THE HAMILTONIAN DYNAMICS
OF THE TWO GYROSTATS PROBLEM

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Abstract. The problem of two gyrostats in a central force field is considered. We prove that the Newton-Euler equations of motion are Hamiltonian with respect to a certain non-canonical structure. The system posseses symmetries. Using them we perform the reduction of the number of degrees of freedom. We show that at every stage of the reduction process, equations of motion are Hamiltonian and give explicit forms corresponding to non-canonical Poissson brackets. Finally, we study the case where one of the gyrostats has null gyrostatic momentum and we study the zero and the second order approximation, showing that all equilibria are unstable in the zero order approximation.

1. Introduction

The problem of roto-traslatory motion of n-rigid bodies has been studied, amongst other authors by Duboshin (1972), Aboelnaga (1979), Barkin (1980), Wang (1990; 1992), and by Maciejewski (1995). They considered the mutual interactions between orbital and rotational motion for artificial and natural bodies in the solar system. But the model of a rigid body to represent celestial bodies implies the absence of internal or relative motions. This is not always suitable, as was shown by Volterra in the study of variation of latitude on the Earth's surface. He explained the anomalies of the free rotation by means of internal or relative motions which do not modify the distribution of masses. A gyrostat is a mechanical system **S** composed of a rigid body $\mathbf{S}'$ and other bodies $\mathbf{S}''$ connected to it; these other bodies are either deformable or rigid, but their motion relative to $\mathbf{S}'$ does not alter the distribution of masses of the system **S**.

In this paper the problem of roto-translatory motion of two gyrostats in a central force field is considered following the papers of Maciejewski (7) and Wang (5). We prove that the Newton-Euler equations of motion are Hamiltonian with respect to a certain non-canonical structure. The system possesses symmetries and using them we perform the reduction of the number of degrees of freedom. We show that on every stage of this reduction process the equations of motion are Hamiltonian and we give the explicit form of corresponding non-canonical Poisson bracket. Finally, we restrict to the case where one of the gyrostats has null gyrostatic momentum and we study the zero and second order approximation. In the first case it is shown that only the Lagrangian equilibria can be obtained and furthermore those equilibria are unstable. In the second case we identify the Lagrangian equilibria. For the non-Lagrangian case we find a rather intricate non-linear systems of equations, where solutions only can be obtained by numerical or perturbative methods in particular cases. We show that apart from the Lagrangian equilibria, it can be found non-

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Lagrangian configurations in contrast with the assert of Maciejweski in the second case (7).

2. Two gyrostats problem

2.1. NOTATION

We will denote by bold italic letters vectors in $\boldsymbol{R}^3$ as geometrical objects,**a**, **b**, **c**. When a reference frame is fixed then the set of components of a vector **x** will be denoted by the corresponding bold Roman letter **x**, and will be considered as one column matrix, it is, $\mathbf{x} = [x_1, x_2, x_3]^T$, where $\mathbf{A}^T$ denotes the transposition of a matrix **A**. We use subscripts for numbering coordinates of a vector, and superscript for distinguishing vectors of different bodies. We will not distinguish the difference between **x** and **x** calling the last object coordinates vector. The standard scalar and vector products in $\boldsymbol{R}^3$, and the length of a vector will be denoted respectively by $< \mathbf{x}, \mathbf{y} >$, $(\mathbf{x} \times \mathbf{y}$ and $\|\mathbf{x}\| = \sqrt{< \mathbf{x}, \mathbf{x} >}$. Gyrostatic motion is described in a chosen reference frame. The reference frames used in this paper are right handed and orthonormal. Here the special orthogonal group $SO(3)$ will be identified with all 3×3 real orthogonal matrices with determinant $+1$. Its Lie algebra, denoted by $so(3)$, is identified with all 3×3 anti-symmetric matrices.

For a vector **x** we denote by $\widehat{\mathbf{x}}$ the image of **x** by the standard isomorphism between the Lie algebra $\boldsymbol{R}^3$ (with the vector product as the algebra multiplication) and $so(3)$.

2.2. EQUATIONS OF MOTION

Let us consider two gyrostats S_1 and S_2 that interact mutually according to the universal law of gravitation. The quantities related to gyrostat (i) will be denoted, in coordinates, by a superscript i, and in vector notation by subscript i. We describe the motion of gyrostats with respect to an arbitrary inertial frame $R = \{u_1, u_2, u_3\}$. The body fixed frame in each gyrostat R_1 and R_2 is attached to its mass center C_i, $i = 1, 2$ and the first frame coincide with the principal axes of inertia of S_1. We denote the versors of the inertial reference frame in the systemsR_i by $\boldsymbol{\alpha}_i$, $\boldsymbol{\beta}_i$ and $\boldsymbol{\gamma}_i$ respectively. Then the matrices

$$\mathbf{A}_i = \begin{bmatrix} \boldsymbol{\alpha}_i^T \\ \boldsymbol{\beta}_i^T \\ \boldsymbol{\gamma}_i^T \end{bmatrix} = \begin{bmatrix} \alpha_1^i & \alpha_2^i & \alpha_3^i \\ \beta_1^i & \beta_2^i & \beta_3^i \\ \gamma_1^i & \gamma_2^i & \gamma_3^i \end{bmatrix} \in SO(3) \quad i = 1, 2, \tag{1}$$

are called the attitude matrices or the direction cosines matrices.

