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PRO-CATEGORIES AND MULTIADJOINT FUNCTORS

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Introduction. For a functor $G:\mathscr{A} \to \mathscr{X}$ and a class $\mathfrak{D}$ of small categories containing the terminal category **1** we form the extension

$$\text{Pro}\,(\mathfrak{D}, G):\text{Pro}\,(\mathfrak{D}, \mathscr{A}) \to \text{Pro}\,(\mathfrak{D}, \mathscr{X})$$

and call G right $\mathfrak{D}$-pro-adjoint if and only if Pro ($\mathfrak{D}$, G) is right adjoint. Here Pro ($\mathfrak{D}$, $\mathscr{A}$) is the completion of $\mathscr{A}$ with respect to $\mathfrak{D}$; it coincides with the usual pro-category of $\mathscr{A}$ in case $\mathfrak{D}$ = directed sets. For this $\mathfrak{D}$ a full embedding G is dense in the sense of Mardešić [**11**] if and only if it is right $\mathfrak{D}$-pro-adjoint in the above sense; this has been proved recently by Stramaccia [**15**]. The most important example is the embedding of the homotopy category of pointed CW-complexes into the homotopy category of pointed topological spaces (cf. [**2**]). In case $\mathfrak{D}$ = all sets (as discrete categories) it turns out that G is right $\mathfrak{D}$-pro-adjoint if and only if it is right multiadjoint in the sense of Diers [**3**]. In particular the theory of multi(co)reflective subcategories has been successfully developed by Salicrup [**12**], [**13**].

In this note we prove some important facts about both, dense and multireflective subcategories in the more general context of right $\mathfrak{D}$-pro-adjoint functors. To be able to do so we provide a simple construction of the category Pro ($\mathfrak{D}$, $\mathscr{A}$) which coincides with the one given by Johnstone and Joyal [**9**] in case $\mathfrak{D}$ = small filtered categories. All properties which, for that $\mathfrak{D}$, were first proved by Grothendieck and Verdier [**6**] hold for all $\mathfrak{D}$ with a certain closedness property under colimits in $\mathscr{C}at$.

The procedure to generalize properties of functors by passing from G to Pro ($\mathfrak{D}$, G) may be applied to other notions like monadicity and semitopologicity. In Section 4 we briefly mention these notions which, however, are beyond the scope of this paper.

1. Relativized pro-categories. Let $\mathfrak{D}$ be a class of small categories containing the terminal category **1**. For a category $\mathscr{X}$ with small hom sets,

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the category Pro $(\mathfrak{D}, \mathscr{K})$ has as objects all contravariant diagrams in $\mathscr{K}$ of type $\mathfrak{D}$, and its hom sets are given by the formula

$$\text{Pro}\,(\mathfrak{D}, \mathscr{K})(\mathbf{X}, \mathbf{Y}) \cong \lim_{j}\ \operatorname{colim}_{i}\ \mathscr{K}(X_i, Y_j).$$

More precisely: objects are all functors $\mathbf{X}:\mathscr{I}^{\mathrm{op}} \to \mathscr{K}$ with $\mathscr{I} \in \mathfrak{D}$; suggestively, but less correctly, we write

$$\mathbf{X} = (X_i)_{i \in \mathrm{Ob}\,\mathscr{I}}$$

where X_i is the value of $\mathbf{X}$ at i; the value in $\mathscr{K}$ of a morphism $\nu:i \to i'$ in $\mathscr{I}$ under $\mathbf{X}$ is again denoted by $\nu:X_{i'} \to X_i$. To define a morphism

$$\mathbf{f}:\mathbf{X} \to \mathbf{Y} = (Y_j)_{j \in \mathrm{Ob}\,\mathscr{J}}$$

one considers, for each j, the smallest equivalence relation $\sim_j$ on $\sum_{i \in \mathrm{Ob}\,\mathscr{I}} \mathscr{K}(X_i, Y_j)$ such that

$$(f, i) \sim_j (f \cdot \nu, i')$$

for all $\nu:i \to i'$ in $\mathscr{I}$. A morphism $\mathbf{X} \to \mathbf{Y}$ is a family $\mathbf{f} = (\mathbf{f}_j)_{j \in \mathrm{Ob}\,\mathscr{J}}$ where each $\mathbf{f}_j$ is an equivalence class with respect to $\sim_j$ such that the coherence condition

(1) $(f, i) \in \mathbf{f}_{j'} \Rightarrow (\mu \cdot f, i) \in \mathbf{f}_j$

holds for all $\mu:j \to j'$ in $\mathscr{J}$. If $\mathbf{g}:\mathbf{Y} \to \mathbf{Z} = (Z_n)_{n \in \mathrm{Ob}\,\mathscr{N}}$ is another morphism the composite $\mathbf{g} \cdot \mathbf{f} = \mathbf{h} = (\mathbf{h}_n)_{n \in \mathrm{Ob}\,\mathscr{N}}$ is defined by

(2) $\mathbf{h}_n = \{ (h, i) \mid \exists (g, j) \in \mathbf{g}_n, (f, i') \in \mathbf{f}_j : (h, i) \sim_n (g \cdot f, i') \}$;

in fact, $\mathbf{h}$ satisfies the coherence condition (1). Sometimes it is more convenient to replace the equivalence class $\mathbf{f}_j$ by a chosen representative (f_j, i_j); then, independently from the choice of the representative, (1) and (2) read as

(1′) $(\mu \cdot f_{j'}, i_{j'}) \sim_j (f_j, i_j)$

(2′) $(h_n, i_n) \sim_n (g_n \cdot f_{j_n}, i_{j_n})$.

