F-1)**,

$$\Delta_\infty^h \phi_{\Omega,\lambda} = -\Lambda_1(\Omega)a(x)\phi_{\Omega,\lambda}^h \ge -\lambda a(x)k(\phi_{\Omega,\lambda}) \ge \lambda f(x, \phi_{\Omega,\lambda}, D\phi_{\Omega,\lambda})$$

in Ω. This shows that $\phi_{\Omega,\lambda}$ is a viscosity subsolution of Problem (1-1). □

Now we establish the existence of a viscosity solution of Problem (1-1) by Perron's method.

PROOF OF THEOREM 1.3. When $\lambda > \Lambda_1(\Omega)k_0^{-1}$, according to Lemma 4.5, we have a viscosity subsolution $\phi_{\Omega,\lambda}$ of Problem (1-1) and $\phi_{\Omega,\lambda} \le 1$ in Ω. Then,

$$\begin{aligned}\Delta_\infty^h \phi_{\Omega,\lambda} &\ge \lambda f(x, \phi_{\Omega,\lambda}, D\phi_{\Omega,\lambda})\\ &\ge -\lambda b(x)g(\phi_{\Omega,\lambda})\\ &\ge -\lambda b(x)\phi_{\Omega,\lambda}^h\Gamma(\phi_{\Omega,\lambda})\\ &\ge -\lambda b(x)\Gamma(\phi_{\Omega,\lambda})\end{aligned}$$

in the viscosity sense, where Γ is the function defined in (4-7).

When $0 < \lambda < \lambda_\Omega^*$, according to Lemma 4.3, we know that $v_{\Omega,\lambda}$ is a viscosity supersolution of Problem (1-1) and satisfies (4-6).

Now we claim that $\phi_{\Omega,\lambda} \le v_{\Omega,\lambda}$ in Ω. Assume that $\phi_{\Omega,\lambda} \le v_{\Omega,\lambda}$ is not valid. Let

$$\mathbb{D} := \{x \in \Omega : \phi_{\Omega,\lambda}(x) > v_{\Omega,\lambda}(x)\}.$$

Since Γ is nonincreasing,

$$\Delta_\infty^h \phi_{\Omega,\lambda} \ge -\lambda b(x)\Gamma(\phi_{\Omega,\lambda}) \ge -\lambda b(x)\Gamma(v_{\Omega,\lambda}) \quad \text{in } \mathbb{D}.$$

Then in $\mathbb{D}$,

$$\Delta_\infty^h \phi_{\Omega,\lambda} \ge -\lambda b(x)\Gamma(v_{\Omega,\lambda}) \quad \text{and} \quad \Delta_\infty^h v_{\Omega,\lambda} \le -\lambda b(x)\Gamma(v_{\Omega,\lambda}).$$

According to the comparison principle (Theorem 1.1), we see that $\phi_{\Omega,\lambda} \le v_{\Omega,\lambda}$ in $\mathbb{D}$, which is a contradiction.

Furthermore, we see that $0 < \phi_{\Omega,\lambda} \le v_{\Omega,\lambda}$ in Ω and $\phi_{\Omega,\lambda}(x) = v_{\Omega,\lambda} = 0$ on $\partial\Omega$. By Corollary 4.2, we conclude that Problem (1-1) admits a viscosity solution $u \in C(\overline{\Omega})$ such that $\phi_{\Omega,\lambda} \le u \le v_{\Omega,\lambda}$ in $\overline{\Omega}$. □

5. Singular problem in the entire Euclidean space

In this section, we consider Problem (1-12) and establish the existence of viscosity solutions in the entire Euclidean space.

First, we show that Condition **(B-w)** holds when $\Omega = \mathbb{R}^n$. To prove this we recall the following lemma in [29].