We have the following equations for the time dependence of the attitude matrices

$$\dot{\mathbf{A}}_i = \mathbf{A}_i \widehat{\boldsymbol{\Omega}}_i \quad i = 1, 2, \tag{2}$$

where $\boldsymbol{\Omega}_i$ is the body angular velocity of S_i. Coordinates of this vector in the inertial frame R are $\omega_i=[\omega_1^i, \omega_2^i, \omega_3^i]^T = \mathbf{A}_i\boldsymbol{\Omega}_i$, $i = 1, 2$ and following equation is satisfied

$$\dot{\mathbf{A}}_i = \hat{\omega}_i \mathbf{A}_i. \tag{3}$$

We will denote by $\mathbf{R}_i$, $i = 1, 2$ the position vector of each mass center C_i in the frame R . If $\mathbf{Q}_i = [Q_1^i, Q_2^i, Q_3^i]^T$ are the coordinates of a generic point of S_i in the inertial frame R and $\mathbf{q}_i = [q_1^i, q_2^i, q_3^i]^T$ are the coordinates in the frames R_i, $i = 1, 2$, then the relations $\mathbf{Q}_i = \mathbf{R}_i + \mathbf{A}_i\mathbf{q}_i$, $i = 1, 2$ are satisfied.

The linear and the total angular momenta of the gyrostats are

$$\mathbf{P}_i = \int_{S_i} \frac{d}{dt}\{\mathbf{R}_i + \mathbf{A}_i\mathbf{q}_i\}\, dm_i = m_i\, \dot{\mathbf{R}}_i, \tag{4}$$

$$\mathbf{L}_i = \int_{S_i} \{\mathbf{R}_i + \mathbf{A}_i\mathbf{q}_i\} \times \frac{d}{dt}\{\mathbf{R}_i + \mathbf{A}_i\mathbf{q}_i\}\, dm_i = \mathbf{R}_i \times \mathbf{P}_i + \boldsymbol{\pi}_i + \mathbf{l}_{r_i}\; i = 1, 2, \tag{5}$$

where

$$m_i = \int_{S_i} dm_i, \quad \boldsymbol{\pi}_i = \mathbf{A}_i\boldsymbol{\Pi}_i, \quad \boldsymbol{\Pi}_i = \mathbf{I}_i\boldsymbol{\Omega}_i,$$

$$\mathbf{I}_i = \int_{S_i} \hat{\mathbf{q}}_i\hat{\mathbf{q}}_i^T\, dm_i, \quad \mathbf{l}_{r_i} = \mathbf{A}_i\mathbf{L}_{r_i}, \quad \mathbf{L}_{r_i} = \int_{S_i} (\mathbf{q}_i \times \dot{\mathbf{q}}_i)\, dm_i, \tag{6}$$

are the masses of the gyrostats, the rotational angular momenta of the gyrostats considered as rigid bodies in the frame R and R_i, $i = 1, 2$, the tensors of inertia and the gyrostatic momenta of the gyrostats in the frame$R$ and R_i (i.e. the relative angular momenta of the mobile parts of the gyrostatsS_i), respectively. In the following we will assume that the $\mathbf{L}_{r_i}$ are known constants.

The kinetic energy of the system is the sum of the kinetic energies of the individual gyrostats

$$\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2,$$

$$\begin{aligned} \mathbf{T}_i &= \frac{1}{2}\int_{S_i} \left\|\dot{\mathbf{q}}_i\right\|^2 dm_i = \mathbf{T}_{1i} + \mathbf{T}_{2i} + \mathbf{T}_{ri}, \\ \mathbf{T}_{1i} &= < \mathbf{L}_{r_i}, \mathbf{I}_i^{-1}\boldsymbol{\Pi}_i >, \mathbf{T}_{2i} = \frac{\|\mathbf{P}_i\|}{2m_i} + \frac{1}{2} < \boldsymbol{\Pi}_i, \mathbf{I}_i^{-1}\boldsymbol{\Pi}_i >, \end{aligned} \tag{7}$$

where $\mathbf{T}_{ri}$ are the relative kinetic energies of the gyrostats S_i and we will assume that $\mathbf{T}_{ri}$ are known functions of time.

The gravitational potential energy of the system is given by

$$\mathbf{U}(\mathbf{R}_1, \mathbf{A}_1; \mathbf{R}_2, \mathbf{A}_2) = -\mathbf{G}\int_{S_1}\int_{S_2} \frac{dm_1\, dm_2}{\|\mathbf{R}_1 + \mathbf{A}_1\mathbf{q}_1 - \mathbf{R}_2 - \mathbf{A}_2\mathbf{q}_2\|}, \tag{8}$$

where $\mathbf{G}$ is the universal gravitational constant, see (Cid and Vigueras, 1995).