Every $\mathscr{K}$-object X can be considered as a $\mathbf{1}$-indexed family. Therefore one has a full embedding $\mathscr{K} \to \text{Pro}\,(\mathfrak{D}, \mathscr{K})$. A Pro $(\mathfrak{D}, \mathscr{K})$-morphism $\mathbf{f}:\mathbf{X} \to Y$ with $Y \in \mathrm{Ob}\,\mathscr{K}$ is a single equivalence class. Every Pro $(\mathfrak{D}, \mathscr{K})$-object $\mathbf{X} = (X_i)_{i \in \mathrm{Ob}\,\mathscr{I}}$ admits, for every $i \in \mathrm{Ob}\,\mathscr{I}$, a canonical morphism $\xi_i:\mathbf{X} \to X_i$ which, as an equivalence class, is generated by $(1_{X_i}, i)$. A Pro $(\mathfrak{D}, \mathscr{K})$-morphism $\mathbf{g}: Y \to \mathbf{X}$ can be completely described by a family

$$(g_i: Y \to X_i)_{i \in \mathrm{Ob}\,\mathscr{I}}$$

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which is natural in i, that is $\nu \cdot g_{i'} = g_i$ for all $\nu: i \to i'$ in $\mathscr{I}$. So $\mathbf{g}$ is nothing but a natural transformation or a cone $\Delta Y \to \mathbf{X}$; that is why one trivially has:

1.1. PROPOSITION. *$\mathscr{K}$ is a coreflective subcategory of* Pro $(\mathfrak{D}, \mathscr{K})$ *if and only if $\mathscr{K}$ is $\mathfrak{D}$-complete, that is, $\mathscr{K}$ is $\mathscr{I}^{\mathrm{op}}$-complete for every $\mathscr{I} \in \mathfrak{D}$. The coreflector is given by forming the limit in $\mathscr{K}$.*

Hence, if $\mathscr{K}$ is $\mathfrak{D}$-complete, the embedding $\mathscr{K} \to$ Pro $(\mathfrak{D}, \mathscr{K})$ preserves all colimits. But this can be easily proved even without the assumption of $\mathfrak{D}$-completeness. We omit the proof since, in the following, we are only interested in the question which limits are preserved. It is well known that, generally, limits of type $\mathscr{I}^{\mathrm{op}}$ with $\mathscr{I} \in \mathfrak{D}$ are not preserved, even in the classical case when $\mathfrak{D}$ is the class of directed sets (cf. [**6**] p. 81; [**14**]). The natural limit type which is preserved is as follows (cf. also [**19**]):

1.2. PROPOSITION. *Let $\mathscr{D}$ be a category such that, in $\mathscr{S}et$, limits of type $\mathscr{D}$ commute with colimits of type $\mathscr{I}$ for all $\mathscr{I} \in \mathfrak{D}$. Then the embedding $\mathscr{K} \to$* Pro *$(\mathfrak{D}, \mathscr{K})$ preserves limits of type $\mathscr{D}$.*

Proof. Let the limit of $H: \mathscr{D} \to \mathscr{K}$ exist in $\mathscr{K}$. The following shows that, when considered in Pro $(\mathfrak{D}, \mathscr{K})$, it is preserved by all covariant hom's of Pro $(\mathfrak{D}, \mathscr{K})$, so it is a limit in Pro $(\mathfrak{D}, \mathscr{K})$:

$$\begin{aligned} \mathrm{Pro}\,(\mathfrak{D}, \mathscr{K})(\mathbf{X}, \lim H) &= \operatorname*{colim}_i \mathscr{K}(X_i, \lim H) \\ &\cong \operatorname*{colim}_i \lim_d \mathscr{K}(X_i, Hd) \\ &\cong \lim_d \operatorname*{colim}_i \mathscr{K}(X_i, Hd) \\ &= \lim_d \mathrm{Pro}\,(\mathfrak{D}, \mathscr{K})(\mathbf{X}, Hd). \end{aligned}$$