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LEMMA 5.1. *For any fixed constant* $a \neq 0$, *the equation* $\Delta_\infty^h w = 2a$ *admits a viscosity solution*

$$v_{x_0,BC}(x) = \frac{1}{2a(h+1)}[2ah|x - x_0| + hB]^{(h+1)/h} + C$$

in $D(x_0, B) = \{x \in \mathbb{R}^n : 2a|x - x_0| > -B \text{ and } x \neq x_0\}$, *where* B, C *are arbitrary constants.*

Now, we establish the existence of the viscosity solution of Problem (1-10) when $\Omega = \mathbb{R}^n$.

LEMMA 5.2. *Let* $b(x) \in C(\mathbb{R}^n)$ *be positive and bounded. Then the problem*

$$\begin{cases} \Delta_\infty^h w = -b(x) & \text{in } \mathbb{R}^n, \\ w(x) \to 0 & \text{as } |x| \to \infty \end{cases} \tag{5-1}$$

admits a viscosity solution $w := w_{\mathbb{R}^n}$.

PROOF. Let $B_r := B(O, r)$ be the ball of radius r centred at the origin O. By [26], we note that the problem

$$\begin{cases} \Delta_\infty^h w = -b(x) & \text{in } B_r, \\ w > 0 & \text{in } B_r, \\ w = 0 & \text{on } \partial B_r \end{cases} \tag{5-2}$$

admits a viscosity solution w_r. Taking $2a = \inf_{\mathbb{R}^n}(-b(x))$ in Lemma 5.1, it is obvious that $v_{x_0,BC}(x)$ is a viscosity supersolution of the equation $\Delta_\infty^h w = -b(x)$ in B_r. Then,

$$0 < w_r \leq v_{x_0,BC}(x) \quad \text{in } B_r.$$

This implies that the sequence $\{w_r\}$ is uniformly bounded in $\mathbb{R}^n$. According to Lemma 2.5, we note that the sequence $\{w_r\}$ is locally uniformly Lipschitz. Therefore, $\{w_r\}$ is equicontinuous. Then we can obtain a subsequence that converges locally uniformly to some w and $0 < w \leq v_{x_0,BC}(x)$ in $\mathbb{R}^n$.

Then we want to show that w is a viscosity solution of $\Delta_\infty^h w = -b(x)$ in $\mathbb{R}^n$. We can use the similar argument to the one in the proof of Theorem 4.1. First, we show $\Delta_\infty^h w \leq -b(x)$ in the viscosity sense.

Let $\varphi \in C^2(\mathbb{R}^n)$ and suppose $w - \varphi$ has a local minimum at some $x_0 \in \mathbb{R}^n$, that is,

$$w(x) - \varphi(x) \geq w(x_0) - \varphi(x_0), \quad x \in B_\delta(x_0) \subset B_s(x_0)$$

for some small $\delta > 0$ and some positive integer s. We take $\varepsilon > 0$ small enough and let $x_r \in \overline{B}_\delta(x_0)$ be a point of minimum of

$$w_r(x) - \Big(\varphi(x) - \frac{\varepsilon}{2}|x - x_0|^2\Big), \quad r \geq s.$$

Then,

$$w_r(x_r) - \Big(\varphi(x_r) - \frac{\varepsilon}{2}|x_r - x_0|^2\Big) \le w_r(x_0) - \varphi(x_0). \tag{5-3}$$

We can assume that $x_r \to \hat{x}$ for some $\hat{x} \in \overline{B}_r(x_0)$. As $r \to \infty$, it follows from (5-3) that

$$w(\hat{x}) - \Big(\varphi(\hat{x}) - \frac{\varepsilon}{2}|\hat{x} - x_0|^2\Big) \le w(x_0) - \varphi(x_0).$$

That is,

$$\frac{\varepsilon}{2}|\hat{x} - x_0|^2 \le w(x_0) - \varphi(x_0) - (w(\hat{x}) - \varphi(\hat{x})) \le 0.$$