In the Newtonian description of motion, we choose as independent variables, the inertial coordinates of radii vectors of mass centers $\mathbf{R}_i$, the linear momenta $\mathbf{P}_i$, the attitude matrices $\mathbf{A}_i$ for $i = 1, 2$, and describe the rotational motions with respect to each gyrostats reference frames using the rotational angular momenta $\boldsymbol{\Pi}_i$. We denote by $\mathbf{f}_{ij}$, $\mathbf{m}_{ij}$ $i, j = 1, 2$ the gravitational forces and the resultant torques acting on the gyrostat S_i expressed in the inertial frame R. Newton equations have the following form

$$\dot{\mathbf{R}}_i = \frac{\mathbf{P}_i}{m_i}, \quad \dot{\mathbf{P}}_i = \mathbf{f}_{ij}, \quad \dot{\boldsymbol{\Pi}}_i = (\boldsymbol{\Pi}_i + \mathbf{L}_{r_i}) \times \mathbf{I}_i^{-1}\boldsymbol{\Pi}_i + \mathbf{M}_{ij}, \tag{9}$$

$$\dot{\mathbf{A}}_i = \mathbf{A}_i\widehat{\boldsymbol{\Omega}}_i \quad i \neq j = 1, 2 \quad \mathbf{M}_{ij} = \mathbf{A}_i^T\mathbf{m}_{ij}. \tag{10}$$

The Newton equations have the classical first integrals

$$\mathcal{H} = \mathbf{T}_{21} + \mathbf{T}_{22} + \mathbf{U}, \quad \mathcal{P} = \mathbf{P}_1 + \mathbf{P}_2, \quad \mathcal{L} = \mathbf{L}_1 + \mathbf{L}_2,$$

$$\text{and } \mathcal{D}_i = \mathbf{A}_i\mathbf{A}_i^T \quad i = 1, 2,$$

where $\mathcal{H}$ is the Jacobi integral, $\mathcal{P}$ and $\mathcal{L}$ are integrals of total linear and angular momenta respectively, and $\mathcal{D}_i$ are geometrical integrals, in fact $\mathcal{D}_i = E$ the identity matrix.

2.3. Hamiltonian Structure of the Equations of Motion

From the expression of attitude matrices (1) we write the gravitational potential in the following form

$$\mathbf{U}(\mathbf{R}_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \boldsymbol{\gamma}_1; \mathbf{R}_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \boldsymbol{\gamma}_2) = \int_{S_1}\int_{S_2} \frac{-\mathbf{G}\, dm_1\, dm_2}{\|\mathbf{R}_1 + \mathbf{A}_1\mathbf{q}_1 - \mathbf{R}_2 - \mathbf{A}_2\mathbf{q}_2\|}. \tag{11}$$

Trivially we obtain

$$\mathbf{f}_{12} = -\frac{\partial \mathbf{U}}{\partial \mathbf{R}_1}, \ \mathbf{f}_{21} = -\frac{\partial \mathbf{U}}{\partial \mathbf{R}_2}, \ \mathbf{M}_{ij} = \boldsymbol{\alpha}_i \times \frac{\partial \mathbf{U}}{\partial \boldsymbol{\alpha}_i} + \boldsymbol{\beta}_i \times \frac{\partial \mathbf{U}}{\partial \boldsymbol{\beta}_i} + \boldsymbol{\gamma}_i \times \frac{\partial \mathbf{U}}{\partial \boldsymbol{\gamma}_i}. \tag{12}$$

Then the Newton-Euler equations (9)-(10) can be expressed as follows

$$\dot{\mathbf{R}}_i = \frac{\mathbf{P}_i}{m_i}, \quad \dot{\mathbf{P}}_i = -\frac{\partial \mathbf{U}}{\partial \mathbf{R}_i}, \tag{13}$$

$$\dot{\boldsymbol{\Pi}}_i = (\boldsymbol{\Pi}_i + \mathbf{L}_{r_i}) \times \mathbf{I}_i^{-1}\boldsymbol{\Pi}_i + \boldsymbol{\alpha}_i \times \frac{\partial \mathbf{U}}{\partial \boldsymbol{\alpha}_i} + \boldsymbol{\beta}_i \times \frac{\partial \mathbf{U}}{\partial \boldsymbol{\beta}_i} + \boldsymbol{\gamma}_i \times \frac{\partial \mathbf{U}}{\partial \boldsymbol{\gamma}_i}, \tag{14}$$

$$\frac{d\boldsymbol{\alpha}_i}{dt} = \boldsymbol{\alpha}_i \times \mathbf{I}_i^{-1}\boldsymbol{\Pi}_i, \quad \frac{d\boldsymbol{\beta}_i}{dt} = \boldsymbol{\beta}_i \times \mathbf{I}_i^{-1}\boldsymbol{\Pi}_i, \quad \frac{d\boldsymbol{\gamma}_i}{dt} = \boldsymbol{\gamma}_i \times \mathbf{I}_i^{-1}\boldsymbol{\Pi}_i. \tag{15}$$

Denote $\mathbf{z}=[\mathbf{z}_1^T,\mathbf{z}_2^T]^T$ with $\mathbf{z}_i = [\mathbf{R}_i^T,\mathbf{P}_i^T,\boldsymbol{\Pi}_i^T,\boldsymbol{\alpha}_i^T,\boldsymbol{\beta}_i^T,\boldsymbol{\gamma}_i^T]^T \in \mathbb{R}^{18}$ $\quad i = 1,2$ and define a twice contravariant skew-symmetric tensor field Λ on $\mathbb{R}^{36}$ that in matrix form is

$$\Lambda[z] = \begin{bmatrix} \Lambda_1[\mathbf{z}] & 0 \\ 0 & \Lambda_2[\mathbf{z}] \end{bmatrix}, \Lambda_i[\mathbf{z}] = \begin{bmatrix} \mathbf{J}_r[\mathbf{z}] & 0 \\ 0 & \mathbf{J_{II}}[\mathbf{z}_i] \end{bmatrix}, \tag{16}$$