Next we will give an explicit construction of limits of type $\mathscr{J}^{\mathrm{op}}$ in Pro $(\mathfrak{D}, \mathscr{K})$ for $\mathscr{J} \in \mathfrak{D}$. It generalizes corresponding constructions by Stramaccia [**15**] in case $\mathfrak{D}$ = directed sets and Johnstone and Joyal [**9**] in case $\mathfrak{D}$ = small filtered categories. So we consider a diagram

$$H: \mathscr{J}^{\mathrm{op}} \to \mathrm{Pro}\,(\mathfrak{D}, \mathscr{K}) \quad \text{with}\, \mathscr{J} \in \mathfrak{D};$$

it is given by Pro $(\mathfrak{D}, \mathscr{K})$-objects

$$Hj = \mathbf{X}^j = (X_i^j)_{i \in \mathrm{Ob}\,\mathscr{I}_j}$$

and Pro $(\mathfrak{D}, \mathscr{K})$-morphisms

$$\mathbf{f}^{\mu} = (\mathbf{f}_i^{\mu})_{i \in \text{Ob}\,\mathcal{I}_j} : \mathbf{X}^{j'} \to \mathbf{X}^j$$

for all $\mu : j \to j'$ in $\mathcal{J}$. From these data one forms the *related category* $\hat{H}$ of H as follows: objects are pairs (i, j) with $j \in \text{Ob}\,\mathcal{J}$ and $i \in \text{Ob}\,\mathcal{I}_j$; a morphism $(f, \mu):(i, j) \to (i', j')$ in $\hat{H}$ consists of a $\mathcal{J}$-morphism $\mu : j \to j'$ and a $\mathcal{K}$-morphism $f : X_{i'}^{j'} \to X_i^j$ such that $(f, i') \in \mathbf{f}_i^{\mu}$; composition is pointwise. In case $\hat{H} \in \mathfrak{D}$ we have the new Pro $(\mathfrak{D}, \mathcal{K})$-object

$$\mathbf{X} = (X_i^j)_{(i,j) \in \text{Ob}\,\hat{H}}$$

which, as a functor $\hat{H}^{\text{op}} \to \mathcal{K}$, maps (f, μ) to f. For every $j \in \text{Ob}\,\mathcal{J}$ there is a functor $L_j : \mathcal{I}_j \to \hat{H}$, $\nu \mapsto (\nu, 1_j)$, with $\mathbf{X} \cdot L_j^{\text{op}} = \mathbf{X}^j$. This yields a Pro $(\mathfrak{D}, \mathcal{K})$-morphism $\Lambda_j : \mathbf{X} \to \mathbf{X}^j$; each of its components is the equivalence class of an identity morphism. It is easy to check that Λ_j is natural in j, so one has a cone $\Lambda : \Delta\mathbf{X} \to H$. In fact, it is also easily proved that it is a limiting cone.

We call $\mathfrak{D}$ *admissible with respect to* $\mathcal{K}$ if, for every $H : \mathcal{J}^{\text{op}} \to \text{Pro}\,(\mathfrak{D}, \mathcal{K})$ with $\mathcal{J} \in \mathfrak{D}$, the related category $\hat{H}$ belongs to $\mathfrak{D}$. Using this phrase we have proved:

1.3. Proposition. *If* $\mathfrak{D}$ *is admissible with respect to* $\mathcal{K}$, *then* Pro $(\mathfrak{D}, \mathcal{K})$ *is* $\mathfrak{D}$*-complete, that is,* $\mathcal{I}^{\text{op}}$*-complete for every* $\mathcal{I} \in \mathfrak{D}$.

An immediate consequence of the above construction is:

1.4. Corollary. *Every* $\mathbf{X} \in$ Ob Pro $(\mathfrak{D}, \mathcal{K})$ *is the limit of*

$$\mathcal{I}^{\text{op}} \xrightarrow{\mathbf{X}} \mathcal{K} \to \text{Pro}\,(\mathfrak{D}, \mathcal{K}).$$

The limit projections are the canonical morphisms $\xi_i : \mathbf{X} \to X_i$ (*cf. before* 1.1).

One can use 1.3 and 1.4 in order to prove:

1.5. Proposition. *Every functor* $\mathcal{K} \to \mathcal{L}$ *into a* $\mathfrak{D}$*-complete category* $\mathcal{L}$ *can be extended to a functor* Pro $(\mathfrak{D}, \mathcal{K}) \to \mathcal{L}$. *If* $\mathfrak{D}$ *is admissible with respect to* $\mathcal{K}$, *it preserves all limits of type* $\mathcal{I}^{\text{op}}$, $\mathcal{I} \in \mathfrak{D}$, *and is, up to natural equivalence, uniquely determined by this property.*

1.6. *Remark*. 1. Propositions 1.3 and 1.5 have been proved before in the dual situation in [**18**] and [**19**], but differently; there Pro $(\mathfrak{D}, \mathcal{K})$ is realized by a full representation in $[\mathcal{K}, \mathcal{S}et]$. Instead of the condition that $\mathfrak{D}$ is admissible with respect to $\mathcal{K}$ Weberpals [**19**] assumes $\mathfrak{D}$ to be "weakly saturated", that is:

(1) $\mathbf{1} \in \mathfrak{D}$,

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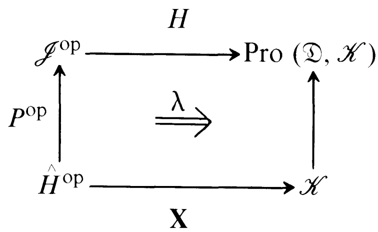
(2) if $\mathscr{I} \to \mathscr{J}$ is a final functor of small categories with $\mathscr{I} \in \mathfrak{D}$ then $\mathscr{J} \in \mathfrak{D}$,

(3) if $H: \mathscr{I} \to \mathscr{C}at$ (= small categories) is a functor with $\mathscr{I} \in \mathfrak{D}$ and all $Hi \in \mathfrak{D}$, $i \in \mathrm{Ob}\,\mathscr{I}$, then colim $H \in \mathfrak{D}$.

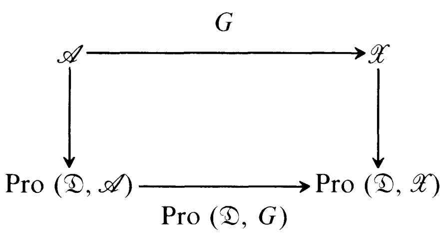
In fact, one can show that the related category $\hat{H}$ as constructed above belongs to $\mathfrak{D}$ if $\mathfrak{D}$ is saturated; hence, in that case, $\mathfrak{D}$ is also admissible with respect to $\mathscr{K}$. For concretely given classes $\mathfrak{D}$ it seems easier to check the latter condition directly.

2. Using the same notation as in 1.3 one gets the following diagram in the 2-category $\mathscr{CAT}$ of all categories:



Here $P: \hat{H} \to \mathscr{J}$ denotes the projection functor, and λ is pointwise a limit projection as in 1.4. We do not know whether this observation leads to a 2-categorical characterization of $\hat{H}$.

2. Pro-adjoint functors. Every functor $G: \mathscr{A} \to \mathscr{X}$ trivially induces a functor Pro ($\mathfrak{D}$, G) rendering the diagram



commutative. From 1.5 it follows that Pro ($\mathfrak{D}$, G) preserves all limits of type $\mathscr{I}^{op}$ for $\mathscr{I} \in \mathfrak{D}$, if $\mathfrak{D}$ is admissible with respect to $\mathscr{A}$.

2.1. *Definition.* G is called *right $\mathfrak{D}$-pro-adjoint* if Pro ($\mathfrak{D}$, G) is right adjoint. If, in addition, G is the inclusion functor of a full subcategory $\mathscr{A}$ is called *$\mathfrak{D}$-pro-reflective in $\mathscr{X}$*. Dually: G is *left $\mathfrak{D}$-pro-adjoint* and $\mathscr{A}$ is *$\mathfrak{D}$-pro-coreflective* in $\mathscr{X}$ if $G^{op}: \mathscr{A}^{op} \to \mathscr{X}^{op}$ is right $\mathfrak{D}$-pro-adjoint and $\mathscr{A}^{op}$ is $\mathfrak{D}$-pro-reflective in $\mathscr{X}^{op}$ respectively.

Since Pro $(\mathfrak{D}, -)$ is functorial, the composition of right $\mathfrak{D}$-pro-adjoint functors is right $\mathfrak{D}$-pro-adjoint. Since a right adjoint functor preserves all limits, a $\mathfrak{D}$-pro-right adjoint functor G preserves those limits of $\mathscr{A}$ which are preserved by $\mathscr{A} \to \text{Pro}(\mathfrak{D}, \mathscr{A})$. Therefore, from 1.2 one obtains:

2.2. THEOREM. *Let $\mathscr{D}$ be a category such that, in $\mathscr{S}et$, limits of type $\mathscr{D}$ commute with colimits of type $\mathscr{I}$ for all $\mathscr{I} \in \mathfrak{D}$. Then every right $\mathfrak{D}$-pro-adjoint functor preserves $\mathscr{D}$-limits.*

The following theorem compares the notions of right adjointness and right $\mathfrak{D}$-pro-adjointness:

2.3. THEOREM. *Let $\mathscr{A}$ be $\mathfrak{D}$-complete. Then $G:\mathscr{A} \to \mathscr{X}$ is right adjoint if and only if G is right $\mathfrak{D}$-pro-adjoint and preserves limits of type $\mathscr{I}^{\text{op}}$ for all $\mathscr{I} \in \mathfrak{D}$.*