This implies $\hat{x} = x_0$. Since w_r is a viscosity solution of (5-2) and x_r is a point of minimum of $w_r(x) - (\varphi(x) - \frac{\varepsilon}{2}|x - x_0|^2)$ in $B_r(x_0)$, we take a test function $\phi_\varepsilon(x) := \varphi(x) - \frac{\varepsilon}{2}|x - x_0|^2$. Then,

$$\Delta_\infty^h \phi_\varepsilon(x_r) \le -b(x_r).$$

Taking the limit as $r \to \infty$,

$$\Delta_\infty^h \phi_\varepsilon(x_0) \le -b(x_0).$$

Letting $\varepsilon \to 0$,

$$\Delta_\infty^h \varphi(x_0) \le -b(x_0),$$

that is, w is a viscosity supersolution of $\Delta_\infty^h w = -b(x)$ in $\mathbb{R}^n$. Similarly, one can prove that w is a viscosity subsolution of $\Delta_\infty^h w = -b(x)$ in $\mathbb{R}^n$. Clearly, $w(x) \to 0$ as $|x| \to \infty$. Therefore, w is a viscosity solution of (5-1). □

By the comparison principle, we have $w_\Omega \le w_{\mathbb{R}^n}$ in Ω. Furthermore, according to Definitions (4-9) and (4-11), we see that $\lambda^*_{\mathbb{R}^n} \le \lambda^*_\Omega$. Then for $0 < \lambda < \lambda^*_{\mathbb{R}^n}$, we have $v_{\Omega,\lambda} \le v_\lambda$ in Ω according to Definition (4-14), where $v_\lambda := v_{\mathbb{R}^n,\lambda}$ is a viscosity supersolution of Problem (1-12) in $\mathbb{R}^n$. Thus, we have $0 < u_{\Omega,\lambda} \le v_\lambda$ in Ω, where $u_{\Omega,\lambda}$ is the viscosity solution of Problem (1-1) in Ω.

Now, we give the proof of the existence result by an approximation procedure in the entire Euclidean space.

Proof of Theorem 1.4. Let $\lambda^* := \lambda^*_{\mathbb{R}^n}$ be the positive constant in Theorem 1.3 corresponding to $\Omega = \mathbb{R}^n$ and $\Lambda_1(B(O,1))k_0^{-1} < \lambda < \lambda^*$. For each positive integer k, let $B_k := B(O,k)$ be the ball of radius k centred at the origin O. Since $\Lambda_1(B(O,k)) \le \Lambda_1(B(O,1))$ and $\lambda^* \le \lambda^*_{B(O,k)}$, we have $\Lambda_1(B(O,k))k_0^{-1} < \lambda < \lambda^*_{B(O,k)}$. Since f satisfies Conditions **(F-1)**, **(F-2)** and b satisfies **(B-w)** in $\Omega = B(O,k)$, by Theorem 1.3, we obtain a viscosity solution $u_k := u_{B_k,\lambda} \in C(\overline{B}_k)(k = 1, 2, \ldots)$ of the singular boundary value problem

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$$\begin{cases} \Delta_\infty^h u = \lambda f(x,u,Du) & \text{in } B_k, \\ u(x) = 0 & \text{on } \partial B_k. \end{cases} \tag{5-4}$$

Letting v_λ be a viscosity supersolution of Problem (1-12) given by the method in Lemma 4.3 in $\Omega = \mathbb{R}^n$,

$$0 < u_k \leq v_\lambda \quad \text{in } B_k. \tag{5-5}$$

Equation (5-5) shows that the sequence $\{u_k\}$ is uniformly bounded in $\mathbb{R}^n$. According to Lemma 2.5, we see that the sequence $\{u_k\}$ is locally uniformly Lipschitz. Thus, $\{u_k\}$ is equicontinuous. Then we can obtain a subsequence that converges locally uniformly to some $u_\lambda \in C(\mathbb{R}^n)$ and $0 \leq u_\lambda \leq v_\lambda$ in $\mathbb{R}^n$.