$$\mathbf{Jr}(\mathbf{z}) = \begin{bmatrix} \mathbf{0} & \mathbf{E} \\ -\mathbf{E} & \mathbf{0} \end{bmatrix} \qquad \mathbf{J_{II}}(\mathbf{z}_i) = \begin{bmatrix} \widehat{\boldsymbol{\Pi}_i + \mathbf{L}_{ri}} & \widehat{\boldsymbol{\alpha}}_i & \widehat{\boldsymbol{\beta}}_i & \widehat{\boldsymbol{\gamma}}_i \\ \widehat{\boldsymbol{\alpha}}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \widehat{\boldsymbol{\beta}}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \widehat{\boldsymbol{\gamma}}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{17}$$

This tensor field allows us to define Poisson bracket on $\mathbb{R}^{36}$ $\{\cdot,\cdot\}$, that for $\mathbf{f},\mathbf{g} \in C^\infty(\mathbb{R}^{36})$ $\quad \{\mathbf{f},\mathbf{g}\}(\mathbf{z}) = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{z}}\right)^T \Lambda[\mathbf{z}] \left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}\right)$. Then $\left(\mathbb{R}^{36}, \{\cdot,\cdot\}_{\mathcal{M}}\right)$ is a Poisson manifold.

Let us consider the Hamiltonian function on $\mathbb{R}^{36}$

$$\mathcal{H}[\mathbf{z}] = \sum_{i=1}^{2} \left\{ \frac{\|\mathbf{P}_i\|^2}{2m_i} + \frac{1}{2} < \boldsymbol{\Pi}_i, \mathbf{I}_i^{-1}\boldsymbol{\Pi}_i > \right\} + \mathbf{U}(\mathbf{z}). \tag{18}$$

Then the Newton-Euler equations of motion (13)-(15) can be written in Hamiltonian form as $\frac{d\mathbf{Z}}{dt} = \{\mathbf{z},\mathcal{H}\}\ (\mathbf{z})$.

2.3.1. Relative coordinates I

In the first step of the reduction process we use the fact that the interaction between gyrostats depends on the relative position of the bodies. We consider the relative variables

$$\mathbf{r}=\mathbf{R}_1 - \mathbf{R}_2, \quad \mathbf{p}=\frac{m_2}{m_1+m_2}\mathbf{P}_1 - \frac{m_1}{m_1+m_2}\mathbf{P}_2.$$

In these variables we can reduce the number of equations by six. The system of 30 equations that we obtain is Hamiltonian with respect to the Poisson bracket in $\mathbb{R}^{30}$ define below.

We now denote $\mathbf{z} = \left[\mathbf{r}^T,\mathbf{p}^T,\mathbf{z}_1^T,\mathbf{z}_2^T\right]^T$, $\mathbf{z}_i = \left[\boldsymbol{\Pi}_i^T,\boldsymbol{\alpha}_i^T,\boldsymbol{\beta}_i^T,\boldsymbol{\gamma}_i^T\right]^T$ $\quad i = 1,2$ and define the Poisson bracket $\{.,.\}_I$ in $\mathbb{R}^{30}$ by

$$\{\mathbf{f},\mathbf{g}\}_I(\mathbf{z}) = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{z}}\right)^T \Lambda_I[\mathbf{z}] \left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}\right) \text{ where} \tag{19}$$

$$\Lambda_I[\mathbf{z}] = \begin{bmatrix} \mathbf{Jr}(\mathbf{z}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{\alpha II}(\mathbf{z}_1) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J_{II}}(\mathbf{z}_2) \end{bmatrix}. \tag{20}$$

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In this Poisson structure the new equations are Hamiltonian in the form

$$\dot{z} = \{\mathbf{z}, \mathcal{H}_I\}_I,$$

with respect to the Hamiltonian

$$\mathcal{H}_I = \frac{\|\mathbf{p}\|^2}{2m} + \frac{1}{2}\sum_{i=1}^{2}\{< \boldsymbol{\Pi}_i, \mathbf{I}_i^{-1}\boldsymbol{\Pi}_i >\} + \mathbf{U}(\mathbf{z}). \tag{21}$$

The Poisson bracket defined by (19)-(20) is just the effect of reduction of the bracket defined by (16)-(17) with respect to the action of the group of traslations on the phase space.

2.3.2. *Relative coordinates II*

In this step of reduction process we will describe the whole motion of the system with respect to the frame fixed in the gyrostat S_2. Then we introduce the new variables

$$\mathbf{R} = \mathbf{A}_2^T\mathbf{r}, \quad \mathbf{P} = \mathbf{A}_2^T\mathbf{p}, \quad \boldsymbol{\Gamma}_1 = \mathbf{A}^T\boldsymbol{\Pi}_1, \quad \boldsymbol{\Gamma}_2 = \boldsymbol{\Pi}_2, \tag{22}$$

$$\mathbf{A} = \mathbf{A}_1^T\mathbf{A}_2, \quad \mathbf{A} = \begin{bmatrix} \boldsymbol{\alpha}^T \\ \boldsymbol{\beta}^T \\ \boldsymbol{\gamma}^T \end{bmatrix}. \tag{23}$$