Proof. To prove that a right adjoint functor is right $\mathfrak{D}$-pro-adjoint is straightforward (cf. [**19**], 2.8(c); also 2.5 below). Also it preserves (in particular) $\mathscr{I}^{\text{op}}$-limits. Vice versa, let us assume that these properties hold true. Let $\bar{F}$ be left adjoint to $\bar{G} = \text{Pro}(\mathfrak{D}, G)$ with unit $\bar{\eta}$ and counit $\bar{\epsilon}$. For every $X \in \text{Ob}\,\mathscr{X}$, there is a limiting cone

$$\lambda_X:\Delta FX \to \bar{F}X;$$

this defines a functor $F:\mathscr{X} \to \mathscr{A}$. Since G preserves this limit there is a unique $\mathscr{X}$-morphism $\eta X:X \to GFX$ such that

$$G\lambda_X \cdot \Delta\eta X = \bar{\eta}X$$

(where $\bar{\eta}X$ is considered as a cone $\Delta X \to G(\bar{F}X) = \bar{G}\,\bar{F}X)$; this defines a natural transformation

$$\eta:\text{Id}_{\mathscr{X}} \to GF.$$

Finally, one defines a natural transformation $\epsilon:FG \to \text{Id}_{\mathscr{A}}$ by

$$\Delta\epsilon A = \bar{\epsilon}A \cdot \lambda_{GA} \quad \text{for all } A \in \text{Ob}\,\mathscr{A}.$$

Immediately from the construction we get

$$G\epsilon A \cdot \eta GA = 1_{GA}.$$

The other equation needed for the adjunction follows from

$$\begin{aligned}\lambda_X \cdot \Delta\epsilon FX \cdot \Delta F\eta X &= \lambda_X \cdot \bar{\epsilon}FX \cdot \bar{F}\eta X \cdot \lambda_X \\ &= \bar{\epsilon}\bar{F}X \cdot \bar{F}\bar{G}\lambda_X \cdot \bar{F}\eta X \cdot \lambda_X \\ &= \bar{\epsilon}\bar{F}X \cdot \bar{F}\bar{\eta}X \cdot \lambda_X = \lambda_X.\end{aligned}$$

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In the following we give a characterization of right $\mathfrak{D}$-pro-adjointness which simplifies checking this property in examples. The equivalence (i) $\Leftrightarrow$ (iii) generalizes the main result of [**15**]:

2.4. THEOREM. *Let $\mathfrak{D}$ be admissible with respect to $\mathscr{A}$ and let $G:\mathscr{A} \to \mathscr{X}$ be a functor, $\bar{G}$ =* Pro *$(\mathfrak{D}, G)$. The following are equivalent*:

(i) *G is right $\mathfrak{D}$-pro-adjoint.*

(ii) *$\bar{G}$ has a partial left adjoint relative to the embedding $\mathscr{X} \to$* Pro *$(\mathfrak{D}, \mathscr{X})$.*

(iii) *For all $X \in$* Ob *$\mathscr{X}$, there is a* Pro *$(\mathfrak{D}, \mathscr{A})$-object* **A** *and a* Pro *$(\mathfrak{D}, \mathscr{X})$-morphism $\mathbf{e}:X \to \bar{G}\mathbf{A}$ such that, for every $g:X \to GB$ in $\mathscr{X}$ with $B \in$* Ob *$\mathscr{A}$, there is a unique* Pro *$(\mathfrak{D}, \mathscr{A})$-morphism $\mathbf{h}:\mathbf{A} \to B$ with $\bar{G}\mathbf{h} \cdot \mathbf{e} = h$.*

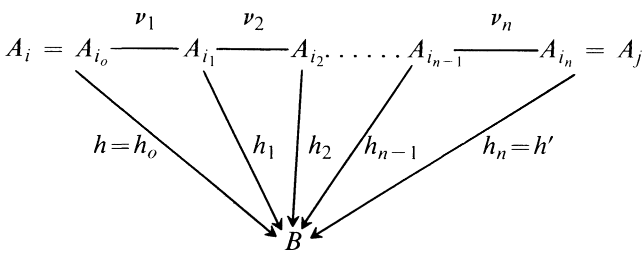
(iv) *For all $X \in$* Ob *$\mathscr{X}$, there is a functor $\mathbf{A}:\mathscr{I}^{\mathrm{op}} \to \mathscr{A}$ with $\mathscr{I} \in \mathfrak{D}$ and a cone*

$$\mathbf{e} = (e_i:X \to GA_i)_{i \in \mathrm{Ob}\,\mathscr{I}}$$

such that, for every $g:X \to GB$ with $B \in$ Ob *$\mathscr{A}$, there are $i \in$* Ob *$\mathscr{I}$ and $h:A_i \to B$ in $\mathscr{A}$ with $Gh \cdot e_i = g$; for any other $j \in$* Ob *$\mathscr{I}$ and $h':A_j \to B$ with $Gh' \cdot e_j = g$ one has $(h, i) \sim (h', j)$, i.e., there are finitely many $i = i_0, i_1, \ldots, i_{n-1}, i_n = j$, $h = h_0, h_1, \ldots, h_{n-1}, h_n = h'$ and $\mathscr{I}$-morphisms*

$$\nu_k \in \mathscr{I}(i_{k-1}, i_k) \cup \mathscr{I}(i_k, i_{k-1}), \quad k = 1, \ldots, n,$$

such that



commutes.

