SELF-ADJOINT SQUARE ROOTS OF POSITIVE SELF-ADJOINT BOUNDED LINEAR OPERATORS

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A corollary of the main theorem presented in this note is a generalisation of the well-known result that a self-adjoint square root of a positive self-adjoint compact linear map in a Hilbert space is itself a compact linear map. The method used here exploits the techniques developed recently in the study of k-set contractions ((**1**), (**2**)).

Before stating our results, it is convenient to recall the relevant definitions. In all that follows H will denote a Hilbert space.

Definition. *The ball measure of non-compactness of a bounded set* $\Omega \subset H$, *denoted by* $\beta(\Omega)$, *is defined by*
$\beta(\Omega) = \inf\{\delta\colon \Omega$ *can be covered by a finite number of balls in* H *with radius* $\delta\}$.

Definition. *A continuous map* $T\colon H \to H$ *is a k-ball contraction provided that*

$$\beta(T(\Omega)) \leqq k\beta(\Omega) \text{ for all bounded sets } \Omega \subset H.$$

Note that a bounded set $\Omega \subset H$ is relatively compact if and only if $\beta(\Omega) = 0$. Hence a map $T\colon H \to H$ is completely continuous if and only if it is a 0-ball contraction. Many results originally obtained for completely continuous maps have now been extended to k-ball contractions, provided $k < 1$.

Turning now to linear maps, we see that, if $T\colon H \to H$ is bounded and linear, then T is a $\| T \|$-ball contraction. However, as is easily seen by considering compact linear maps, $\| T \|$ need not equal

$$\gamma(T) = \inf\{k\colon T \text{ is a } k\text{-ball contraction}\}.$$

In fact, $\gamma(T) = 0$ if and only if T is compact. It is easily checked that γ defines a seminorm on the linear space of all bounded linear maps from H into itself. Just as $\| TS \| \leqq \| T \| \, \| S \|$ for bounded linear maps on H, the above seminorm has the property that $\gamma(TS) \leqq \gamma(T)\gamma(S)$. Concerning the involution $*$, denoting the adjoint, we recall that $\| T \| = \| T^* \| = \| T^*T \|^{\frac{1}{2}}$. Our main result shows that the seminorm γ has a similar property.

Theorem. *Let* H *be a Hilbert space and* $A\colon H \to H$ *a bounded linear map. Then*

$$\gamma(A) = \gamma(A^*) = \{\gamma(A^*A)\}^{\frac{1}{2}},$$

where $A^*\colon H \to H$ *denotes the adjoint of* A.

Proof. As is shown in (2), $\gamma(A) = \gamma(A^*)$. Furthermore, from what has been said above,

$$\gamma(A^*A) \leqq \gamma(A^*)\gamma(A) = \{\gamma(A)\}^2.$$

Hence it is sufficient to prove that

$$\{\gamma(A^*A)\}^{\frac{1}{2}} \geqq \gamma(A).$$

With this in mind, let $\gamma(A^*A) = k$. We shall now complete the proof by showing that A is a $k^{\frac{1}{2}}$-ball contraction.

We give the proof for a real Hilbert space; but, *mutatis mutandis*, it will establish the result for complex Hilbert spaces.

It is enough to show that, if $D = S(z, d)$ (the closed ball in H with centre z and radius d), then given any $\varepsilon > 0$, $A(D)$ can be covered by finitely many balls of radius less than or equal to $k^{\frac{1}{2}}d + \sqrt{2\varepsilon d}$.

Now $A^*A(S(0, 1))$ can be covered by finitely many balls of radius $k + \varepsilon/d$. Suppose that

$$A^*A(S(0, 1)) \subset \bigcup_{j=1}^{N} S(x_j, k+\varepsilon/d).$$

Since D is bounded, $\{(x_j, y)\colon y \in D\}$ is a relatively compact subset of the real line, for each $j \in \{1, \ldots, N\}$. Hence $\{(x_j, y)\colon y \in D\}$ can be covered by a finite number, M_j (say), of closed intervals S_i^j each of length less than or equal to ε, for $i \in \{1, \ldots, M_j\}$ and $j \in \{1, \ldots, N\}$. Let $p = (p_1, \ldots, p_N)$ where $p_j \in \{1, \ldots, M_j\}$ and set

$$E_p = \{y \in D\colon (x_j, y) \in S_{p_j}^j \text{ for each } j \in \{1, \ldots, N\}\}.$$

Clearly $A(D) = \bigcup A(E_p)$. Since this is a finite union the proof will be complete if we show that $A(E_p)$ is contained in a ball of radius $k^{\frac{1}{2}}d + \sqrt{2\varepsilon d}$.

With this in mind, we note that each E_p is closed and convex. Suppose now that E_p is non-empty. Then let z_p denote the unique nearest point of E_p to z. It follows that

$$\| z_p - y \| \leqq \| z - y \| \leqq d \quad \text{for all } y \in E_p. \tag{1}$$

We shall now show that $A(E_p) \subset S(Az_p, k^{\frac{1}{2}}d + \sqrt{2\varepsilon d})$. Let $y \in E_p$. Then

$$\begin{aligned} \| Ay - Az_p \|^2 &= (A(y-z_p), A(y-z_p)) \\ &= (A^*A(y-z_p), y-z_p) \\ &\leqq \| A^*A(y-z_p)\| \, \| y-z_p \| \\ &\leqq \| A^*A(y-z_p)\| \, d \qquad \text{by (1).} \end{aligned}$$

Now,

$$\begin{aligned} \| A^*A(y-z_p)\| &= \sup_{x \in S(0,1)} |(x, A^*A(y-z_p))| \\ &= \sup_{x \in S(0,1)} |(A^*Ax, y-z_p)|. \end{aligned}$$

But, for $x \in S(0, 1)$, there exists $j \in \{1, \ldots, N\}$ such that $\| A^*Ax - x_j \| \leqq k + \varepsilon/d$, and so

$$\begin{aligned} \|(A^*Ax, y-z_p)\| &\leqq |(A^*Ax - x_j, y-z_p)| + |(x_j, y-z_p)| \\ &\leqq (k+\varepsilon/d)d + |(x_j, y-z_p)| \quad \text{by (1)} \\ &\leqq kd + \varepsilon + \varepsilon. \end{aligned}$$

(Observe that, since y and $z_p \in E_p$, we have (x_j, y) and (x_j, z_p) both belong to the interval $S^j_{p_j}$ which has length less than or equal to ε.)

Therefore $\| A^*A(y-z_p)\| \leqq kd + 2\varepsilon$, and so

$$\begin{aligned} \| Ay - Az_p \| &\leqq (kd+2\varepsilon)d = kd^2 + 2\varepsilon d \\ &\leqq (k^{\frac{1}{2}}d + \sqrt{2\varepsilon d})^2. \end{aligned}$$

Hence $\| Ay - Az_p \| \leqq k^{\frac{1}{2}}d + \sqrt{2\varepsilon d}$, and the proof is complete.

Corollary. *Let H be a Hilbert space and A: $H \rightarrow H$ be a positive self-adjoint bounded linear map. Then, for any self-adjoint square root, $A^{\frac{1}{2}}$, of A, we have*

$$\gamma(A^{\frac{1}{2}}) = \{\gamma(A)\}^{\frac{1}{2}}.$$

Remark. Clearly this corollary has the classical result for compact linear maps as a special case.

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