Now we denote $\mathbf{v}\left[\mathbf{R}^T, \mathbf{P}^T, \boldsymbol{\Gamma}_1^T, \boldsymbol{\Gamma}_2^T, \boldsymbol{\alpha}^T, \boldsymbol{\beta}^T, \boldsymbol{\gamma}^T\right]^T \in \mathbb{R}^{21}$ and define the following Poisson bracket $\{.,.\}_{II}$ in $\mathbb{R}^{21}$, $\{\mathbf{f},\mathbf{g}\}_{II}(\mathbf{v}) = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{v}}\right)^T \boldsymbol{\Delta}[\mathbf{v}] \left(\frac{\partial \mathbf{g}}{\partial \mathbf{v}}\right)$ for $\mathbf{f}, \mathbf{g} \in C^\infty\left(\mathbb{R}^{21}\right)$ where $\boldsymbol{\Delta}[\mathbf{v}]$ is the Poisson tensor defined as follows

$$\boldsymbol{\Delta}[\mathbf{v}] = \begin{bmatrix} \mathbf{0} & \mathbf{E} & \mathbf{0} & \widehat{\mathbf{R}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{E} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{P}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \widehat{\mathbf{R}} & \widehat{\mathbf{P}} & \mathbf{A}^T\widehat{\mathbf{L}_{r_1}} - \boldsymbol{\Gamma}_1 & \widehat{\boldsymbol{\Gamma}_1} & -\widehat{\boldsymbol{\alpha}} & -\widehat{\boldsymbol{\beta}} & -\widehat{\boldsymbol{\gamma}} \\ \mathbf{0} & \mathbf{0} & \widehat{\boldsymbol{\Gamma}_1} & \widehat{\boldsymbol{\Gamma}_2 + \mathbf{L}_{r_2}} & \widehat{\boldsymbol{\alpha}} & \widehat{\boldsymbol{\beta}} & \widehat{\boldsymbol{\gamma}} \\ \mathbf{0} & \mathbf{0} & -\widehat{\boldsymbol{\alpha}} & \widehat{\boldsymbol{\alpha}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\widehat{\boldsymbol{\beta}} & \widehat{\boldsymbol{\beta}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\widehat{\boldsymbol{\gamma}} & \widehat{\boldsymbol{\gamma}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

In this Poisson structure the equations of motion described in the new variables (22)-(23) are Hamiltonian with respect the Hamiltonian $\mathcal{H}_{II}$ defined by

$$\mathcal{H}_{II} = \frac{\|\mathbf{P}\|^2}{2m} + \mathbf{H}_r + \mathbf{U}(\mathbf{v}), \tag{24}$$

where $\mathbf{U}(\mathbf{v}) = -\mathbf{G}\int_{S_1}\int_{S_2} \frac{dm_1\, dm_2}{\left\|\mathbf{R}+\mathbf{A}^T\mathbf{q}_1-\mathbf{q}_2\right\|}$

and $\mathbf{H}_r = \frac{1}{2} < \boldsymbol{\Gamma}_1, \mathbf{A}^T\mathbf{I}_1^{-1}\mathbf{A}\boldsymbol{\Gamma}_1 > +\frac{1}{2} < \boldsymbol{\Gamma}_2, \mathbf{I}_2^{-1}\boldsymbol{\Gamma}_2>$. Then in $\left(\boldsymbol{R}^{21}, \{\cdot,\cdot\}_{II}\right)$ the induced Hamiltonian equations become $\frac{d\mathbf{v}}{dt} = \{\mathbf{z}, \mathcal{H}_{II}\}_{II}\,(\mathbf{v})$.

Finally, we note that the Poisson bracket $\{\cdot,\cdot\}_{II}$ is degenerated, i. e., there are non-constant Casimir functions: six geometrical integrals and the total angular momentum of the system

$$\mathcal{C}_1 = \frac{1}{2}\boldsymbol{\alpha}^T\boldsymbol{\alpha}, \quad \mathcal{C}_2 = \boldsymbol{\alpha}^T\boldsymbol{\beta}, \quad \mathcal{C}_3 = \boldsymbol{\alpha}^T\boldsymbol{\gamma}, \quad \mathcal{C}_4 = \frac{1}{2}\boldsymbol{\beta}^T\boldsymbol{\beta}, \quad \mathcal{C}_5 = \boldsymbol{\beta}^T\boldsymbol{\gamma},$$

$$\mathcal{C}_6 = \frac{1}{2}\boldsymbol{\gamma}^T\boldsymbol{\gamma}, \mathcal{F} = \frac{1}{2}\mathbf{L}\mathbf{L}^T, \ \mathbf{L} = \boldsymbol{\Gamma}_1 + \mathbf{A}^T\mathbf{L}_{r_1} + \boldsymbol{\Gamma}_2 + \mathbf{R}\times\mathbf{P}.$$