Proof. Trivially (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). (iv) is just another formulation of (iii) avoiding the explicit use of pro-categories, so (iii) $\Leftrightarrow$ (iv).

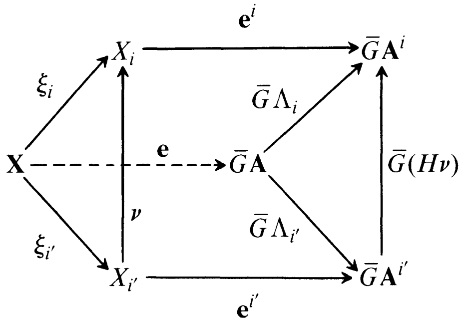
(iii) $\Rightarrow$ (i): For $\mathbf{X} = (X_i)_{i \in \mathrm{Ob}\,\mathscr{I}} \in$ Ob Pro $(\mathfrak{D}, \mathscr{X})$ and each $i \in$ Ob $\mathscr{I}$ one has $\mathbf{e}^i:X_i \to \bar{G}\mathbf{A}^i$ with the universal property described in (iii). By

$$Hi = \mathbf{A}^i \quad \text{and} \quad \bar{G}(H\nu) \cdot \mathbf{e}^{i'} = \mathbf{e}^i \cdot \nu$$

for $\nu:i \to i'$ in $\mathscr{I}$ one defines a functor

$$H: \mathscr{I}^{\mathrm{op}} \to \mathrm{Pro}\,(\mathfrak{D}, \mathscr{A}).$$

By 1.3, there is a limiting cone $(\Lambda_i: \mathbf{A} \to \mathbf{A}^i)_{i \in \mathrm{Ob}\,\mathscr{I}}$ which is preserved by $\bar{G}$. The family $(\mathbf{e}^i \cdot \xi_i)_{i \in \mathrm{Ob}\,\mathscr{I}}$ forms a cone $\Delta\mathbf{X} \to \bar{G}H$ (where $\xi_i: \mathbf{X} \to X_i$ is the limit presentation of $\mathbf{X}$). Hence there is a unique $\mathbf{e}$ rendering the diagram



commutative for every $\nu: i \to i'$ in $\mathscr{I}$. We consider a Pro $(\mathfrak{D}, \mathscr{X})$-morphism $\mathbf{g}: \mathbf{X} \to \bar{G}\mathbf{B}$, $\mathbf{B} = (B_j)_{j \in \mathrm{Ob}\,\mathscr{J}}$ with limit projections $\beta_j: \mathbf{B} \to B_j$. For every $j \in \mathrm{Ob}\,\mathscr{J}$ there are $i_j \in \mathrm{Ob}\,\mathscr{I}$ and $g_j: X_{i_j} \to GB_j$ such that

$$g_j \cdot \xi_{i_j} = \bar{\mathbf{G}}\beta_j \cdot \mathbf{g},$$

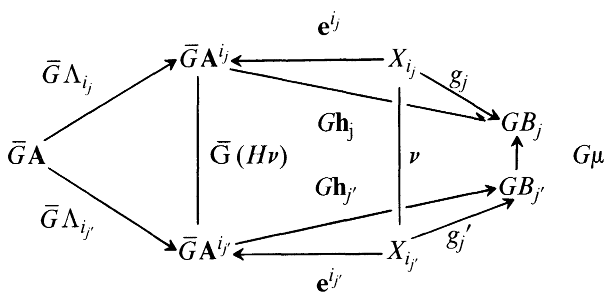
so there is a unique $\mathbf{h}_j$: $\mathbf{A}^{i_j} \to B_j$ with $\bar{G}\mathbf{h}_j \cdot \mathbf{e}^{i_j} = g_j$. We claim that the family $(\mathbf{h}_j \cdot \Lambda_{i_j})_{j \in \mathrm{Ob}\,\mathscr{J}}$ forms a cone $\Delta\mathbf{A} \to \mathbf{B}$: for $\mu: j \to j'$ in $\mathscr{J}$ one has

$$(G\mu \cdot g_{j'}, i_{j'}) \sim_j (g_j, i_j);$$

without loss of generality we assume that there is

$$\nu \in \mathscr{I}(i_j, i_{j'}) \cup \mathscr{I}(i_{j'}, i_j)$$

such that the right trapezium of the following diagram commutes:



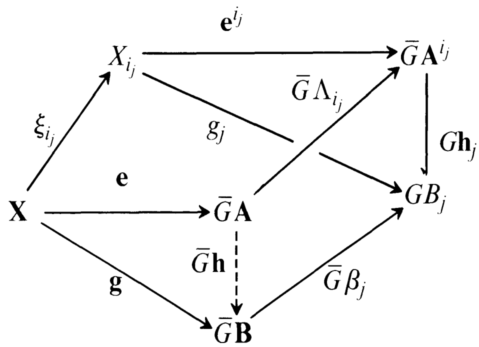
Since the middle square and the left triangle (without $\bar{G}$) also commute we obtain from the uniqueness property of $\mathbf{h}_j$ or $\mathbf{h}_{j'}$, that

$$\mathbf{h}_j \cdot \Lambda_{i_j} = \mu \cdot \mathbf{h}_{j'} \cdot \Lambda_{i_{j'}}.$$