Next, for a given positive integer l, let $\varphi_l := \varphi_{B_l}$ be an eigenfunction of Problem (1-11) corresponding to the principal eigenvalue $\Lambda_1(B(O,l))$ on the ball $B_l := B(O,l)$. For any viscosity solution u_k of (5-4), we want to show that

$$u_k \geq \varphi_l \quad \text{in } B(O,l) \text{ for all } k > l. \tag{5-6}$$

For each $(x,t,p) \in \mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{R}^n$, let

$$\hat{f}(x,t,p) = \frac{f(x,e^t,e^t p)}{a(x)(e^t)^h}.$$

For convenience, we set $\Lambda_1 := \Lambda_1(B(O,1))$ and $\Lambda_l := \Lambda_1(B(O,l))$. Note that

$$\lambda > \Lambda_1 k_0^{-1},$$

where k_0 is defined in (1-9) and k_0^{-1} is interpreted to be zero if $k_0 = \infty$. Fix τ such that

$$\Lambda_1 < \tau < k_0 \lambda. \tag{5-7}$$

Then by the definition of k_0, there exists $0 < t^* := t^*(\lambda) \leq t_0$ such that $k(t) \geq (\frac{\tau}{\lambda}) t^h$ for $0 < t < t^*$. Thus,

$$f(x,t,p) \leq -a(x)k(t) \leq -\frac{\tau}{\lambda} a(x) t^h, \quad (x,t,p) \in \mathbb{R}^n \times (0,t^*) \times \mathbb{R}^n.$$

Therefore,

$$\hat{f}(x,t,p) \leq -\frac{\tau}{\lambda}, \quad (x,t,p) \in \mathbb{R}^n \times (-\infty, \ln t^*) \times \mathbb{R}^n. \tag{5-8}$$

Suppose that φ_l has been normalized so that $0 < \varphi_l \leq t^*$ in B_l. Suppose that Inequality (5-6) is not true. Since $u_k \geq \varphi_l$ on ∂B_l, it follows that $\frac{\varphi_l}{u_k}$ attains its maximum in B_l. Since u_k and φ_l are positive in B_l, we see that

$$\ln\left(\frac{\varphi_l}{u_k}\right) = \ln \varphi_l - \ln u_k$$

attains its positive maximum at some point $x_0 \in B_l$. Set

$$\alpha(x) = \ln \varphi_l(x) \quad \text{and} \quad \beta(x) = \ln u_k(x).$$

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By direct calculations,

$$\Delta_\infty^h \alpha = -\frac{1}{\varphi_l^{h+1}}|D\varphi_l|^{h+1} + \frac{1}{\varphi_l^h}\Delta_\infty^h \varphi_l = -|D\alpha|^{h+1} - \Lambda_l a, \tag{5-9}$$

$$\Delta_\infty^h \beta = -\frac{1}{u_k^{h+1}}|Du_k|^{h+1} + \frac{1}{u_k^h}\Delta_\infty^h u_k = -|D\beta|^{h+1} + \frac{\lambda f(x, e^\beta, e^\beta D\beta)}{(e^\beta)^h}. \tag{5-10}$$

Therefore,

$$\Delta_\infty^h \alpha + |D\alpha|^{h+1} = -a\Lambda_l \quad \text{and} \quad \Delta_\infty^h \beta + |D\beta|^{h+1} = \frac{\lambda f(x, e^\beta, e^\beta D\beta)}{(e^\beta)^h}.$$

Note that

$$\alpha - \beta = \ln\left(\frac{\varphi_l}{u_k}\right)$$

has a positive maximum in $\overline{B}_l$, that is, $M := \max_{\overline{B}_l}(\alpha - \beta) > 0$. Define

$$\Phi_\varepsilon(x, y) = \alpha(x) - \beta(y) - \frac{1}{2\varepsilon}|x - y|^2, \quad (x, y) \in \overline{B}_l \times \overline{B}_l.$$

Let

$$M_\varepsilon := \Phi_\varepsilon(x_\varepsilon, y_\varepsilon) = \max_{\overline{B}_l \times \overline{B}_l} \Phi_\varepsilon(x, y) \quad \text{for some } (x_\varepsilon, y_\varepsilon) \in \overline{B}_l \times \overline{B}_l.$$