3. Relative Equilibria

The relative equilibria of our system are the equilibria for the Hamiltonian dynamic generated by $\mathcal{H}_{II}$ on $\mathcal{M}_{II}$, i.e. $\mathbf{v}_e \in \mathcal{M}_{II}$ such that $\{\mathbf{v}, \mathcal{H}_{II}\}_{\mathcal{M}_{II}}(\mathbf{v}_e) = \mathbf{0}$. Using vector algebra in the first equations of the Poisson system we show that there exists $\lambda \in \boldsymbol{R}$ such that we can write the conditions for the equilibria as

$$\|\boldsymbol{\Omega}\|^2\mathbf{R} - < \mathbf{R}, \boldsymbol{\Omega} > \boldsymbol{\Omega} - \frac{1}{m}\frac{\partial \mathbf{U}}{\partial \mathbf{R}} = \mathbf{0}, \tag{25}$$

$$(\boldsymbol{\Gamma}_2 + \mathbf{L}_{r_2}) \times \boldsymbol{\Omega} + \boldsymbol{\mu}_2 = \mathbf{0}, \tag{26}$$

$$\boldsymbol{\Gamma}_1 + \mathbf{A}^T\mathbf{L}_{r_1} + \boldsymbol{\Gamma}_2 + \mathbf{L}_{r_2} - m < \mathbf{R}, \boldsymbol{\Omega} > \boldsymbol{\Omega} - \lambda\boldsymbol{\Omega} = 0, \tag{27}$$

$$\boldsymbol{\alpha}^T\boldsymbol{\alpha} = \boldsymbol{\beta}^T\boldsymbol{\beta} = \boldsymbol{\gamma}^T\boldsymbol{\gamma} = 1, \ \boldsymbol{\alpha}^T\boldsymbol{\beta} = \boldsymbol{\alpha}^T\boldsymbol{\gamma} = \boldsymbol{\beta}^T\boldsymbol{\gamma} = 0, \tag{28}$$

with $\boldsymbol{\mu}_2 = -G\int_{S_1}\int_{S_2} q_2 \times \left((q_2 - \mathbf{R} - \mathbf{A}^T q_1)/\left\|\mathbf{R} + \mathbf{A}^T q_1 - q_2\right\|^3\right) dm(q_1)dm(q_2)$ and $\mathbf{P} = m\boldsymbol{\Omega} \times \mathbf{R}$, $\boldsymbol{\Omega}_1 = \mathbf{A}\boldsymbol{\Omega}_2$ and $\boldsymbol{\Omega} = \boldsymbol{\Omega}_2$.

Using the expression of $\mathbf{P}$ in the equilibria we obtain trivially $\mathbf{L} = \left(\lambda + m\,\|\mathbf{R}\|^2\right)\boldsymbol{\Omega}$, so $\mathbf{L}$ is parallel to $\boldsymbol{\Omega}$.

There are two types of equilibria, the Lagrangian equilibria where $\mathbf{R}$ is orthogonal to $\boldsymbol{\Omega}$ and the non-Lagrangian equilibria where $< \mathbf{R}, \boldsymbol{\Omega} > \neq 0$. In the first case we have $\|\mathbf{r}\| = \|\mathbf{R}\| = cte$ and $< \mathbf{r}, \boldsymbol{\omega} > = < \mathbf{R}, \boldsymbol{\Omega} > = 0$. Then the gyrostat S_1 describes a circular orbit with respect to the gyrostat S_2 in a plane orthogonal to the vector $\boldsymbol{\omega}$. In the non-Lagrangian case $\|\mathbf{r}\| = \|\mathbf{R}\| = cte$ and $< \mathbf{r}, \boldsymbol{\omega} > = < \mathbf{R}, \boldsymbol{\Omega} > \neq 0$, so the projection of $\mathbf{r}$ over the vector $\boldsymbol{\omega}$ is constant. Then the gyrostat S_1 describes, with respect to the gyrostat S_2, a circular orbit in a cone of axis $\boldsymbol{\omega}$.

Finally, we note that we have adopted the names of Lagrangian and non-Lagrangian equilibria as Maciejewski suggests in (Maciejewski, 1995). The results obtained here for the two gyrostats problem are similar to the ones obtained by Maciejewski for the two rigid bodies.

4. Approximations

In the present section we consider two approximations of the gravitational potential **U** based on Taylor series in a neighborhood of ε =(nominal dimension of gyrostats)/(orbital radius). The Taylor series expression of **U** is

$$\begin{aligned}\mathbf{U}(\mathbf{v}) &= \mathbf{U}_0+\mathbf{U}_1+o\left(\|\mathbf{R}\|^5\right), \quad \mathbf{U}_0=\frac{-Gm_1m_2}{\|\mathbf{R}\|},\\ \mathbf{U}_1 &= -\frac{G}{2\|\mathbf{R}\|^3}\{m_2Tr\mathbf{I}_1+m_1Tr\mathbf{I}_2\}+\\ &\quad \frac{3G}{2\|\mathbf{R}\|^5}\{m_2<\mathbf{AR},\mathbf{I}_1\mathbf{AR}>+m_1<\mathbf{R},\mathbf{I}_2\mathbf{R}>\}.\end{aligned}$$

We will consider therefore two approximate Hamiltonians

$$\mathcal{H}_0=\frac{\|\mathbf{P}\|^2}{2m}+\mathbf{H}_r+\mathbf{U}_0, \quad \mathcal{H}_1=\mathcal{H}_0+\mathbf{U}_1. \tag{29}$$

In the following we will consider that the gyrostatic momentum of S_2 is null, i. e., $\mathbf{L}_{r_2}=\mathbf{0}$.