Hence we obtain a unique $\mathbf{h}:\mathbf{A} \to \mathbf{B}$ such that

$$\beta_j \cdot \mathbf{h} = \mathbf{h}_j \cdot \Lambda_{i_j}.$$

From the limit property of $(\bar{G}\beta_j)_{j \in \mathrm{Ob}\mathscr{J}}$ we finally obtain $\bar{G}\mathbf{h} \cdot \mathbf{e} = \mathbf{g}$, and this factorization is obviously unique.



If $\mathfrak{D}' \subset \mathfrak{D}$, then Pro $(\mathfrak{D}', \mathscr{A})$ is a full subcategory of Pro $(\mathfrak{D}, \mathscr{A})$. From the characterization 2.4 (iii) we obtain:

2.5. COROLLARY. *Every right $\mathfrak{D}'$-pro-adjoint functor is right $\mathfrak{D}$-pro-adjoint for all* $\{\mathbf{1}\} \subset \mathfrak{D}' \subset \mathfrak{D}$.

In case $\mathfrak{D}' = \{\mathbf{1}\}$ this proves the "only if" part of 2.3.

3. Special classes $\mathfrak{D}$. We consider some special classes $\mathfrak{D}$; each of them is admissible with respect to every category.

3.1. $\mathfrak{D} = \{\mathbf{1}\}$. This case gives nothing new: Pro $(\mathfrak{D}, \mathscr{K}) = \mathscr{K}$ for every category $\mathscr{K}$, and right $\mathfrak{D}$-pro-adjointness means right adjointness.

3.2. $\mathfrak{D}$ = all sets = small discrete categories. Then Pro $(\mathfrak{D}, \mathscr{K})$ is the formal product completion of $\mathscr{K}$: objects are small families $(X_i)_{i \in I}$ of $\mathscr{K}$-objects; a morphism

$$(f_j, \varphi)_{j \in J}:(X_i)_{i \in I} \to (Y_j)_{j \in J}$$

consists of a mapping $\varphi:J \to I$ and morphisms

$$f_j:X_{\varphi(j)} \to Y_j.$$

The equivalence relations $\sim_j$ are discrete. This shows that the factorization $Gh \cdot e_i = g$ in 2.4(iv) holds for a unique index i and a unique morphism h. Hence right $\mathfrak{D}$-pro-adjointness means right multi-adjointness (i.e., existence of a left multi-adjoint) in the sense of Diers [**3**]. The small types $\mathscr{D}$ of limits which commute with coproducts (= discrete colimits) in $\mathscr{S}et$ are precisely the small (non-void) connected categories $\mathscr{D}$. Hence 2.2 means that right multi-adjoint functors preserve (non-void) connected colimits: this is Theorem 20 in [**1**] and Proposition 3.5.1 in [**3**].

3.3. $\mathfrak{D}$ = all (non-void) directed sets, considered as small (filtered) categories. Then Pro $(\mathfrak{D}, \mathscr{K})$ is the usual pro-category of $\mathscr{K}$: Pro $(\mathfrak{D}, \mathscr{K})$ = Pro-$\mathscr{K}$. So here Theorem 2.4 is Stramaccia's result [**15**] which tells us that, for G the embedding of a full subcategory, $\mathfrak{D}$-pro-reflectivity means density in sense of Mardešić [**11**]. Since this terminology can be confused with the usual notion of density in Category Theory I should strongly suggest to call a right $\mathfrak{D}$-pro-adjoint functor just *right pro-adjoint*. So a dense subcategory in the sense of [**11**], [**5**], [**15**] should be called *pro-reflective*. Dual notion: *pro-coreflective*. Since finite products commute with filtered colimits in $\mathscr{S}et$, from 2.2 we obtain that a right pro-adjoint functor preserves finite products. Application of Theorem 2.3 gives us Giuli's result [**5**], Theorem 2.3 for arbitrary functors instead of just subcategories: if $\mathscr{A}$ has inverse limits, then $G:\mathscr{A} \to \mathscr{X}$ is right adjoint if and only if it is right pro-adjoint and preserves inverse limits.