Without loss of generality, we can assume that the sequence $\{(x_\varepsilon, y_\varepsilon)\}$ converges to $(z, z) \in \overline{B}_l \times \overline{B}_l$ as $\varepsilon \to 0$, where $M = (\alpha - \beta)(z)$. Since $M > 0$, we must have $z \in B_l$. Then we can suppose that $(x_\varepsilon, y_\varepsilon) \in B_l$ for sufficiently small $\varepsilon > 0$. According to the maximum principle in [18, 23], there are $n \times n$ symmetric matrices $X_\varepsilon, Y_\varepsilon \in \mathbb{S}$ such that

$$(\eta_\varepsilon, X_\varepsilon) \in \overline{\mathcal{J}}^{2,+}\alpha(x_\varepsilon), \quad (\eta_\varepsilon, Y_\varepsilon) \in \overline{\mathcal{J}}^{2,-}\beta(y_\varepsilon)$$

and

$$-\frac{3}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X_\varepsilon & 0 \\ 0 & -Y_\varepsilon \end{pmatrix} \le \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{5-11}$$

where $\eta_\varepsilon := \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon)$. Following from Inequality (5-11), we have $X_\varepsilon \le Y_\varepsilon$. Applying the definitions of the viscosity subsolution and viscosity supersolution to (5-9) and (5-10), respectively,

$$-\Lambda_l a(x_\varepsilon) \le \langle X_\varepsilon \eta_\varepsilon, \eta_\varepsilon\rangle |\eta_\varepsilon|^{h-3} + |\eta_\varepsilon|^{h+1} \quad \text{and}$$
$$\langle Y_\varepsilon \eta_\varepsilon, \eta_\varepsilon\rangle |\eta_\varepsilon|^{h-3} + |\eta_\varepsilon|^{h+1} \le \frac{\lambda f(y_\varepsilon, e^\beta, e^\beta \eta_\varepsilon)}{(e^\beta)^h}.$$

Since $X_\varepsilon \le Y_\varepsilon$,

$$\frac{\lambda f(y_\varepsilon, e^\beta, e^\beta \eta_\varepsilon)}{(e^\beta)^h} \ge -\Lambda_l a(x_\varepsilon).$$

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As $\varepsilon \to 0$, we get the inequality

$$\frac{\lambda f(z, e^{\beta}(z), e^{\beta}(z)\eta_{\varepsilon})}{(e^{\beta}(z))^{h}} \geq -\Lambda_{l} a(z).$$

Therefore,

$$\lambda \hat{f}(z, e^{\beta}(z), e^{\beta}(z)\eta_{\varepsilon})) \geq -\Lambda_{l} \geq -\Lambda_{1}.$$

By the assumption $\varphi_l(z) > u_k(z)$ and since $\hat{f}(x, t, p)$ is nondecreasing in t, we conclude that

$$\lambda \hat{f}(z, e^{\alpha}(z), e^{\alpha}(z)\eta_{\varepsilon})) \geq -\Lambda_{1}.$$

Together with Inequality (5-8),

$$\tau \leq \Lambda_{1}. \tag{5-12}$$

Inequality (5-12) is an obvious contradiction to (5-7). Thus, the claim that $u_k \geq \varphi_l$ in $B(O, l)$ for all $k > l$ holds.

Then,

$$0 < \varphi_l \leq u_k \quad \text{in } B(O, l) \text{ for all } k \geq l.$$

As a consequence of this inequality, we have $0 < \varphi_l \leq u_\lambda$ in $B(O, l)$. Thus,

$$0 < u_\lambda \leq v_\lambda \quad \text{in } B(O, l).$$

Since l is arbitrary, we conclude that $0 < u_\lambda \leq v_\lambda$ in $\mathbb{R}^n$. By a similar argument to the one used in the proof of Theorem 4.1, we can show that u_λ is a viscosity solution of Problem (1-12). Clearly, $u(x) \to 0$ as $|x| \to \infty$. □

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