4.1. Zero order approximation

Let us consider the dynamic generated by $\mathcal{H}_0$. Then the condition for equilibria (25)-(28) can be written

$$\|\boldsymbol{\Omega}\|^2\mathbf{R}\text{-} <\mathbf{R},\boldsymbol{\Omega}>\boldsymbol{\Omega}=\frac{Gm_1m_2}{\|\mathbf{R}\|}\mathbf{R}, \boldsymbol{\Gamma}_2\times\boldsymbol{\Omega}=\mathbf{0},$$
$$\boldsymbol{\Gamma}_1+\mathbf{A}^T\mathbf{L}_{r_1}+\boldsymbol{\Gamma}_2-m<\mathbf{R},\boldsymbol{\Omega}>\boldsymbol{\Omega}-\lambda\boldsymbol{\Omega}=0, \mathbf{A}\in SO(3).$$

Taking cross product in the above first equation by **R** we obtain that **R** is orthogonal to $\boldsymbol{\Omega}$ in the equilibria (because we exclude the trivial case **P=0**), then in the zero order approximation only the Lagrangian equilibria are possible. Now from the second above equation we deduce that $\boldsymbol{\Gamma}_2$ is parallel to $\boldsymbol{\Omega}$, and so $\boldsymbol{\Omega}$ is an eigenvector of $\mathbf{I}_2$. Then **R, P** and $\boldsymbol{\Omega}$ form a triad in the equilibria, and without loss of generality we assign

$$\boldsymbol{\Omega}_e=\|\boldsymbol{\Omega}_e\|\,\mathbf{e}_1^2, \quad \mathbf{R}_e=\|\mathbf{R}_e\|\,\mathbf{e}_3^2, \quad \mathbf{P}_e=m\,\|\boldsymbol{\Omega}_e\|\,\|\mathbf{R}_e\|\,\mathbf{e}_1^2\times\mathbf{e}_3^2,$$

where $\{\mathbf{e}_1^2,\mathbf{e}_2^2,\mathbf{e}_3^2\}$ is the basis of the frame R_2. We describe the equilibria in this basis and trivialy we have

$$\boldsymbol{\Gamma}_{2e}=\|\boldsymbol{\Omega}_e\|\,I_1^2\mathbf{e}_1^2, \quad \boldsymbol{\Gamma}_{1e}=\|\boldsymbol{\Omega}_e\|\,\mathbf{A}_e^T\mathbf{I}_1\mathbf{A}_e\mathbf{e}_1^2, \quad \mathbf{I}_2=\begin{bmatrix} I_1^2 & 0 & 0\\ 0 & I_2^2 & J\\ 0 & J & I_3^2\end{bmatrix},$$

such that $(\mathbf{I}_1\mathbf{A}_e\mathbf{e}_1^2 + \mathbf{L}_{r1}) \times \mathbf{A}_e\mathbf{e}_1^2 = \mathbf{0}$, $\quad m\,\|\boldsymbol{\Omega}_e\|^2 = \frac{Gm_1m_2}{\|\mathbf{R}_e\|}$ where $\mathbf{A}_e \in SO(3)$.

Proceeding as Wang (1990), let us consider now the following solutions of the Hamiltonian system $d\mathbf{v}/dt = \{\mathbf{v}, \mathcal{H}_0\}_{\mathbf{II}}(\mathbf{v})$

$$\boldsymbol{\Gamma}_1(t) = \boldsymbol{\Gamma}_{1e},\ \boldsymbol{\Gamma}_2(t) = \boldsymbol{\Gamma}_{2e},\ \mathbf{A}(tA\alpha) = \boldsymbol{A}_e, \tag{30}$$

$$\mathbf{R}(t) = \exp\left(t \cdot \frac{\mathbf{k}}{\left\|\mathbf{I}_2^{-1}\boldsymbol{\Gamma}_{2e}\right\|}\widehat{\mathbf{I}_2^{-1}\boldsymbol{\Gamma}_{2e}}\right)\mathbf{R}_e, \tag{31}$$

$$\mathbf{P}(t) = m\left(1 + \frac{\mathbf{k}}{\left\|\mathbf{I}_2^{-1}\boldsymbol{\Gamma}_{2e}\right\|}\right)\exp\left(t \cdot \frac{\mathbf{k}}{\left\|\mathbf{I}_2^{-1}\boldsymbol{\Gamma}_{2e}\right\|}\widehat{\mathbf{I}_2^{-1}\boldsymbol{\Gamma}_{2e}}\right)\widehat{\mathbf{I}_2^{-1}\boldsymbol{\Gamma}}_{2e}\mathbf{R}_e, \tag{32}$$

verifying the modified formula $\left(\mathbf{k}+\left\|\mathbf{I}_2^{-1}\boldsymbol{\Gamma}_{2e}\right\|\right)^2 = \frac{Gm_1m_2}{\|\mathbf{R}_e\|}$.

The above solutions represent a pertubation of the above zero order relative equilibria and it is easy to prove that the solutions escape from any small neigborhood of the relative equilibria in finite time. Therefore we have instability.