3.4. $\mathfrak{D}$ = all small categories. Then Pro $(\mathfrak{D}, \mathscr{K})$ is the usual completion of $\mathscr{K}$ with respect to all small limits (cf. [**10**]). According to 2.5 for this $\mathfrak{D}$ we get the weakest notion of right pro-adjointness. From the characterization 2.4(iv) it is clear that a right $\mathfrak{D}$-pro-adjoint functor satisfies the Solution Set Condition of Freyd's Adjoint Functor Theorem. The converse assertion is not true, as is demonstrated by the following example:

Let $\mathscr{A}$ consist of (pairwise different) objects A, B, C_α and (non-identical) morphisms $a_\alpha:A \to C_\alpha$, $b_\alpha:B \to C_\alpha$ where α runs through a proper class Ω; let $\mathscr{X}$ consist of objects X, U, V, Z_α and morphisms

$$u:X \to U,\ v:X \to V,\ u_\alpha:U \to Z_\alpha,\ v_\alpha:V \to Z_\alpha$$

and

$$x_\alpha = u_\alpha \cdot u = v_\alpha \cdot v:X \to Z_\alpha, \quad \alpha \in \Omega.$$

The functor $G:\mathscr{A} \to \mathscr{X}$ with $Ga_\alpha = u_\alpha$, $Gb_\alpha = v_\alpha$ obviously satisfies the solution set condition. But condition 2.4 (iv) does not hold true for $X \in \text{Ob}\,\mathscr{X}$; otherwise a cone $\mathbf{e}$ with the property 2.4 (iv) must contain u, v, but, from smallness reasons, it cannot contain all the x_α's, and these admit two different factorizations through $\mathbf{e}$ which cannot be connected.

Since $\mathfrak{D}$-pro-adjointness is slightly stronger than the solution set condition the assertion of Theorem 2.3 is slightly weaker than Freyd's Adjoint Functor Theorem in case $\mathfrak{D}$ = all small categories. But $\mathfrak{D} = \{1\}$ shows that Theorem 2.3 cannot be sharpened in this general form.

4. Remarks on further developments. It seems worth to generalize other notions than adjointness like we did in Section 2: if $\mathscr{E}$ is a property of functors, then we say that $G:\mathscr{A} \to \mathscr{X}$ has the property $\mathfrak{D}$-pro-$\mathscr{E}$ if and only if Pro ($\mathfrak{D}$, G) has the property $\mathscr{E}$. This procedure leads to known notions at least in case $\mathfrak{D}$ = all sets.

For instance, in [**4**] Diers has introduced the notion of a multimonadic functor and gives the following characterization ([**4**], Theorem 3.1): $G:\mathscr{A} \to \mathscr{X}$ is *multimonadic* if and only if G has a left multiadjoint, reflects isomorphisms, and those pairs of parallel morphism of $\mathscr{A}$ whose image by G has a split coequalizer have a coequalizer preserved by G. Straightforward computation shows that this is equivalent to the following properties:

$\bar{G}$ = Pro ($\mathfrak{D}$, G) (with $\mathfrak{D}$ = all sets) has a left adjoint, reflects isomorphisms, and those pairs of parallel morphisms of Pro ($\mathfrak{D}$, $\mathscr{A}$) whose image by $\bar{G}$ has a split coequalizer have a coequalizer preserved by $\bar{G}$. This proves:

4.1. THEOREM. *G is multimonadic in the sense of Diers if and only if G is $\mathfrak{D}$-pro-monadic for $\mathfrak{D}$ = all sets.*

Monadicity is understood in the weak sense that the comparison functor is an equivalence rather than an isomorphism.

A corresponding observation holds for semitopologicity (cf. [**16**]). In [**17**] the author introduced the notion of a localizing semitopological functor and gave several characterization theorems. One of them ([**17**], Proposition 6.1) precisely means that G is a localizing semitopological functor if and only if G is $\mathfrak{D}$-pro-semitopological with $\mathfrak{D}$ = all sets. So the best name of those functors now seems to be multi-semitopological. The main result of [**17**] (multi-semitopological functors are precisely the restrictions of topological functors to multi-reflective subcategories) can be proved for all classes $\mathfrak{D}$ which are admissible with respect to $\mathscr{A}$:

4.2. THEOREM. *$G:\mathscr{A} \to \mathscr{X}$ is $\mathfrak{D}$-pro-semitopological if and only if there is a topological functor $T:\mathscr{B} \to \mathscr{X}$ and a full $\mathfrak{D}$-pro-reflective embedding $E:\mathscr{A} \to \mathscr{B}$ with $G = TE$.*

Proof. The procedure is the same as in [**17**]: $\bar{G}$ = Pro ($\mathfrak{D}$, G) is semitopological, hence there is a topological functor $\hat{T}:\hat{\mathscr{B}} \to$ Pro ($\mathfrak{D}$, $\mathscr{X}$)

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and a full reflective embedding $\hat{E}$:Pro $(\mathfrak{D}, \mathscr{A}) \rightarrow \hat{\mathscr{B}}$. The pullback of $\hat{T}$ along $\mathscr{X} \rightarrow$ Pro $(\mathfrak{D}, \mathscr{X})$ is a topological functor $T:\mathscr{B} \rightarrow \mathscr{X}$, and the induced functor $E:\mathscr{A} \rightarrow \mathscr{B}$ is a full $\mathfrak{D}$-pro-reflective embedding by Theorem 2.4.

4.3. COROLLARY. *Every $\mathfrak{D}$-pro-semitopological functor admits a MacNeille completion* (*cf.* **[8]**).

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