4.2. Second order approximation

We now take the dynamic generated by the Hamiltonian $\mathcal{H}_1$. The equilibria conditions (25)-(28) read now

$$m\,\|\boldsymbol{\Omega}\|^2\mathbf{R} - m < \mathbf{R}, \boldsymbol{\Omega} > \boldsymbol{\Omega} = \frac{Gm_1m_2}{\|\mathbf{R}\|}\mathbf{R} +$$

$$\frac{3G}{2\,\|\mathbf{R}\|^3}\{\{m_2Tr\mathbf{I}_1 + m_1Tr\mathbf{I}_2\}\mathbf{R} + 2m_2\mathbf{A}^T\mathbf{I}_1\mathbf{A}\mathbf{R} + 2m_1\mathbf{I}_2\mathbf{R} -$$

$$\frac{15G}{2\,\|\mathbf{R}\|^5}\{m_2 <, I\alpha \boldsymbol{AR}, \boldsymbol{I}_1 > \alpha \boldsymbol{AR} > + m_1 < \mathbf{R}, \mathbf{I}_2\mathbf{R} >\}\mathbf{R}, \tag{33}$$

$$\boldsymbol{\Gamma}_2 \times \boldsymbol{\Omega} + \boldsymbol{\mu}_2 = \mathbf{0} \quad \text{with} \quad \boldsymbol{\mu}_2 = \frac{3Gm_2}{2\,\|\mathbf{R}\|^5}\mathbf{R} \times \mathbf{I}_2\mathbf{R}, \tag{34}$$

$$[\mathbf{A}^T\mathbf{I}_1\mathbf{A}\boldsymbol{\Omega} + \mathbf{A}^T\mathbf{L}_{r_1} + \mathbf{I}_2\boldsymbol{\Omega} - m < \mathbf{R}, \boldsymbol{\Omega} > \boldsymbol{\Omega}]\lambda\boldsymbol{\Omega} = 0. \tag{35}$$

The above system is a cumbersome non-linear system. Nevertheless, we can always find numerically solutions for specific values fixed of the parameters, or introducing a small parameter, we can use perturbative techniques. An example of Non-Lagrangian equilibria was found numerically by Wang (1990) in the case of a rigid body in a central Newtonian force field. Here, we will restrict to the Lagrangian case.

As in the previous subsection we will denote the equilibria solutions with subscripts e. We choose $\{\boldsymbol{\Omega}_e/\,||\boldsymbol{\Omega}_e||\,, \mathbf{P}_e/\,||\mathbf{P}_e||\,, \mathbf{R}_e/\,||\mathbf{R}_e||\}$ as the reference frame R_2. Using the equations (34) it is easy to show that the inertial tensor $\mathbf{I}_2$ is diagonal on this basis.

Now from equation (35) we have that $\mathbf{I}_1\mathbf{A}_e\boldsymbol{\Omega}_e$+$\mathbf{L}_{r_1}$ is parallel to $\mathbf{A}\boldsymbol{\Omega}_e$. Taking cross product in (33) by $\mathbf{R}$ we obtain $\mathbf{I}_1\mathbf{A}_e\mathbf{R}_e$ parallel to $\mathbf{A}_e\mathbf{R}_e$. Since $\mathbf{A}_e \in SO(3)$ and takes eigenvectors of $\mathbf{I}_2$ into eigenvectors of $\mathbf{I}_1$ we deduce $\mathbf{A}_e$ belong to

$$\left\{\begin{bmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1 \end{bmatrix}, a^2+b^2=1;\ a,b \in \boldsymbol{R}\right\}. \tag{36}$$

Taking scalar product in (34) by $\mathbf{R}_e$, and since $\mathbf{A}_e$ takes $\mathbf{e}_3$ into $\mathbf{e}_j$ $j \in \{1,2,3\}$, we obtain

$$\begin{aligned} ||\boldsymbol{\Omega}_e||^2 \;=\; & \frac{G}{||\mathbf{R}_e||^3}\left\{m_1m_2 + m_2I_3^1 + m_1I_3^2\right\} + \\ & \frac{3G}{2\,||\mathbf{R}_e||^5}\left\{m_2(Tr\mathbf{I}_1 - 5I_j^1) + m_1(Tr\mathbf{I}_2 - 5I_j^2)\right\}. \end{aligned} \tag{37}$$

Then if $||\mathbf{R}_e||$ is large enough it is clear that there exist solutions of the above equation. Denoting $\mathbf{L}_{r_1} = [a_1, b_1, c_1]^T$, and since $\mathbf{I}_1\mathbf{A}_e\boldsymbol{\Omega}_e + \mathbf{L}_{r_1}$ is parallel to $\mathbf{A}\boldsymbol{\Omega}_e$ we obtain $||\boldsymbol{\Omega}_e||\,(I_2^2 - I_1^1)ab - b_1a - a_1b = 0, \quad c_1 = 0$.

If we choose $\{\mathbf{P}_e/\,||\mathbf{P}_e||\,, \boldsymbol{\Omega}_e/\,||\boldsymbol{\Omega}_e||\,, \mathbf{R}_e/\,||\mathbf{R}_e||\}$ as the reference frame R_2 we obtain

$$\mathbf{A}_e = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \mathbf{L}_{r_1} = [0, b_1, 0]^T,$$

verifying the formula (37) as in the previous case.

Finally, we observe that in the case of two rigid bodies (see (7)) the equilibrium configurations are all the possible combinations between the vector $\mathbf{R}_e$, $\mathbf{P}_e$, and $\boldsymbol{\Omega}_e$ as elements of the reference frame R_2; and the matrix $\mathbf{A}_e$ can be any of the matrices which represent a permutation between the elements of the base. However, in the case considered in this paper, for each choice of the base only special cases of the matrix $\mathbf{A}_e$ can be choosen.

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